Equivalence and Invariants: an Overview

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The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two p-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Classical Geometry — *F. Klein*

- **Euclidean group:** $$G = \begin{cases} \mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m \\ \mathrm{E}(m) = \mathrm{O}(m) \ltimes \mathbb{R}^m \end{cases}$$

  $\boxed{z \longmapsto A \cdot z + b}$ $\quad A \in \mathrm{SO}(m)$ or $\mathrm{O}(m)$, $\quad b \in \mathbb{R}^m$, $\quad z \in \mathbb{R}^m$

  $\Rightarrow$ isometries: rotations, translations , (reflections)

- **Equi-affine group:** $G = \mathrm{SA}(m) = \mathrm{SL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{SL}(m)$ — volume-preserving

- **Affine group:** $G = \mathrm{A}(m) = \mathrm{GL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{GL}(m)$

- **Projective group:** $G = \mathrm{PSL}(m+1)$

  acting on $\mathbb{R}^m \subset \mathbb{RP}^m$

  $\Longrightarrow$ Applications in computer vision

Tennis, Anyone?

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

- multiplier representation of GL(2)
- modular forms

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)$$

Transformation group:

$$g\colon (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}\,,\ \frac{u}{(\gamma x + \delta)^n}\right)$$

Equivalence of functions $\iff$ equivalence of graphs

$$\Gamma_Q = \{\, (x, u) = (x, Q(x)) \,\} \subset \mathbb{C}^2$$

Calculus of Variations

$$\int L(x,u,p)dx \quad \Longrightarrow \quad \int \overline{L}(\bar{x},\bar{u},\bar{p})d\bar{x}$$

Standard Equivalence:

$$L = \overline{L}\, D_x\bar{x} = \overline{L}\, \left(\frac{\partial \bar{x}}{\partial x} + p\, \frac{\partial \bar{x}}{\partial u} \right)$$

Divergence Equivalence:

$$L = \overline{L}\, D_x\bar{x} + D_x B$$

Allowed Changes of Variables

$\Longrightarrow$ Lie pseudo-groups

- Fiber-preserving transformations

$$\bar{x} = \varphi(x) \qquad \bar{u} = \psi(x,u) \qquad \bar{p} = \chi(x,u,p) = \frac{\alpha\, p + \beta}{\delta}$$

- Point transformations

$$\bar{x} = \varphi(x,u) \qquad \bar{u} = \psi(x,u) \qquad \bar{p} = \chi(x,u,p) = \frac{\alpha\, p + \beta}{\gamma\, p + \delta}$$

$$\alpha = \frac{\partial \varphi}{\partial u} \qquad \beta = \frac{\partial \varphi}{\partial x} \qquad \gamma = \frac{\partial \varphi}{\partial u} \qquad \delta = \frac{\partial \varphi}{\partial x}$$

- Contact transformations

$$\bar{x} = \varphi(x,u,p) \qquad \bar{u} = \psi(x,u,p) \qquad \bar{p} = \chi(x,u,p)$$

$$d\bar{u} - \bar{p}\, d\bar{x} = \lambda(du - p\, dx) \qquad \lambda \neq 0$$

Ordinary Differential Equations

$$\frac{d^2u}{dx^2} = F\left(x, u, \frac{du}{dx}\right) \quad \Longrightarrow \quad \frac{d^2\bar{u}}{d\bar{x}^2} = \overline{F}\left(\bar{x}, \bar{u}, \frac{d\bar{u}}{d\bar{x}}\right)$$

$\Longrightarrow$ Reduce an equation to a solved form,

e.g., linearization, Painlevé, ...

Control Theory

$$\frac{dx}{dt} = F(t, x, u) \quad \Longrightarrow \quad \frac{d^2\bar{x}}{dt^2} = \overline{F}\,(\,t, \bar{x}, \bar{u}\,)$$

Equivalence map: $\bar{x} = \varphi(x)$ $\bar{u} = \psi(x, u)$

$\Longrightarrow$ Feedback linearization, normal forms, ...

Differential Operators

$$\mathcal{D} = \sum_{i=0}^{n} a_i(x) D^i$$

- Linear o.d.e.: $\mathcal{D}[u] = 0$
- Eigenvalue problem: $\mathcal{D}[u] = \lambda\, u$
- Evolution or Schrödinger equation: $u_t = \mathcal{D}[u]$

Equivalence map: $\bar{x} = \varphi(x) \qquad \bar{u} = \psi(x) u \qquad \overline{\mathcal{D}} = \begin{cases} \mathcal{D} \cdot \psi \\[1ex] \dfrac{1}{\psi} \cdot \mathcal{D} \cdot \psi \end{cases}$

$\Longrightarrow$ exactly and quasi-exactly solvable quantum operators, ...

Equivalence & Invariants

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  must have the same invariants: $\; I = \bar{I}$.

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Constant invariants provide immediate information:

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Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

However, a functional dependency or syzygy among the invariants *is* intrinsic:

e.g. $$\kappa_s = \kappa^3 - 1 \qquad \Longleftrightarrow \qquad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

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$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

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- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Theorem. (Cartan) Two submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

$$C \subset \mathbb{R}^2$$

G — transitive, ordinary Lie group action (no pseudo-stabilization)

κ — unique (up to functions thereof) differential invariant of lowest order — curvature

ds — unique (up to multiple) contact-invariant one-form of lowest order — arc length element

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Theorem. Every differential invariant of plane curves under ordinary Lie group actions is a function of the curvature invariant and its derivatives with respect to arc length:

$$I = F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots \ , \kappa_m)$$

Orbits

If κ is constant, then all the higher order differential invariants are also constant:

$$\kappa = c, \qquad\qquad 0 = \kappa_s = \kappa_{ss} = \;\cdots$$

Theorem. κ is constant if and only if the curve is a (segment of) an orbit of a one-parameter subgroup.

- Euclidean plane geometry: $G = \mathrm{E}(2)$ — circles, lines
- Equi-affine plane geometry: $G = \mathrm{SA}(2)$ — conic sections
- Projective plane geometry: $G = \mathrm{PSL}(2)$ — W curves (*Lie & Klein*)

Suppose κ is *not* constant, and assume $\kappa_s \neq 0$.

Then every syzygy is, locally, equivalent to one of the form

$$\frac{d^m \kappa}{ds^m} = H_m(\kappa) \qquad m = 1, 2, 3, \ \ldots$$

★ ★ If we know

$$\kappa_s = H_1(\kappa) \tag{$*$}$$

then we can determine all higher order syzygies:

$$\kappa_{ss} = \frac{d}{ds} H_1(\kappa) = H_1'(\kappa)\,\kappa_s = H_1'(\kappa)\,H_1(\kappa) \equiv H_2(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy

$$\kappa_s = H_1(\kappa) \qquad (*)$$

★★ For plane curves under an ordinary transformation group, we need only know a single syzygy between κ and κ_s in order to establish equivalence!

Reconstruction

When $H_1 \not\equiv 0$, the generating syzygy equation

$$\kappa_s = H_1(\kappa) \tag{$*$}$$

is an example of an automorphic differential equation, meaning that it admits G as a symmetry group, and, moreover, all solutions are obtained by applying group transformations to a single fixed solution: $u = g \cdot u_0$

$\Longrightarrow$ Rob Thompson's 2013 thesis.

Example. The Euclidean syzygy equation

$$\kappa_s = H_1(\kappa) \tag{$*$}$$

is the following third order ordinary differential equation:

$$\frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3} = H_1\left(\frac{u_{xx}}{(1+u_x^2)^{3/2}}\right)$$

It admits $G = \mathrm{SE}(2)$ as a symmetry group.

If $H_1 \not\equiv 0$, then given any one solution $u = f_0(x)$, every other solution is obtained by applying a rigid motion to its graph.

On the other hand, if $H_1 \equiv 0$, then the solutions are all the circles and straight lines, being the graphs of one-parameter subgroups.

Question for the audience: SE(2) is a 3 parameter Lie group, but the initial data $(x^0, u^0, u_x^0, u_{xx}^0)$ for $(*)$ depends upon 4 arbitrary constants. Reconcile these numbers.

The Signature Map

In general, the generating syzygies are encoded by the signature map

$$\sigma : N \quad \longrightarrow \quad \mathbb{R}^l$$

of the submanifold N, which is parametrized by a finite collection of fundamental differential invariants:

$$\sigma(x) = (I_1(x), \ldots, I_l(x))$$

The image

$$\Sigma = \operatorname{Im} \sigma \subset \mathbb{R}^l$$

is the signature subset (or submanifold) of N.

Equivalence & Signature

Theorem. Two regular submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\overline{\Sigma} = \Sigma$$

Signature Curves

Definition. The *signature curve* $\Sigma \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ under an ordinary transformation group G is parametrized by the two lowest order differential invariants:

$$\Sigma = \left\{ \left(\kappa \,,\, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

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$$\Sigma = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Theorem. Two regular curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C} \qquad \text{for} \qquad g \in G$$

if and only if their signature curves are identical:

$$\overline{\Sigma} = \Sigma$$

Other Signatures

Euclidean space curves: $\mathcal{C} \subset \mathbb{R}^3$

- κ — curvature, τ — torsion

$$\Sigma = \{\, (\,\kappa\,,\,\kappa_s\,,\,\tau\,)\,\} \quad \subset \quad \mathbb{R}^3$$

Euclidean surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

- H — mean curvature, K — Gauss curvature

$$\Sigma = \left\{ \left(H\,,\,K\,,\,H_{,1}\,,\,H_{,2}\,,\,K_{,1}\,,\,K_{,2} \right) \right\} \quad \subset \quad \mathbb{R}^6$$

$$\widetilde{\Sigma} = \left\{ \left(H\,,\,H_{,1}\,,\,H_{,2}\,,\,H_{,1,1} \right) \right\} \quad \subset \quad \mathbb{R}^4$$

Equi–affine surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

- P — Pick invariant

$$\Sigma = \left\{ \left(P\,,\,P_{,1}\,,\,P_{,2},\,P_{,1,1} \right) \right\} \quad \subset \quad \mathbb{R}^3$$

Symmetry and Signature

Theorem. The dimension of the (local) symmetry group

$$G_N = \{\, g \mid g \cdot N = N \,\}$$

of a nonsingular submanifold $N \subset M$ equals the codimension of its signature:

$$\dim G_N \;=\; \dim N - \dim \Sigma$$

Corollary. For a nonsingular submanifold $N \subset M$,

$$0 \;\leq\; \dim G_N \;\leq\; \dim N$$

$\Longrightarrow$ Totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature Σ degenerates to a point: $\dim \Sigma = 0$
- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a (nonsingular) p-dimensional subgroup $H \subset G$

Discrete Symmetries

Definition. The index of a submanifold N equals the number of points in N which map to a generic point of its signature:

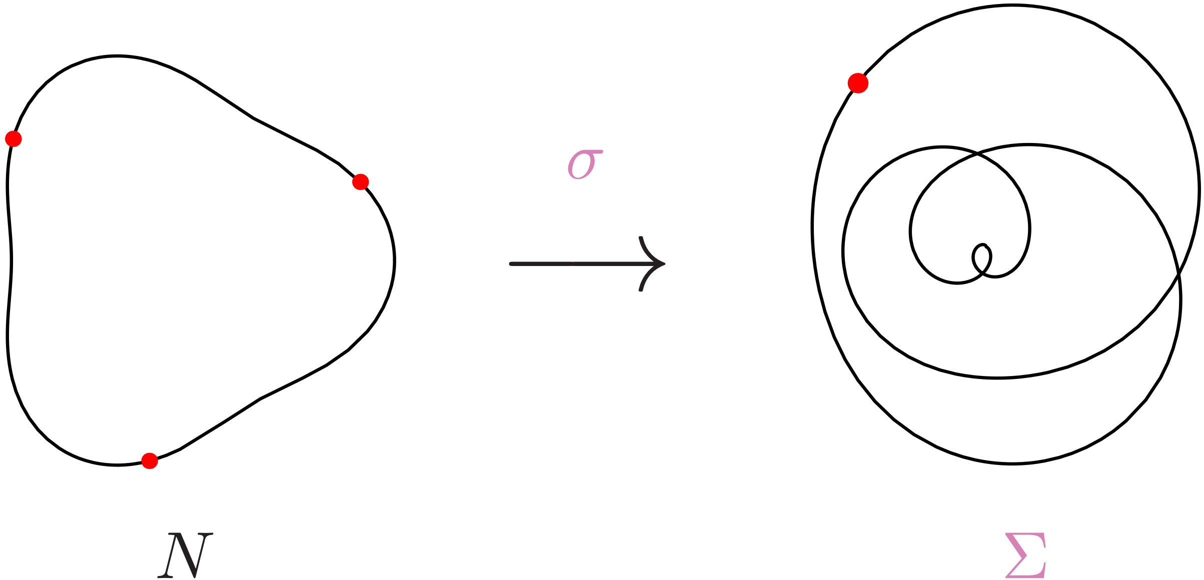
$$\iota_N = \min\left\{ \,\#\,\sigma^{-1}\{w\} \;\middle|\; w \in \Sigma \,\right\}$$

$\Longrightarrow$ Self–intersections

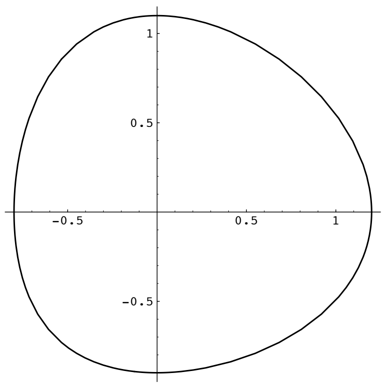
Theorem. The number of local symmetries of a submanifold at a generic point $z \in N$ equals its index ι_z.

$\Longrightarrow$ Approximate symmetries

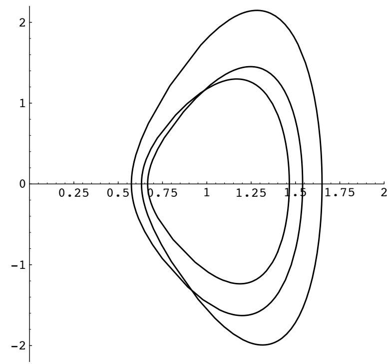
The Index



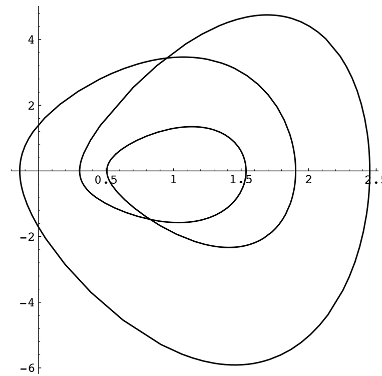
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

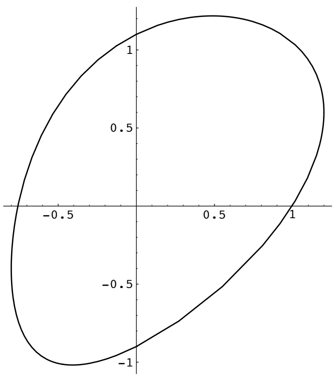


Euclidean Signature

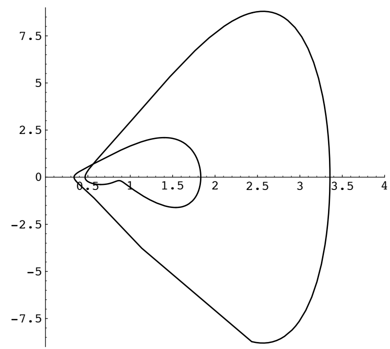


Equi-affine Signature

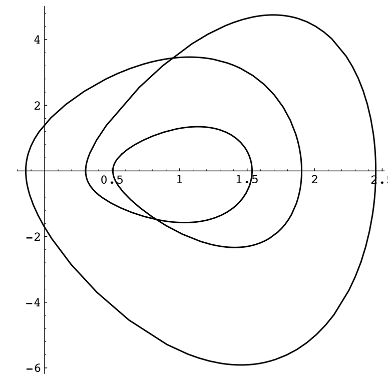
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

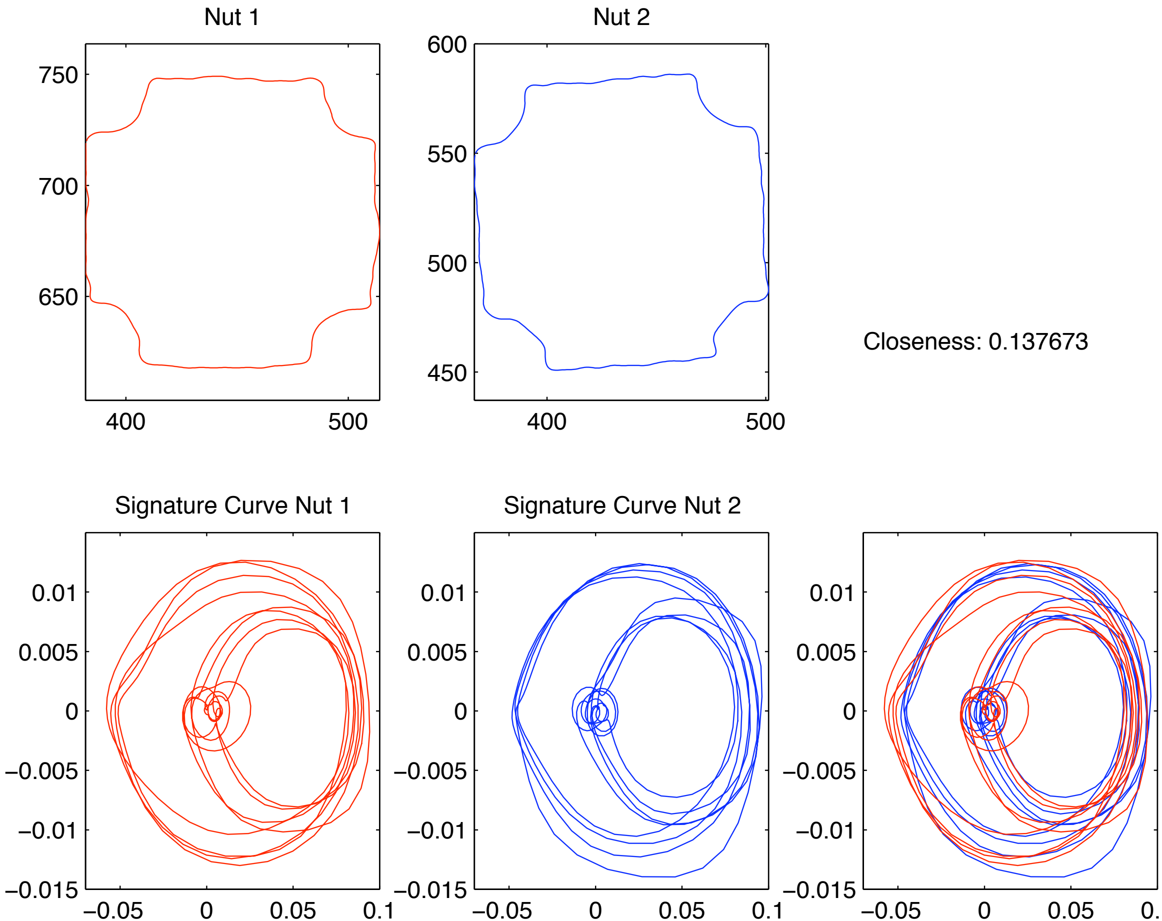


Euclidean Signature



Equi-affine Signature

$\Longrightarrow$ Steve Haker


Nut 1
Nut 2
750
700
650
400
500
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
0.01
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0
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−0.01
−0.015
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0
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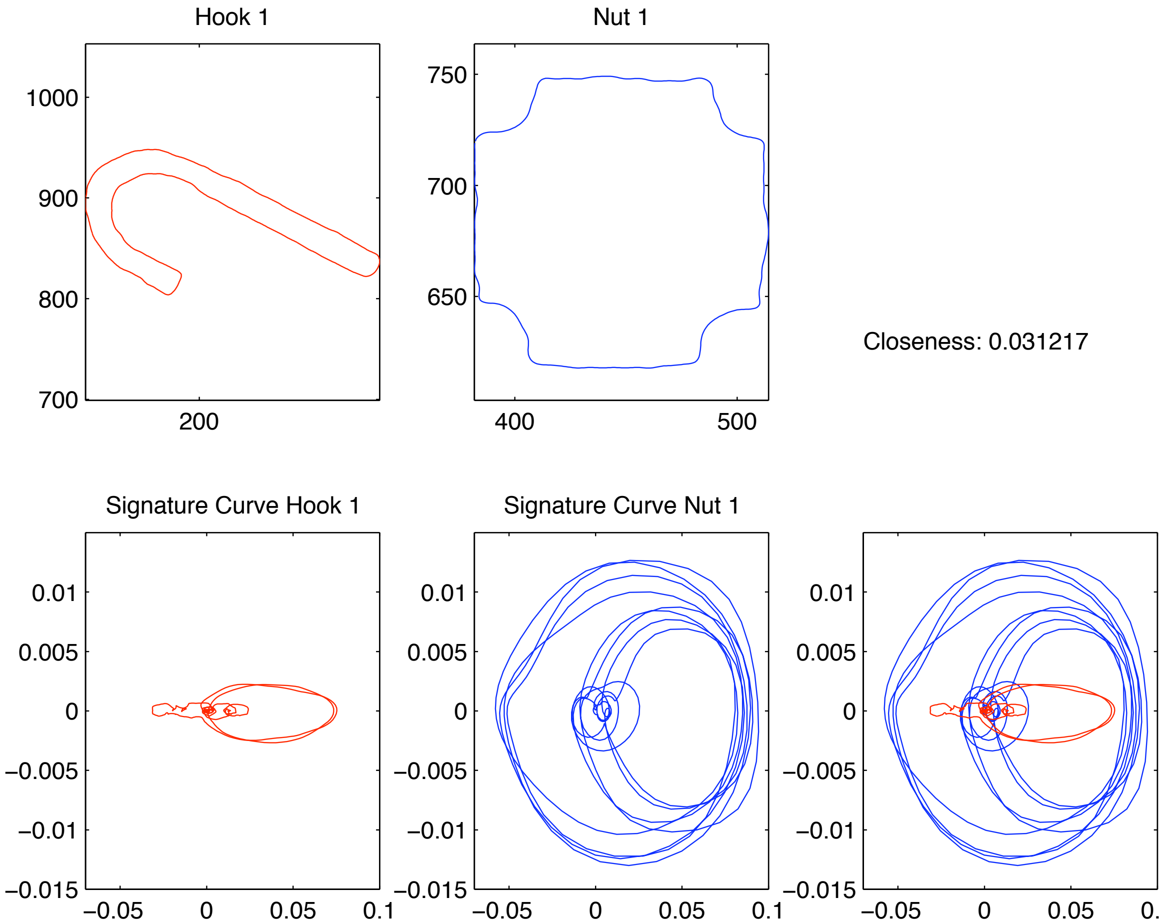
Hook 1
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1000
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Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
0.01
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−0.005
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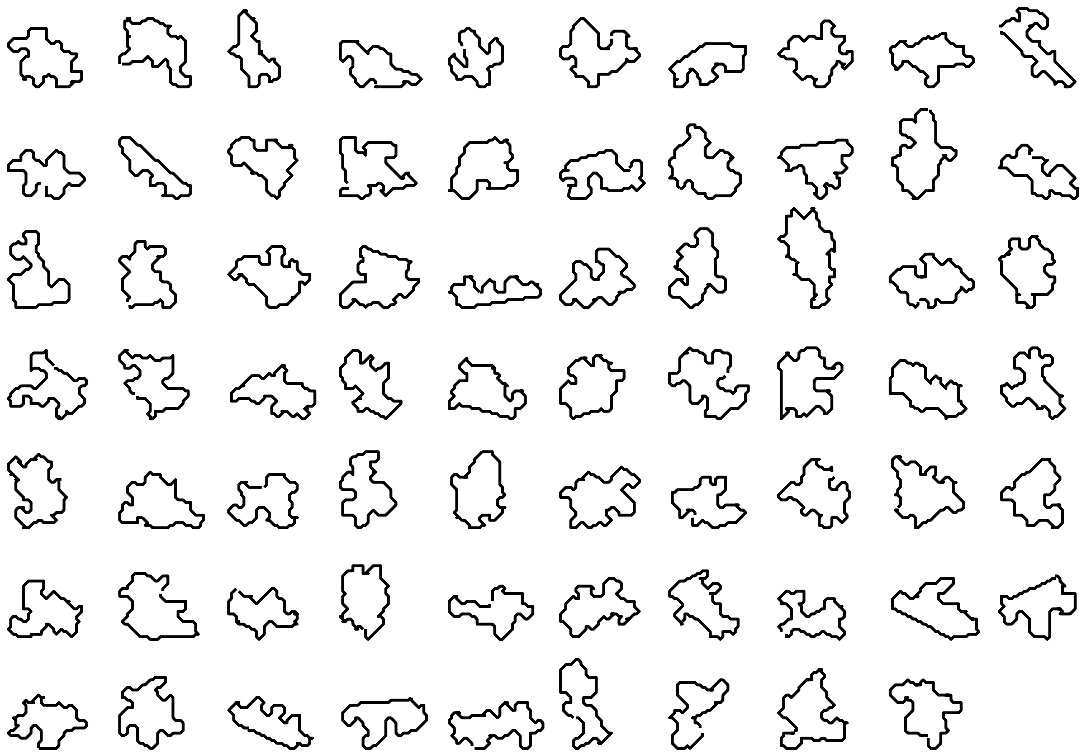
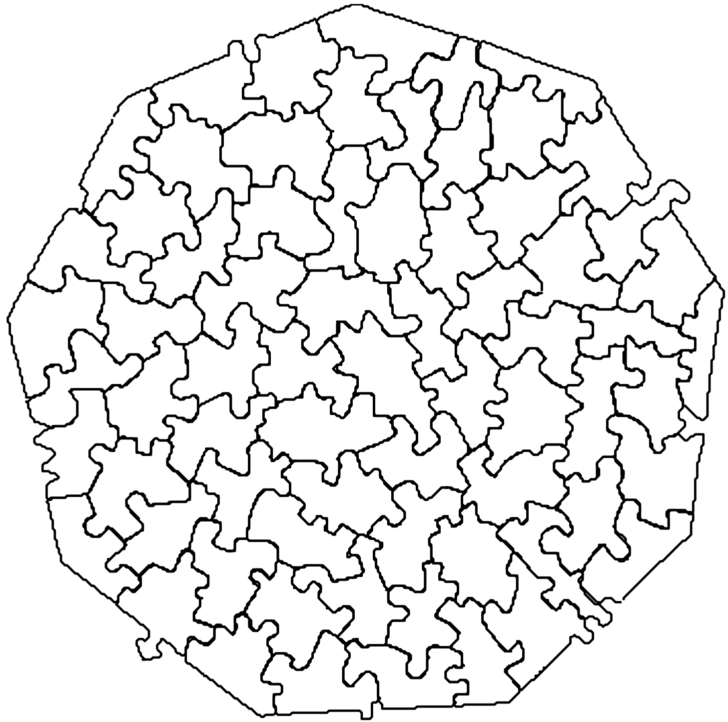
−0.015
−0.05
0
0.05
0.1

The Baffler Jigsaw Puzzle

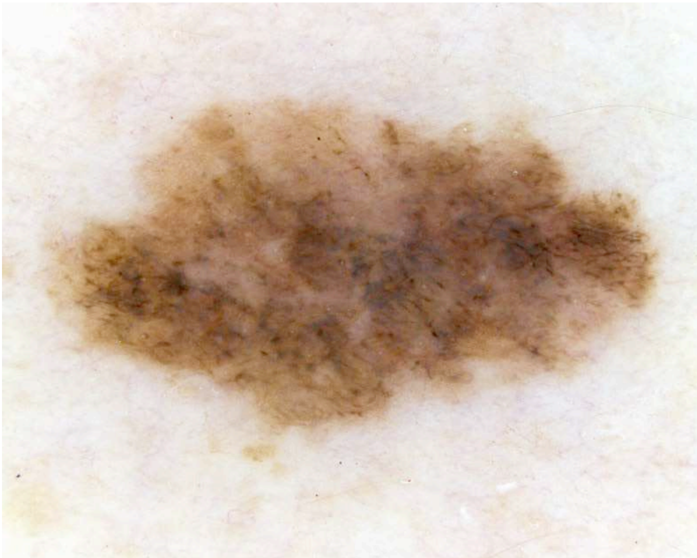


The Baffler Solved



$\Longrightarrow$ Dan Hoff

Distinguishing Melanomas from Moles



Melanoma

Mole

$\Longrightarrow$ A. Grim, A. Rodriguez, C. Shakiban, J. Stangl

Classical Invariant Theory

$$M = \mathbb{R}^2 \setminus \{ u = 0 \}$$

$$G = \mathrm{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \Delta = \alpha\delta - \beta\gamma \neq 0 \right\}$$

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

Differential invariants:

$$\kappa = \frac{T^2}{H^3} \qquad\qquad \kappa_s \approx \frac{U}{H^2}$$

$$\Longrightarrow \quad \text{absolute rational covariants}$$

Hessian:

$$H = \tfrac{1}{2}(u,\, u)^{(2)} = n(n-1)u\, u_{xx} - (n-1)^2 u_x^2 \neq 0$$

Higher transvectants (Jacobian determinants):

$$T = (u,\, H)^{(1)} = (2n-4)u_x H - n u\, H_x$$

$$U = (u,\, T)^{(1)} = (3n-6)Q_x T - n Q\, T_x$$

Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves, parametrized by (κ, κ_s), are identical.

Symmetries of Binary Forms

Theorem. The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree n is:

- A two-parameter group if and only if $H \equiv 0$ if and only if Q is equivalent to a constant. $\Longrightarrow$ totally singular
- A one-parameter group if and only if $H \not\equiv 0$ and $T^2 = c\,H^3$ if and only if Q is complex-equivalent to a monomial x^k, with $k \neq 0, n$. $\Longrightarrow$ maximally symmetric
- In all other cases, a finite group whose cardinality equals the index of the signature curve, and is bounded by

$$\iota_Q \quad \leq \quad \begin{cases} 6n - 12 & U = cH^2 \\ 4n - 8 & \text{otherwise} \end{cases}$$

Cartan's Main Idea

⋆⋆ Recast the equivalence problem for submanifolds under a (pseudo-)group action, in the geometric language of differential forms.

Then reduce the equivalence problem to the most fundamental equivalence problem:

⋆ Equivalence of coframes.

Coframes

Let M be an m-dimensional manifold, e.g., $M \subset \mathbb{R}^m$.

Definition. A coframe on M is a linearly independent system of one-forms $\theta = \{\theta^1, \ldots, \theta^m\}$ forming a basis for its cotangent space $T^*M|_z$ at each point $z \in M$.

In other words

$$\theta^i = \sum_{j=1}^{m} h^i_j(x)\, dx^j, \qquad \det(\, h^i_j(x)\,) \neq 0$$

Equivalence of Coframes

Definition. Two coframes θ on M and $\overline{\theta}$ on $\overline{M}$ are equivalent if there is a diffeomorphism $\Phi\colon M \longrightarrow \overline{M}$ such that

$$\Phi^* \overline{\theta}^i = \theta^i \qquad i = 1, \ldots, m$$

Since the exterior derivative d commutes with pull-back,

$$\Phi^*(d\overline{\theta}^i) = d\theta^i \qquad i = 1, \ldots, m$$

Structure equations

$$\boxed{d\theta^i = \sum_{j<k} I^i_{jk}\, \theta^j \wedge \theta^k}$$

$\Longrightarrow$ The torsion coefficients are invariant: $\overline{I}^i_{jk}(\bar{x}) = I^i_{jk}(x)$

Covariant derivatives

$$dF = \frac{\partial F}{\partial \theta^1}\,\theta^1 + \;\cdots\; + \frac{\partial F}{\partial \theta^m}\,\theta^m$$

If I_j is invariant, so are all its derived invariants:

$$I_{j,k} = \frac{\partial I_j}{\partial \theta^k} \qquad I_{j,k,l} = \frac{\partial I_{j,k}}{\partial \theta^l} \qquad \ldots$$

★ We now have a potentially infinite collection of invariants!

Rank and Order of a Coframe

$r_n = \#$ functionally independent invariants of order $\leq n$:

$$r_0 = \operatorname{rank}\{\, I_j \,\} \qquad\qquad r_1 = \operatorname{rank}\{\, I_j, I_{j,k} \,\} \qquad\qquad \ldots$$

$$r_0 < r_1 < \ \cdots \ < r_s = r_{s+1} = r_{s+2} = \ \cdots$$

$$\text{Order} = s$$

$$\text{Rank} = r = r_s$$

The Order 0 Case

$$s = 0 \qquad r = r_0 = r_1 = \ \cdots$$

Syzygies:

$$I_{j,k} = F_{jk}(I_1, \ldots, I_r)$$

★ ★ Signature: parametrized by $I_j, I_{j,k}$.

Equivalence of Coframes

Cartan's Theorem: Two order 0 coframes are equivalent if and only if

- Their ranks are the same
- Their signature manifolds are identical
- The invariant equations $\bar{I}_j(\bar{x}) = I_j(x)$ have a common real solution.

⋆ Any solution to the invariant equations determines an equivalence between the two coframes.

Symmetry Groups of Coframes

Theorem. Let θ be an invariant coframe of rank r on an m-dimensional manifold M. Then θ admits an $(m - r)$-dimensional (local) symmetry group.

Cartan's Graphical Proof Technique

The graph of the equivalence map

$$\psi : M \longrightarrow \overline{M}$$

can be viewed as a transverse m-dimensional integral submanifold

$$\Gamma_\psi \subset M \times \overline{M}$$

for the involutive differential system generated by the one-forms and functions

$$\overline{\theta}^i - \theta^i \qquad\qquad \bar{I}_j - I_j$$

Existence of suitable integrable submanifolds determining equivalence maps is guaranteed by the Frobenius Theorem, which is, at its heart, an existence theorem for ordinary differential equations, and hence valid in the smooth category.

Extended Coframes

Definition. An extended coframe $\{\theta, J\}$ on M consists of

- a coframe $\theta = \{\theta^1, \ldots, \theta^m\}$ and
- a collection of functions $J = (J_1, \ldots, J_l)$.

Two extended coframes are equivalent if there is a diffeomorphism Φ such that

$$\Phi^* \overline{\theta}^k = \theta^k \qquad\qquad \Phi^* \overline{J}_i = J_i$$

The solution to the equivalence of extended coframes is a straightforward extension of that of coframes. One merely adds the extra invariants J_i to the collection of torsion invariants I^i_{jk} to form the basic invariants, and then applies covariant differentiation to all of them to produce the higher order invariants.

Determining the Invariant (Extended) Coframe

There are now two methods for explicitly determining the invariant (extended) coframe associated with a given equivalence problem.

- The Cartan Equivalence Method
- Equivariant Moving Frames

Either will produce the fundamental differential invariants required to construct a signature and thereby effectively solve the equivalence problem.

- Infinitesimal methods (solve PDEs)

The Cartan Equivalence Method

(1) Reformulate the problem as an equivalence problem for G-valued coframes, for some structure group G

(2) Calculate the structure equations by applying d

(3) Use absorption of torsion to determine the essential torsion

(4) Normalize the group-dependent essential torsion coefficients to reduce the structure group

(5) Repeat the process until the essential torsion coefficients are all invariant

(6) Test for involutivity

(7) If not involutive, prolong (à la EDS) and repeat until involutive

The result is an invariant coframe that completely encodes the equivalence problem, perhaps on some higher dimensional space. The structure invariants for the coframe are used to parametrize the signature.

Equivariant Moving Frames

(1) Prolong (à la jet bundle) the (pseudo-)group action to the jet bundle of order n where the action becomes (locally) free

(2) Choose a cross-section to the group orbits and solve the normalization equations to determine an equivariant moving frame map $\rho \colon \mathrm{J}^n \to G$

(3) Use invariantization to determine the normalized differential invariants of order $\leq n+1$ and invariant differential forms; invariant differential operators; . . .

(4) Apply the recurrence formulae to determine higher order differential invariants, and the structure of the differential invariant algebra

★ Step *(4)* can be done completely symbolically, using only linear algebra, independent of the explicit formulae in step *(3)*

The Recurrence Formulae

The moving frame recurrence formulae enable one to determine the generating differential invariants and hence the invariants $I_1, \ldots, I_l$ required for constructing a signature. The extended coframe used to prove equivalence consists of the pulled-back Maurer–Cartan forms $\nu^i = \rho^*(\mu^I)$ along with the generating differential invariants I_j and their differentials dI_j.

$\Longrightarrow$ It is not (yet) known how to construct the recurrence formulae through the Cartan equivalence method!

$\Longrightarrow$ See Francis Valiquette's recent paper for an alternative method for solving Cartan equivalence problems using the moving frame approach for Lie pseudo-groups.

The Basis Theorem

Theorem. Given a Lie group (or Lie pseudo-group*) acting on p-dimensional submanifolds, the corresponding differential invariant algebra $\mathcal{I}_G$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_k$$

and p invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_i = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_i .$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov, Fels–O*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Kruglikov–Lychagin, Muñoz–Muriel–Rodríguez, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for the invariant differential operators:

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} Y_{jk}^i \, \mathcal{D}_i$$

$$\Longrightarrow \quad \text{Non-commutative differential algebra}$$

- Syzygies (functional relations) among

the differentiated invariants:

$$\Phi(\;\ldots\; \mathcal{D}_J I_\kappa \;\ldots\;) \equiv 0$$

Recurrence Formulae

$$\boxed{\mathcal{D}_i\, \iota(F) = \iota(D_i F) \;+\; \sum_{\kappa=1}^{r} R_i^\kappa\, \iota(\mathbf{v}_\kappa^{(n)}(F))}$$

ι	—	invariantization map
$F(x, u^{(n)})$	—	differential function
$I = \iota(F)$	—	differential invariant
D_i	—	total derivative with respect to x^i
$\mathcal{D}_i = \iota(D_i)$	—	invariant differential operator
$\mathbf{v}_\kappa^{(n)}$	—	infinitesimal generators of prolonged action of G on jets
R_i^κ	—	Maurer–Cartan invariants (coefficients of pulled-back Maurer–Cartan forms)

Recurrence Formulae

$$\boxed{\mathcal{D}_i \, \iota(F) = \iota(D_i F) \; + \; \sum_{\kappa=1}^{r} R_i^\kappa \, \iota(\mathbf{v}_\kappa^{(n)}(F))}$$

♠ If $\iota(F) = c$ is a phantom differential invariant coming from the moving frame cross-section, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_i^κ.

♡ Once the Maurer–Cartan invariants R_i^κ are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}_G$!

Euclidean Surfaces

Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ acts on surfaces $S \subset \mathbb{R}^3$.

For simplicity, we assume the surface is (locally) the graph of a function

$$z = u(x, y)$$

Infinitesimal generators:

$$\mathbf{v}_1 = -y\,\partial_x + x\,\partial_y, \qquad \mathbf{v}_2 = -u\,\partial_x + x\,\partial_u, \qquad \mathbf{v}_3 = -u\,\partial_y + y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u.$$

- The translations $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ will be ignored, as they play no role in the higher order recurrence formulae.

Cross-section (Darboux frame):

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0$$

Principal curvatures

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Higher order differential invariants — invariantized jet coordinates:

$$I_{jk} = \iota(u_{jk}) \qquad \text{where} \qquad u_{jk} = \frac{\partial^{j+k} u}{\partial x^j \partial y^k}$$

★ ★ Nondegeneracy condition: non-umbilic point $\kappa_1 \neq \kappa_2$.

Algebra of Euclidean Differential Invariants

Principal curvatures:

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Invariant differentiation operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y)$$

$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

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$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

The recurrence formulae enable one to express the higher order differential invariants in terms of the principal curvatures, or, equivalently, the mean and Gauss curvatures, and their invariant derivatives:

$$I_{jk} = \iota(u_{jk}) = \widetilde{\Phi}_{jk}(\kappa_1, \kappa_2, \mathcal{D}_1\kappa_1, \mathcal{D}_2\kappa_1, \mathcal{D}_1\kappa_2, \mathcal{D}_2\kappa_2, \mathcal{D}_1^2\kappa_1, \ \ldots\ )$$

$$= \Phi_{jk}(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )$$

Recurrence Formulae

$$\boxed{\iota(D_i u_{jk}) = \mathcal{D}_i \, \iota(u_{jk}) - \sum_{\kappa=1}^{3} R_i^\kappa \, \iota[\, \varphi_\kappa^{jk}(x, y, u^{(j+k)}) \,], \qquad j + k \geq 1}$$

$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

Recurrence Formulae

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$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

$\varphi_\kappa^{jk}(0,0,I^{(j+k)}) = \iota[\, \varphi_\kappa^{jk}(x,y,u^{(j+k)})\,]$

— invariantized prolonged infinitesimal generator coefficients.

$$\boxed{\begin{aligned} I_{j+1,k} &= \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_1^\kappa \\ I_{j,k+1} &= \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_2^\kappa \end{aligned}}$$

Prolonged infinitesimal generators:

$$\begin{aligned}
\mathrm{pr}\,\mathbf{v}_1 &= -y\,\partial_x + x\,\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y} \\
&\quad - 2\,u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\,\partial_{u_{xy}} - 2\,u_{xy}\partial_{u_{yy}} + \ \cdots\ , \\
\mathrm{pr}\,\mathbf{v}_2 &= -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + u_x\,u_y\partial_{u_y} \\
&\quad + 3\,u_x\,u_{xx}\partial_{u_{xx}} + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\partial_{u_{xy}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\,\partial_{u_{yy}} + \ \cdots\ , \\
\mathrm{pr}\,\mathbf{v}_3 &= -u\,\partial_y + y\,\partial_u + u_x\,u_y\partial_{u_x} + (1 + u_y^2)\,\partial_{u_y} \\
&\quad + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\partial_{u_{xx}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\,\partial_{u_{xy}} + 3\,u_y\,u_{yy}\partial_{u_{yy}} + \ \cdots\ .
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&\qquad - 2\,u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\,\partial_{u_{xy}} - 2\,u_{xy}\partial_{u_{yy}} + \;\cdots\;, \\
\mathrm{pr}\,\mathbf{v}_2 &= -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + u_x\,u_y\partial_{u_y} \\
&\qquad + 3\,u_x\,u_{xx}\partial_{u_{xx}} + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\partial_{u_{xy}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\,\partial_{u_{yy}} + \;\cdots\;, \\
\mathrm{pr}\,\mathbf{v}_3 &= -u\,\partial_y + y\,\partial_u + u_x\,u_y\partial_{u_x} + (1 + u_y^2)\,\partial_{u_y} \\
&\qquad + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\partial_{u_{xx}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\,\partial_{u_{xy}} + 3\,u_y\,u_{yy}\partial_{u_{yy}} + \;\cdots\;.
\end{aligned}$$

$$I_{jk} = \iota(u_{jk})$$

Phantom differential invariants:

$$I_{00} = I_{10} = I_{01} = I_{11} = 0$$

Principal curvatures:

$$I_{20} = \kappa_1 \qquad I_{02} = \kappa_2$$

Phantom recurrence formulae:

$$\begin{aligned}
\kappa_1 &= I_{20} = \mathcal{D}_1 I_{10} - R_1^2 = -R_1^2, \\
0 &= I_{11} = \mathcal{D}_1 I_{01} - R_1^3 = -R_1^3, \\
I_{21} &= \mathcal{D}_1 I_{11} - (\kappa_1 - \kappa_2)\, R_1^1 = -(\kappa_1 - \kappa_2)\, R_1^1, \\
0 &= I_{11} = \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2, \\
\kappa_2 &= I_{02} = \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3, \\
I_{12} &= \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.
\end{aligned}$$

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$$0 = I_{11} = \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2,$$

$$\kappa_2 = I_{02} = \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3,$$

$$I_{12} = \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.$$

Maurer–Cartan invariants:

$$R_1^1 = -Y_1, \qquad R_1^2 = -\kappa_1, \qquad R_1^3 = 0,$$

$$R_1^2 = -Y_2, \qquad R_2^2 = 0, \qquad R_3^2 = -\kappa_2.$$

Commutator invariants:

$$Y_1 = \frac{I_{21}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad\qquad Y_2 = \frac{I_{12}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

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$$\boxed{[\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1\, \mathcal{D}_2 - \mathcal{D}_2\, \mathcal{D}_1 = Y_2\, \mathcal{D}_1 - Y_1\, \mathcal{D}_2,}$$

Third order recurrence relations:

$$I_{30} = \mathcal{D}_1 \kappa_1 = \kappa_{1,1}, \quad I_{21} = \mathcal{D}_2 \kappa_1 = \kappa_{1,2}, \quad I_{12} = \mathcal{D}_1 \kappa_2 = \kappa_{2,1}, \quad I_{03} = \mathcal{D}_2 \kappa_2 = \kappa_{2,2},$$

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Fourth order recurrence relations:

$$\begin{aligned}
I_{40} &= \kappa_{1,11} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3, \\
I_{31} &= \kappa_{1,12} - \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} && = \kappa_{1,21} + \frac{\kappa_{1,1}\kappa_{1,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2}, \\
I_{22} &= \kappa_{1,22} + \frac{\kappa_{1,1}\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 && = \kappa_{2,11} - \frac{\kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2, \\
I_{13} &= \kappa_{2,21} + \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} && = \kappa_{2,12} - \frac{\kappa_{2,1}\kappa_{2,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2}, \\
I_{04} &= \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3.
\end{aligned}$$

★ The two expressions for I_{31} and I_{13} follow from the commutator formula.

Fourth order recurrence relations

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$$I_{04} = \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3.$$

$\star\star$ The two expressions for I_{22} imply the Codazzi syzygy

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\,\kappa_{2,1} + \kappa_{1,2}\,\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1\,\kappa_2\,(\kappa_1 - \kappa_2) = 0,$$

which can be written compactly as

$$K = \kappa_1\kappa_2 = -\,(\mathcal{D}_1 + Y_1)\,Y_1 - (\mathcal{D}_2 + Y_2)\,Y_2.$$

$\Longrightarrow$ Gauss' Theorema Egregium

Generating Differential Invariants

♡ From the general structure of the recurrence relations, one proves that the Euclidean differential invariant algebra $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by the principal curvatures κ_1, κ_2 or, equivalently, the mean and Gauss curvatures, H, K, through the process of invariant differentiation:

$$I = \Phi(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots \ )$$

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♢ Remarkably, for suitably generic surfaces, the Gauss curvature can be written as a universal rational function of the mean curvature and its invariant derivatives of order ≤ 4:

$$K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1^2 H, \ \ldots\ , \mathcal{D}_2^4 H)$$

and hence $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by mean curvature alone!

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♠ To prove this, given

$$K = \kappa_1 \kappa_2 = -\,(\mathcal{D}_1 + Y_1)\, Y_1 - (\mathcal{D}_2 + Y_2)\, Y_2$$

it suffices to write the commutator invariants Y_1, Y_2 in terms of H.

The Commutator Trick

$$\boxed{K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1) Y_1 - (\mathcal{D}_2 + Y_2) Y_2}$$

To determine the commutator invariants:

$$\begin{aligned} \mathcal{D}_1 \mathcal{D}_2 H - \mathcal{D}_2 \mathcal{D}_1 H &= Y_2 \, \mathcal{D}_1 H - Y_1 \, \mathcal{D}_2 H \\ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_J H - \mathcal{D}_2 \mathcal{D}_1 \mathcal{D}_J H &= Y_2 \, \mathcal{D}_1 \mathcal{D}_J H - Y_1 \, \mathcal{D}_2 \mathcal{D}_J H \end{aligned} \qquad (*)$$

Non-degeneracy condition:

$$\det \begin{pmatrix} \mathcal{D}_1 H & \mathcal{D}_2 H \\ \mathcal{D}_1 \mathcal{D}_J H & \mathcal{D}_2 \mathcal{D}_J H \end{pmatrix} \neq 0,$$

Solve $(*)$ for Y_1, Y_2 in terms of derivatives of H, producing a universal formula

$$\boxed{K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots \ )}$$

for the Gauss curvature as a rational function of the mean curvature and its invariant derivatives!

Definition. A surface $S \subset \mathbb{R}^3$ is mean curvature degenerate if, near any non-umbilic point $p_0 \in S$, there exist scalar functions $F_1(t), F_2(t)$ such that

$$\mathcal{D}_1 H = F_1(H), \qquad \mathcal{D}_2 H = F_2(H).$$

- surfaces with symmetry: rotation, helical;
- minimal surfaces;
- constant mean curvature surfaces;
- ???

Theorem. If a surface is mean curvature non-degenerate then the algebra of Euclidean differential invariants is generated entirely by the mean curvature and its successive invariant derivatives.

Minimal Generating Invariants

A set of differential invariants is a generating system if all other differential invariants can be written in terms of them and their invariant derivatives.

Euclidean curves $C \subset \mathbb{R}^3$:	curvature κ and torsion τ
Equi–affine curves $C \subset \mathbb{R}^3$:	affine curvature κ and torsion τ
Euclidean surfaces $S \subset \mathbb{R}^3$:	mean curvature H
Equi–affine surfaces $S \subset \mathbb{R}^3$:	Pick invariant P.
Conformal surfaces $S \subset \mathbb{R}^3$:	third order invariant J_3.
Projective surfaces $S \subset \mathbb{R}^3$:	fourth order invariant K_4.

$\Longrightarrow$ For any $n \geq 1$, there exists a Lie group G_N acting on surfaces $S \subset \mathbb{R}^3$ such that its differential invariant algebra requires n generating invariants!

♠ Finding a minimal generating set appears to be a very difficult problem.
(No known bound on order of syzygies.)