Dispersive Quantization — the Talbot Effect

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Hong Kong, December, 2014

Happy 70th, Roderick!

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In appreciation for being my father's greatest fan and supporter . . .

Peter J. Olver

Introduction to Partial Differential Equations

Undergraduate Texts, Springer, 2014

— now in print!

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—, Dispersive quantization, *Amer. Math. Monthly* **117** (2010) 599–610.

Gong Chen & —, Dispersion of discontinuous periodic waves, *Proc. Roy. Soc. London A* **469** (2012), 20120407.

Gong Chen & —, Numerical simulation of nonlinear dispersive quantization, *Discrete Cont. Dyn. Syst. A* **34** (2013), 991–1008.

Dispersion

Definition. A linear partial differential equation is called dispersive if the different Fourier modes travel unaltered but at different speeds.

Substituting

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)}$$

produces the dispersion relation

$$\omega = \omega(k)$$

relating frequency ω and wave number k.

Phase velocity: $$c_p = \frac{\omega(k)}{k}$$

Group velocity: $$c_g = \frac{d\omega}{dk}$$ (stationary phase)

A Simple Linear Dispersive Wave Equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$\Longrightarrow$ linearized Korteweg–deVries equation

Dispersion relation: $\omega = k^3$

Phase velocity: $c_p = \dfrac{\omega}{k} = k^2$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 3\,k^2$

Thus, wave packets (and energy) move *faster* (to the right) than the individual waves.

Linear Dispersion on the Line

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0, x) = f(x)$$

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$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,\mathrm{i}\,(k\,x - k^3\,t)}\, dk$$

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Fundamental solution $\qquad u(0,x) = \delta(x)$

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\,\mathrm{i}\,(k\,x - k^3\,t)}\, dk = \frac{1}{\sqrt[3]{3\,t}}\ \mathrm{Ai}\left(-\,\frac{x}{\sqrt[3]{3\,t}} \right)$$

Linear Dispersion on the Line

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$$u(t,x) = \frac{1}{\sqrt[3]{3t}} \int_{-\infty}^{\infty} f(\xi)\ \mathrm{Ai}\left(\frac{\xi - x}{\sqrt[3]{3t}}\right) d\xi$$

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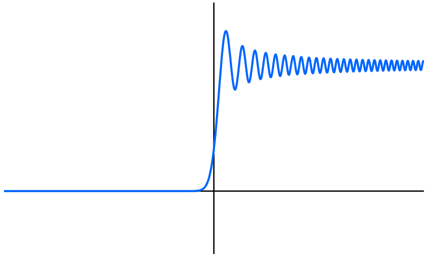
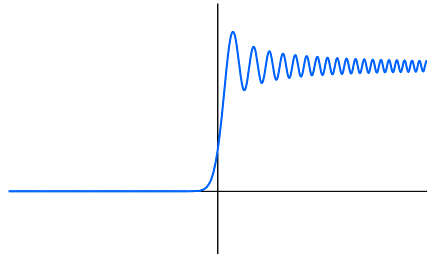
$$u(t,x) = \frac{1}{3} - H\left(- \frac{x}{\sqrt[3]{3t}} \right)$$

$$H(z) = \frac{z\,\Gamma\left(\frac{2}{3}\right)\, {}_1F_2\left(\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; \frac{1}{9} z^3\right)}{3^{5/3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)} - \frac{z^2\,\Gamma\left(\frac{2}{3}\right)\, {}_1F_2\left(\frac{2}{3}; \frac{4}{3}, \frac{5}{3}; \frac{1}{9} z^3\right)}{3^{7/3}\,\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}$$

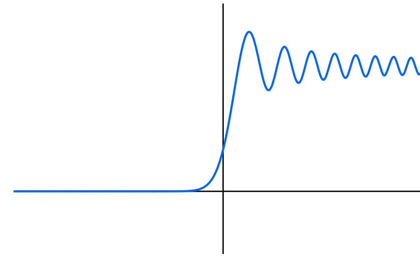
$\Longrightarrow$ Mathematica — via Meijer G functions



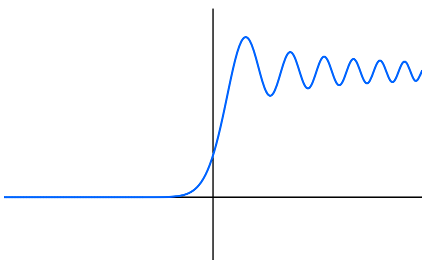
$t = .005$



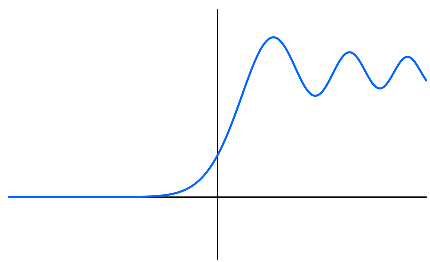
$t = .01$



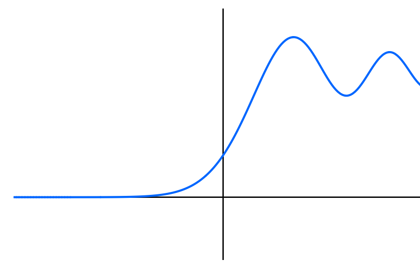
$t = .05$



$t = .1$



$t = .5$



$t = 1.$

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,-\pi) = \frac{\partial^2 u}{\partial x^2}(t,\pi)$$

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Step function initial data:

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

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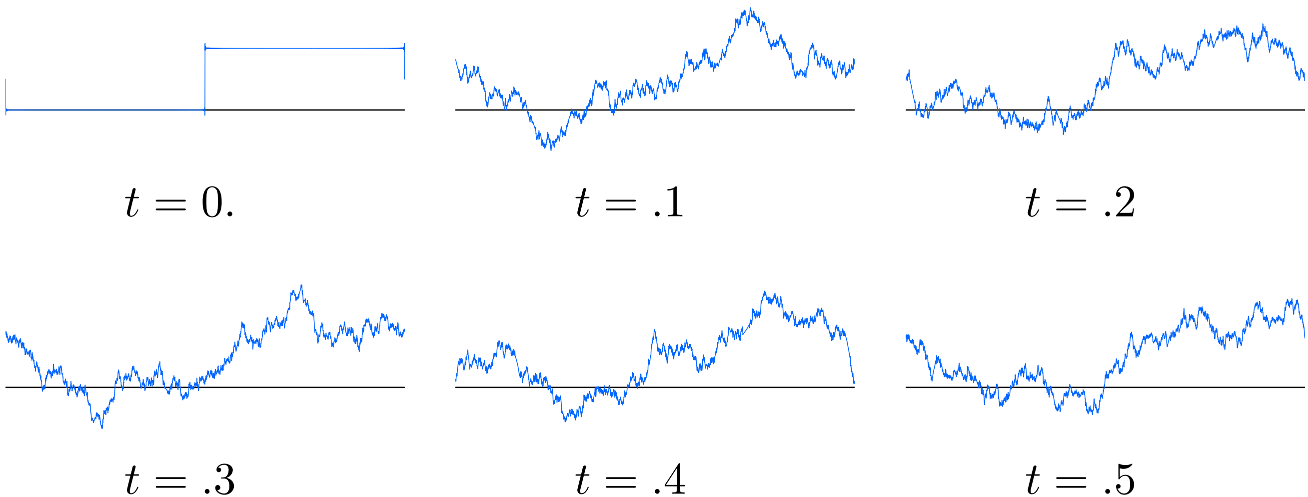
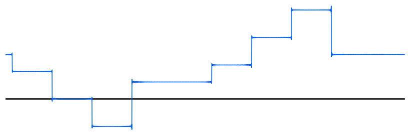
Step function initial data:

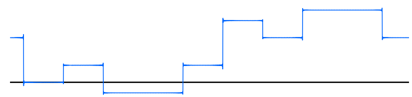
$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

$$u^\star(t,x) \ \sim \ \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin(\,(2\,j+1)\,x - (2\,j+1)^3\,t\,)}{2\,j+1}\,.$$

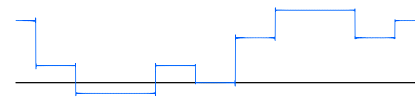

$t = 0.$
$t = .1$
$t = .2$
$t = .3$
$t = .4$
$t = .5$



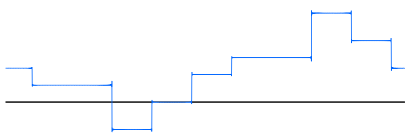
$$t = \frac{1}{30}\pi$$



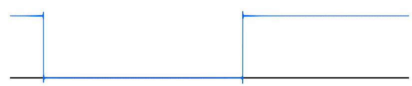
$$t = \frac{1}{15}\pi$$



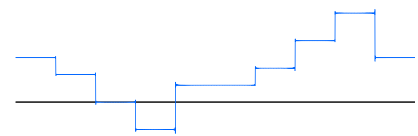
$$t = \frac{1}{10}\pi$$



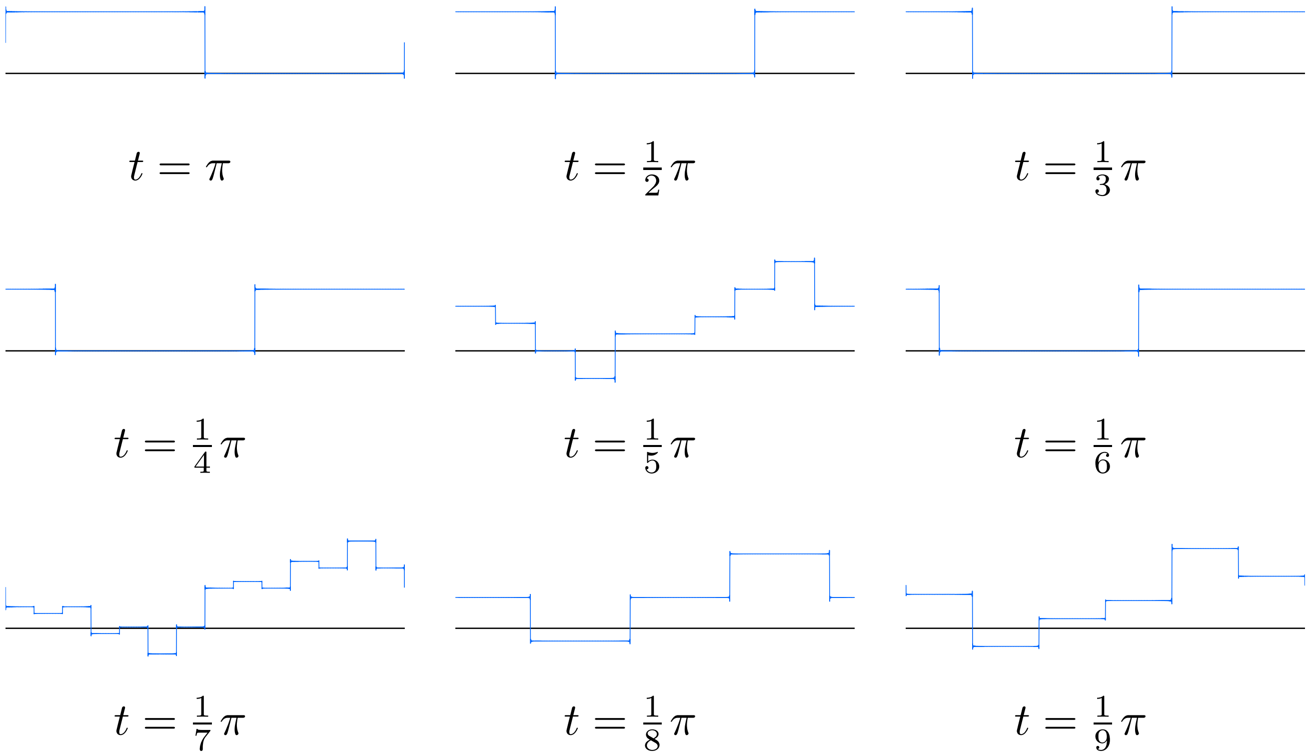
$$t = \frac{2}{15}\pi$$



$$t = \frac{1}{6}\pi$$



$$t = \frac{1}{5}\pi$$


$t = \pi$
$t = \frac{1}{2}\pi$
$t = \frac{1}{3}\pi$
$t = \frac{1}{4}\pi$
$t = \frac{1}{5}\pi$
$t = \frac{1}{6}\pi$
$t = \frac{1}{7}\pi$
$t = \frac{1}{8}\pi$
$t = \frac{1}{9}\pi$

Theorem. At rational time $t = 2\pi p/q$, the solution $u^\star(t,x)$ is constant on every subinterval $2\pi j/q < x < 2\pi(j+1)/q$. At irrational time $u^\star(t,x)$ is a non-differentiable continuous function.

Lemma.

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k \, e^{\,\mathrm{i}\,k\,x}$$

is piecewise constant on intervals $2\pi j/q < x < 2\pi(j+1)/q$ if and only if

$$\widehat{c}_k = \widehat{c}_l, \quad k \equiv l \not\equiv 0 \mod q, \qquad \widehat{c}_k = 0, \quad 0 \neq k \equiv 0 \mod q.$$

where

$$\widehat{c}_k = \frac{2\pi\,k\,c_k}{\mathrm{i}\,q\,(e^{-2\,\mathrm{i}\,\pi\,k/q} - 1)} \qquad k \not\equiv 0 \mod q.$$

$\Longrightarrow$ DFT

The Fourier coefficients of the solution $u^\star(t,x)$ at rational time $t = 2\pi\, p/q$ are

$$c_k = b_k\, e^{-2\pi\,\mathrm{i}\,k^3\,p/q} \tag{$*$}$$

where, for the step function initial data,

$$b_k = \begin{cases} -\,\mathrm{i}\,/(\pi\,k), & k \text{ odd},\\ 1/2, & k = 0,\\ 0, & 0 \neq k \ \text{ even}. \end{cases}$$

Crucial observation: $\boxed{\text{if } k \equiv l \mod q \text{ then } k^3 \equiv l^3 \mod q}$

which implies

$$e^{-2\pi\,\mathrm{i}\,k^3\,p/q} = e^{-2\pi\,\mathrm{i}\,l^3\,p/q}$$

and hence the Fourier coefficients $(*)$ satisfy the condition in the Lemma. *Q.E.D.*

The Fundamental Solution: $F(0, x) = \delta(x)$

Theorem. At rational time $t = 2\pi p/q$, the fundamental solution $F(t, x)$ is a linear combination of finitely many periodically extended delta functions, based at $2\pi j/q$ for integers $-\frac{1}{2}q < j \leq \frac{1}{2}q$.

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Corollary. At rational time, any solution profile $u(2\pi p/q, x)$ to the periodic initial-boundary value problem is a linear combination of $\leq q$ translates of the initial data, namely $f(x + 2\pi j/q)$, and hence its value depends on only finitely many values of the initial data.

★★ The same quantization/fractalization phenomenon appears in any linearly dispersive equation with "integral polynomial" dispersion relation:

$$\omega(k) = \sum_{m=0}^{n} c_m k^m$$

where

$$c_m = \alpha\, n_m \qquad n_m \in \mathbb{Z}$$

Linear Free-Space Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

Dispersion relation: $\omega = k^2$

Phase velocity: $c_p = \dfrac{\omega}{k} = k$

Group velocity: $c_g = \dfrac{d\omega}{dk} = 2\,k$

Periodic Linear Schrödinger Equation

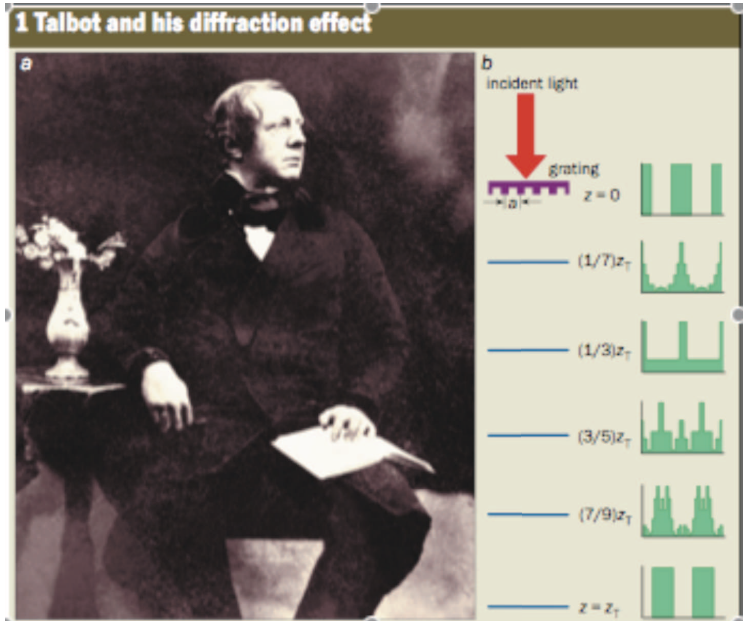
$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi)$$

- Michael Berry, et. al.
- Bernd Thaller, *Visual Quantum Mechanics*
- Oskolkov
- Kapitanski, Rodnianski

  "Does a quantum particle know the time?"
- Michael Taylor

William Henry Fox Talbot (1800–1877)



- ⋆ Talbot’s 1835 image of a latticed window in Lacock Abbey

$\Longrightarrow$ oldest photographic negative in existence.

The Talbot Effect

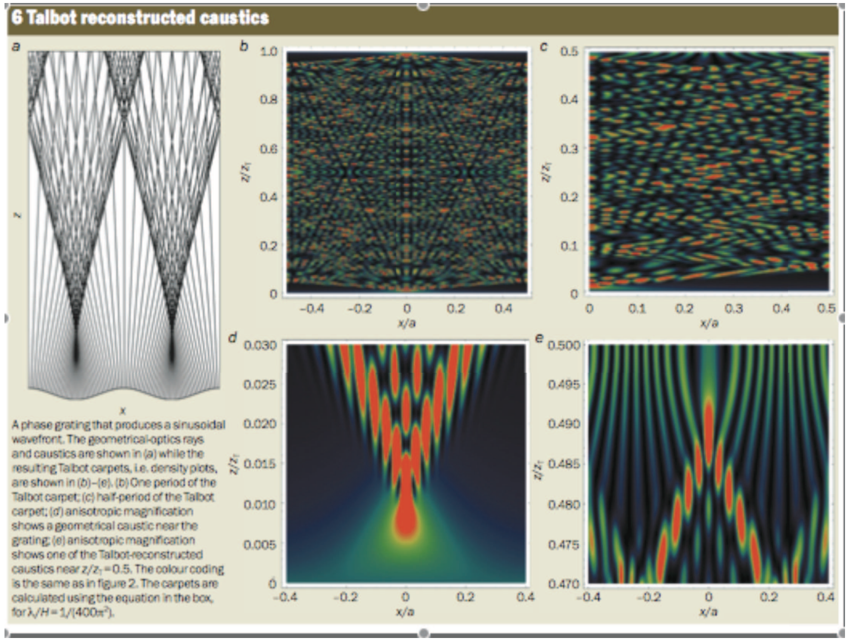
Fresnel diffraction by periodic gratings (1836)

"It was very curious to observe that though the grating was greatly out of the focus of the lens . . . the appearance of the bands was perfectly distinct and well defined . . . the experiments are communicated in the hope that they may prove interesting to the cultivators of optical science."

— *Fox Talbot*

$\Longrightarrow$ Lord Rayleigh calculates the Talbot distance (1881)

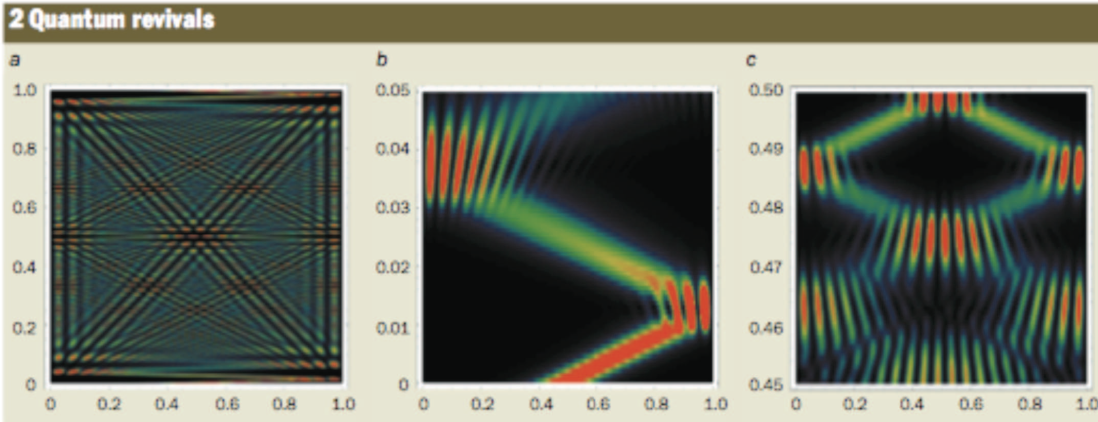
The Quantized/Fractal Talbot Effect



A phase grating that produces a sinusoidal wavefront. The geometrical-optics rays and caustics are shown in (*a*) while the resulting Talbot carpets, i.e. density plots, are shown in (*b*)–(*e*). (*b*) One period of the Talbot carpet; (*c*) half-period of the Talbot carpet; (*d*) anisotropic magnification shows a geometrical caustic near the grating; (*e*) anisotropic magnification shows one of the Talbot-reconstructed caustics near $z/z_T = 0.5$. The colour coding is the same as in figure 2. The carpets are calculated using the equation in the box, for $\lambda/H = 1/(400\pi^2)$.

- Optical experiments — Berry & Klein
- Diffraction of matter waves (helium atoms) — Nowak et. al.

Quantum Revival



- Electrons in potassium ions — Yeazell & Stroud
- Vibrations of bromine molecules — Vrakking, Villeneuve, Stolow

Periodic Linear Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\,\frac{\partial^2 u}{\partial x^2}$$

$$u(t,-\pi) = u(t,\pi) \qquad\qquad \frac{\partial u}{\partial x}(t,-\pi) = \frac{\partial u}{\partial x}(t,\pi)$$

Integrated fundamental solution:

$$u(t,x) = \frac{1}{2\,\pi} \sum_{0\neq k=-\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x-k^2t)}}{k}\,.$$

For $x/t \in \mathbb{Q}$, this is known as a Gauss (or, more generally, Weyl) sum, of importance in number theory

$\star\star$ Riemann Hypothesis!

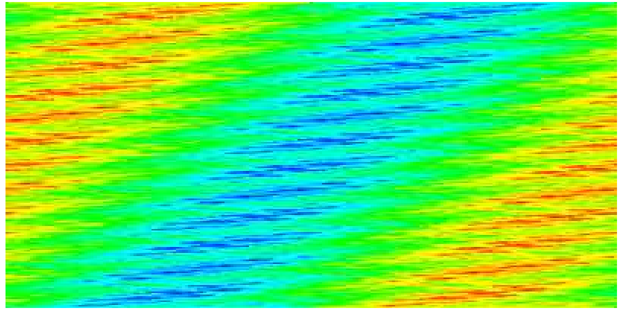
$\Longrightarrow$ Hardy, Littlewood, Weil, I. Vinogradov, etc.

Integrated fundamental solution:

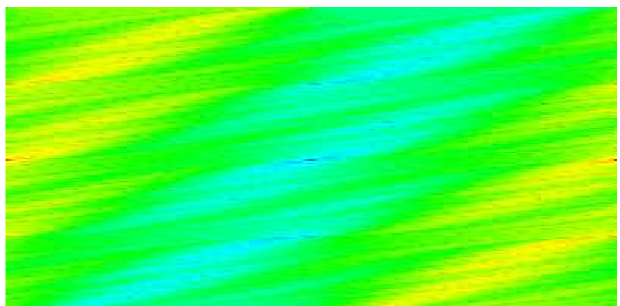
$$u(t,x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x - k^2 t)}}{k}\,.$$

Theorem.

- The fundamental solution $\partial u / \partial x$ is a Jacobi theta function. At rational times $t = 2\pi p/q$, it linear combination of delta functions concentrated at rational nodes $x_j = 2\pi j/q$.
- At irrational times t, the integrated fundamental solution is a continuous but nowhere differentiable function. (*Claim*: The fractal dimension of its graph is $\frac{3}{2}$.)



Dispersive Carpet



Schrödinger Carpet

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = L(D_x)\, u, \qquad u(t, x + 2\pi) = u(t, x)$$

Dispersion relation:

$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)} \quad \Longrightarrow \quad \omega(k) = -\,\mathrm{i}\,L(-\,\mathrm{i}\,k) \qquad$ assumed real

Riemann problem: step function initial data

$$u(0,x) = \sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

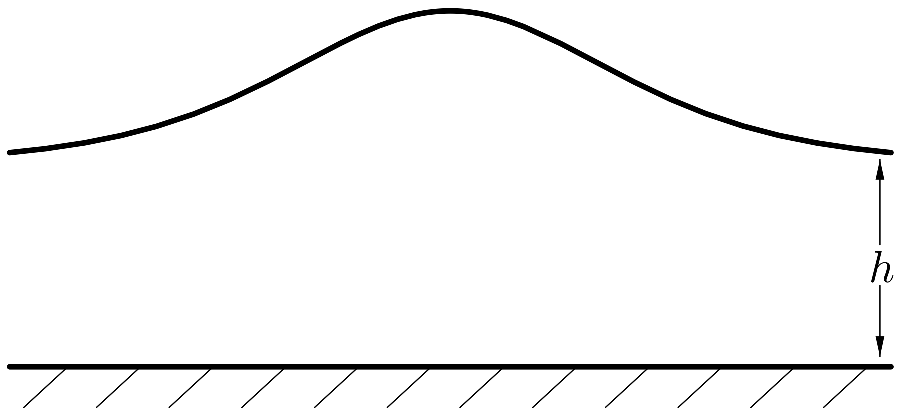
Solution:

$$u(t,x) \;\sim\; \frac{1}{2} + \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin[\,(2\,j+1)\,x - \omega(k)\,t\,]}{2\,j+1}\,.$$

$\star\star\;\; \omega(-\,k) = -\,\omega(k)$ odd

Polynomial dispersion, rational $t \;\Longrightarrow\;$ Weyl exponential sums

2D Water Waves



$y = h + \eta(t, x)$ surface elevation

$\phi(t, x, y)$ velocity potential

2D Water Waves

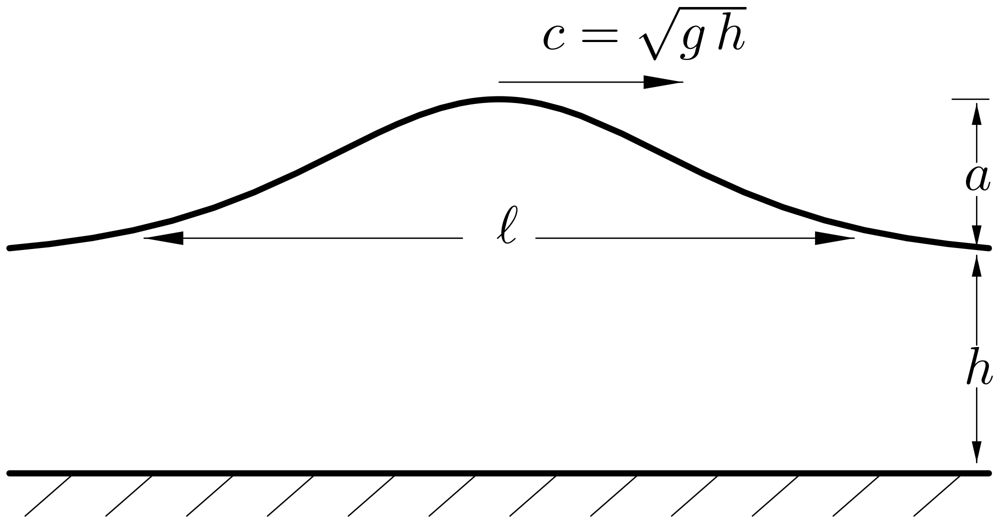
- Incompressible, irrotational fluid.
- No surface tension

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}\,\phi_x^2 + \tfrac{1}{2}\,\phi_y^2 + g\,\eta = 0 \\ \eta_t = \phi_y - \eta_x\phi_x \end{aligned}\right\} \qquad y = h + \eta(t,x)$$

$$\phi_{xx} + \phi_{yy} = 0 \qquad\qquad 0 < y < h + \eta(t,x)$$

$$\phi_y = 0 \qquad\qquad y = 0$$

- Wave speed (maximum group velocity): $\quad c = \sqrt{g\,h}$
- Dispersion relation: $\quad \sqrt{g\,k\,\tanh(h\,k)} = c\,k - \frac{1}{6}\,c\,h^2k^3 + \;\cdots$



Small parameters — long waves in shallow water (KdV regime)

$$\alpha = \frac{a}{h} \qquad\qquad \beta = \frac{h^2}{\ell^2} = O(\alpha)$$

Rescale:

$$x \longmapsto \ell\, x \qquad y \longmapsto h\, y \qquad t \longmapsto \frac{\ell\, t}{c}$$

$$\eta \longmapsto a\, \eta \qquad \phi \longmapsto \frac{g\, a\, \ell\, \phi}{c} \qquad c = \sqrt{g\, h}$$

Rescaled water wave system:

$$\left.\begin{aligned} \phi_t + \frac{\alpha}{2}\, \phi_x^2 + \frac{\alpha}{2\,\beta}\, \phi_y^2 + \eta = 0 \\ \eta_t = \frac{1}{\beta}\, \phi_y - \alpha\, \eta_x\, \phi_x \end{aligned}\right\} \qquad y = 1 + \alpha\, \eta$$

$$\beta\, \phi_{xx} + \phi_{yy} = 0 \qquad 0 < y < 1 + \alpha\, \eta$$

$$\phi_y = 0 \qquad y = 0$$

Boussinesq expansion

Set

$$\psi(t,x) = \phi(t,x,0) \qquad u(t,x) = \phi_x(t,x,\theta) \qquad 0 \le \theta \le 1$$

Solve Laplace equation:

$$\phi(t,x,y) = \psi(t,x) - \tfrac{1}{2}\beta^2 y^2 \psi_{xx} + \tfrac{1}{4!}\beta^4 y^4 \psi_{xxxx} + \cdots$$

Plug expansion into free surface conditions: To first order

$$\psi_t + \eta + \tfrac{1}{2}\alpha\,\psi_x^2 - \tfrac{1}{2}\beta\,\psi_{xxt} = 0$$

$$\eta_t + \psi_x + \alpha\,(\eta\psi_x)_x - \tfrac{1}{6}\beta\,\psi_{xxxx} = 0$$

Bidirectional Boussinesq systems:

$$u_t + \eta_x + \alpha\, u\, u_x - \tfrac{1}{2}\beta\,(\theta^2 - 1)\, u_{xxt} = 0$$

$$\eta_t + u_x + \alpha\,(\eta\, u)_x - \tfrac{1}{6}\beta\,(3\,\theta^2 - 1)\, u_{xxx} = 0$$

$\star\star$ at $\theta = 1$ this system is integrable

$\Longrightarrow$ Kaup, Kupershmidt

Boussinesq equation

$$u_{tt} = u_{xx} + \tfrac{1}{2}\alpha\,(u^2)_{xx} - \tfrac{1}{6}\beta\, u_{xxxx}$$

Regularized Boussinesq equation

$$u_{tt} = u_{xx} + \tfrac{1}{2}\alpha\,(u^2)_{xx} - \tfrac{1}{6}\beta\, u_{xxtt}$$

$\Longrightarrow$ DNA dynamics (Scott)

Unidirectional waves:

$$u = \eta - \tfrac{1}{4}\,\alpha\,\eta^2 + \left(\tfrac{1}{3} - \tfrac{1}{2}\,\theta^2\right)\beta\,\eta_{xx} + \;\cdots$$

Korteweg-deVries (1895) equation:

$$\eta_t + \eta_x + \tfrac{3}{2}\,\alpha\,\eta\,\eta_x + \tfrac{1}{6}\,\beta\,\eta_{xxx} = 0$$

$\Longrightarrow$ Due to Boussinesq in 1877!

Benjamin–Bona–Mahony (BBM) equation:

$$\eta_t + \eta_x + \tfrac{3}{2}\,\alpha\,\eta\,\eta_x - \tfrac{1}{6}\,\beta\,\eta_{xxt} = 0$$

Shallow Water Dispersion Relations

Water waves	$\pm\sqrt{k\,\tanh k}$
Boussinesq system	$\pm\dfrac{k}{\sqrt{1+\frac{1}{3}k^2}}$
Boussinesq equation	$\pm k\sqrt{1+\frac{1}{3}k^2}$
Korteweg–deVries	$k-\frac{1}{6}k^3$
BBM	$\dfrac{k}{1+\frac{1}{6}k^2}$

Dispersion Asymptotics

⋆ The qualitative behavior of the solution to the periodic problem depends crucially on the asymptotic behavior of the dispersion relation $\omega(k)$ for large wave number $k \to \pm\infty$.

$$\omega(k) \sim k^{\alpha}$$

- $\alpha = 0$ — large scale oscillations
- $0 < \alpha < 1$ — dispersive oscillations
- $\alpha = 1$ — traveling waves
- $1 < \alpha < 2$ — oscillatory becoming fractal
- $\alpha \geq 2$ — fractal/quantized

Periodic Korteweg–deVries equation

$$\frac{\partial u}{\partial t} = \alpha\, \frac{\partial^3 u}{\partial x^3} + \beta\, u\, \frac{\partial u}{\partial x} \qquad u(t, x + 2\ell) = u(t, x)$$

Zabusky–Kruskal (1965)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \cos \pi x.$$

Lax–Levermore (1983) — small dispersion

$$\alpha \longrightarrow 0, \qquad \beta = 1.$$

Gong Chen (2011)

$$\alpha = 1, \qquad \beta = .000484, \qquad \ell = 1, \qquad u(0, x) = \sigma(x).$$

Periodic Korteweg–deVries Equation

Analysis of nonsmooth initial data:

Estimates, existence, well-posedness, stability, . . .

- Kato
- Bourgain
- Kenig, Ponce, Vega
- Colliander, Keel, Staffilani, Takaoka, Tao
- Oskolkov
- D. Russell, B–Y Zhang
- Erdoğan, Tzirakis

Operator Splitting

$$u_t = \alpha\, u_{xxx} + \beta\, u u_x = L[u] + N[u]$$

Flow operators: $\Phi_L(t), \quad \Phi_N(t)$

Godunov scheme:

$$u_\Delta^G(t_n) \simeq \left(\, \Phi_L(\Delta t)\, \Phi_N(\Delta t)\,\right)^n u_0$$

Strang scheme:

$$u_\Delta^S(t_n) \simeq \left(\, \Phi_N(\tfrac{1}{2}\Delta t\,)\, \Phi_L(\Delta t)\, \Phi_N(\tfrac{1}{2}\Delta t\,)\,\right)^n u_0$$

Numerical implementation:

- FFT for Φ_L — linearized KdV
- FFT + convolution for Φ_N — conservative version of inviscid Burgers', using Backward Euler + fixed point iteration to overcome mild stiffness. Shock dynamics doesn't complicate due to small time stepping.

Convergence of Operator Splitting

★ Holden, Karlsen, Risebro and Tao prove:

First order convergence of the Godunov scheme

$$u_\Delta^G(t_n) \simeq \left(\Phi_L(\Delta t)\, \Phi_N(\Delta t) \right)^n u_0$$

for initial data $u_0 \in H^s$ for $s \geq 5$:

$$\| u(t_n) - u_\Delta^G(t_n) \| \leq C\, \Delta t$$

Second order convergence of the Strang scheme

$$u_\Delta^S(t_n) \simeq \left(\Phi_N(\tfrac{1}{2}\Delta t)\, \Phi_L(\Delta t)\, \Phi_N(\tfrac{1}{2}\Delta t) \right)^n u_0$$

for initial data $u_0 \in H^s$ for $s \geq 17$:

$$\| u(t_n) - u_\Delta^S(t_n) \| \leq C\, (\Delta t)^2$$

Convergence for Rough Data?

However, subtle issues prevent us from establishing convergence of the operator splitting method for rough initial data.

- Bourgain proves well-posedness of the periodic KdV flow in L^2
- Conservation of the L^2 norm establishes well-posedness in L^2 of the linearized flow Φ_L
- Thus, if the solution has bounded L^∞ norm, then the linearized flow is L^1 contractive
- Oskolkov proves that is the initial data is has bounded BV norm, then the resulting solution to the periodic linearized KdV equation is uniformly bounded in L^∞
- Unfortunately, Oskolkov's bound depends on the BV and L^∞ norms of the initial data. Moreover, at irrational times, the solution is nowhere differentiable and has unbounded BV norm
- Also, we do not have good control of the BV norm of the nonlinear inviscid Burgers' flow Φ_N
- ????

Periodic Nonlinear Schrödinger Equation

$$\mathrm{i}\, u_t + u_{xx} + |\, u\,|^p\, u = 0, \qquad x \in \mathbb{R}/2\pi\mathbb{Z}$$

$$u(0,x) = g(x)$$

Theorem. (Erdoğan, Tzirakis) Suppose $p = 2$ and $g \in \mathrm{BV}$.

(*i*) $u(t,\cdot)$ is continuous at irrational times $t/(2\pi) \notin \mathbb{Q}$

(*ii*) $u(t,\cdot)$ is bounded with at most countably many discontinuities at rational times $t \notin \mathbb{Q}$

(*iii*) When the initial data is sufficiently "rough", $g \notin \cup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real or imaginary part of the graph of $u(t,\cdot)$ has fractal (upper Minkowski) dimension $\frac{3}{2}$.

Periodic Linear Dispersive Equations

$\Longrightarrow$ Chousionis, Erdoğan, Tzirakis

Theorem. Suppose $3 \leq k \in \mathbb{Z}$ and $g \in \mathrm{BV}$:

$$\mathrm{i}\, u_t + (-\mathrm{i}\, \partial_x)^k u = 0, \qquad x \in \mathbb{R}/2\pi\mathbb{Z}, \qquad u(0,x) = g(x) \in \mathrm{BV}$$

(i) $u(t,\cdot)$ is continuous for almost all t

(ii) When $g \notin \cup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real and imaginary parts of the graph of $u(t,\cdot)$ has fractal dimension $1 + 2^{1-k} \leq D \leq 2 - 2^{1-k}$.

Theorem. For the periodic Korteweg–deVries equation

$$u_t + u_{xxx} + u\, u_x = 0, \qquad x \in \mathbb{R}/2\pi\mathbb{Z}, \qquad u(0,x) = g(x) \in \mathrm{BV}$$

(i) $u(t,\cdot)$ is continuous for almost all t

(ii) When $g \notin \cup_{\epsilon>0} H^{1/2+\epsilon}$, then, at almost all t, the real and imaginary parts of the graph of $u(t,\cdot)$ has fractal dimension $\frac{5}{4} \leq D \leq \frac{7}{4}$.

Vortex Filament Equation

a.k.a. Localized Induction Approximation (LIA) or binormal flow

$\Longrightarrow$ Da Rios (1906)

$$\boxed{\gamma_t = \gamma_s \times \gamma_{ss} = \kappa\, \mathbf{b}}$$

$\gamma(t,s) \;\in\; \mathbb{R}^3$ at time t represents the vortex filament — a space curve parametrized by arc length — that moves in an incompressible fluid flow with vorticity concentrated on the filament.

Frenet frame: $\mathbf{t}, \mathbf{n}, \mathbf{b}$ — unit tangent, normal, binormal

κ — curvature $\qquad\qquad$ τ — torsion

$$\boxed{\gamma_t = \gamma_s \times \gamma_{ss}}$$

Hasimoto transformation:

$$u = \kappa \exp\left(\mathrm{i} \int \tau \, ds \right)$$

solves the integrable nonlinear Schrödinger equation:

$$\mathrm{i}\, u_t = u_{xx} + |\, u \,|^2 \, u$$

de la Hoz and Vega (2013): If the initial data is a closed polygon, then at rational times the curve is a polygon, whereas at irrational times it is a fractal.

Chousionis, Erdoğan, Tzirakis (2014): further results on fractal behavior for some smooth initial data

Future Directions

- General dispersion behavior
- Other boundary conditions (Fokas' Method)
- Higher space dimensions and other domains (tori, spheres, . . .)
- Dispersive nonlinear partial differential equations
- Discrete systems: Fermi–Pasta–Ulam
- Numerical solution techniques?
- Experimental verification in dispersive media?