Moving Frames

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Collaborators

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Moving Frames

Classical contributions:

M. Bartels ($\sim$1800), J. Serret, J. Frénet, C. Jordan, G. Darboux, É. Cotton, **É. Cartan**

Modern contributions:

P. Griffiths, M. Green, G. Jensen

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear."

"Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

— Hermann Weyl

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Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint Invariants and Semi-Differential Invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory
- Computer vision
    - object recognition
    - symmetry detection
- Invariant numerical methods
- Poisson geometry & solitons
- Lie pseudogroups
- Invariants of Killing tensors
- Invariants of Lie algebras

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two n-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Self-equivalence or *self-congruence*:

$$N = g \cdot N$$

Classical Geometry

Equivalence Problem: Determine whether or not two given submanifolds N and $\overline{N}$ are congruent under a group transformation: $\overline{N} = g \cdot N$.

Symmetry Problem: Given a submanifold N, find all its symmetries (belonging to the group).

- *Euclidean group* — $G = \mathrm{SE}(n)$ or $\mathrm{E}(n)$
  - $\Rightarrow$ isometries of Euclidean space
  - $\Rightarrow$ translations, rotations (& reflections)

$$z \longmapsto R \cdot z + a \qquad \begin{cases} R \in \mathrm{SO}(n) \text{ or } \mathrm{O}(n) \\ a \in \mathbb{R}^n \\ z \in \mathbb{R}^n \end{cases}$$

- *Equi-affine group*: $G = \mathrm{SA}(n)$
  $R \in \mathrm{SL}(n)$ — area-preserving
- *Affine group*: $G = \mathrm{A}(n)$
  $R \in \mathrm{GL}(n)$
- *Projective group*: $G = \mathrm{PSL}(n)$
  acting on $\mathbb{RP}^{n-1}$

$\Longrightarrow$ Applications in computer vision

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q} \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

$\Rightarrow$ multiplier representation of GL(2)
$\Rightarrow$ modular forms

Transformation group:

$$g\colon (x,u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta} \, , \, \frac{u}{(\gamma x + \delta)^n} \right)$$

Equivalence of functions $\Longleftrightarrow$ equivalence of graphs

$$N_Q = \{ \, (x,u) = (x, Q(x)) \, \} \subset \mathbb{C}^2$$

Moving Frames

Definition.

A *moving frame* is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

$G_z = \{ g \mid g \cdot z = z \}$ $\Longrightarrow$ Isotropy subgroup

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:
  $\Longrightarrow$ $G_z = \{e\}$ for all $z \in M$.

- locally free — the orbits all have the same dimension as G:
  $\Longrightarrow$ G_z is a discrete subgroup of G.

- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  $\not\approx$ irrational flow on the torus

- effective — the only group element $g \in G$ which fixes *every* point $z \in M$ is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

$$G_M = \bigcap_{z \in M} G_z = \{e\}$$

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Necessity: Let $z \in M$.

Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

Regularity: Suppose

$$z_n = g_n \cdot z \longrightarrow z \qquad \text{as} \qquad n \to \infty$$

By continuity,

$$\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$$

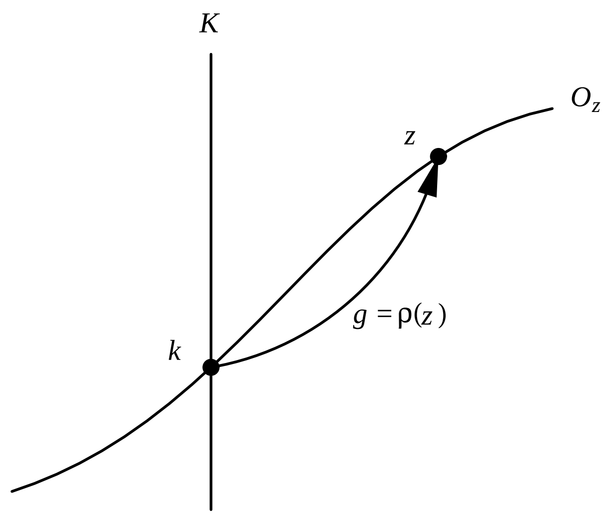
Hence $g_n \longrightarrow e$ in G.

Sufficiency: By construction — "normalization".

Q.E.D.

Geometrical Construction

Normalization = choice of cross-section to the group orbits



K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Construction of Moving Frames

$$r = \dim G \ \leq \ m = \dim M$$

Coordinate cross-section

$$K = \{ \, z_1 = c_1, \ \ldots \ , z_r = c_r \, \}$$

left	right
$w(g,z) = g^{-1} \cdot z$	$w(g,z) = g \cdot z$

Choose $r = \dim G$ components to *normalize*:

$$w_1(g,z) = c_1 \qquad \ldots \qquad w_r(g,z) = c_r$$

The solution

$$g = \rho(z)$$

is a (local) moving frame.

$\Longrightarrow$ Implicit Function Theorem

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants:

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ldots, I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame ρ is the the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

$$\iota\,[\,F(z_1, \ldots, z_m)\,] \;=\; F(c_1, , \ldots c_r, I_1(z), \ldots, I_{m-r}(z))$$

Invariantization amounts to restricting F to the cross-section

$$I \mid K = F \mid K$$

and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota : \quad \text{functions} \quad \longmapsto \quad \text{invariants}$$

The Rotation Group

$$G = \mathrm{SO}(2) \qquad \text{acting on} \qquad \mathbb{R}^2$$

$$z = (x, u) \longmapsto g \cdot z = (\, x \cos\theta - u \sin\theta \;,\; x \sin\theta + u \cos\theta \,)$$

$$\Longrightarrow \quad \text{Free on } M = \mathbb{R}^2 \setminus \{0\}$$

Left moving frame:

$$w(g, z) = g^{-1} \cdot z = (y, v)$$

$$y = x \cos\theta + u \sin\theta \qquad v = -\, x \sin\theta + u \cos\theta$$

Cross-section

$$K = \{\, u = 0, \; x > 0 \,\}$$

Normalization equation

$$v = -\, x \sin\theta + u \cos\theta = 0$$

Left moving frame:

$$\theta = \tan^{-1} \frac{u}{x} \quad \Longrightarrow \quad \theta = \rho(x, u) \in \mathrm{SO}(2)$$

Fundamental invariant

$$r = \iota(x) = \sqrt{x^2 + u^2}$$

Invariantization

$$\iota[\, F(x, u) \,] = F(r, 0)$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

Jet Space

- Although in use since the time of Lie and Darboux, jet space was first formally defined by Ehresmann in 1950.

- Jet space is the proper setting for the geometry of partial differential equations.

M — smooth m-dimensional manifold

$$1 \leq p \leq m - 1$$

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — (extended) jet bundle

$\Longrightarrow$ Defined as the space of equivalence classes of p-dimensional submanifolds under the equivalence relation of n^{th} order contact at a single point.

$\Longrightarrow$ Can be identified as the space of n^{th} order Taylor polynomials for submanifolds given as graphs $u = f(x)$

Local Coordinates on Jet Space

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — n^{th} extended jet bundle for
p-dimensional submanifolds $N \subset M$

Local coordinates:

Assume $N = \{ u = f(x) \}$ is a graph (section).

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$$p + q = m = \dim M$$

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

$$u^\alpha_J = \partial_J u^\alpha \qquad 0 \le \# J \le n$$

— induced jet coordinates

- No bundle structure assumed on M.
- Projective completion of $\mathrm{J}^n E$ when $E \to X$ is a bundle.

Prolongation of Group Actions

G — transformation group acting on M

$\Longrightarrow$ G maps submanifolds to submanifolds and preserves the order of contact

$G^{(n)}$ — prolonged action of G on the jet space J^n

The prolonged group formulae

$$w^{(n)} = (y, v^{(n)}) = g^{(n)} \cdot z^{(n)}$$

are obtained by implicit differentiation:

$$dy^i = \sum_{j=1}^{p} P^i_j(g, z^{(1)})\, dx^j$$

$$D_{y^j} = \sum_{i=1}^{p} Q^i_j(g, z^{(1)})\, D_{x^i} \qquad \Longrightarrow \qquad Q = P^{-T}$$

$$v^\alpha_J = D_{y^{j_1}} \cdots D_{y^{j_k}}(v^\alpha)$$

Differential invariant $\quad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$$

$\Longrightarrow$ curvatures

Freeness

Theorem. If G acts (locally) effectively on M, then G acts (locally) freely on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$ for $n \gg 0$.

Definition. $N \subset M$ is *regular* at order n if $\mathrm{j}_n N \subset \mathcal{V}^n$.

Corollary. Any regular submanifold admits a (local) moving frame.

Theorem. A submanifold is totally singular, $\mathrm{j}_n N \subset \mathrm{J}^n \setminus \mathcal{V}^n$ for all n, if and only if its symmetry group

$$G_N = \{ \, g \mid g \cdot N \subset N \, \}$$

does not act freely on N.

Moving Frames on Jet Space

$$w^{(n)} = (y, v^{(n)}) = \begin{cases} g^{(n)} \cdot z^{(n)} & \text{right} \\ (g^{(n)})^{-1} \cdot z^{(n)} & \text{left} \end{cases}$$

Choose $r = \dim G$ jet coordinates

$$z_1, \ldots, z_r \qquad\qquad x^i \text{ or } u^\alpha_J$$

Coordinate cross-section $K \subset \mathrm{J}^n$

$$z_1 = c_1 \quad \ldots \quad z_r = c_r$$

Corresponding lifted differential invariants:

$$w_1, \ldots, w_r \qquad\qquad y^i \text{ or } v^\alpha_J$$

Normalization Equations

$$w_1(g, x, u^{(n)}) = c_1 \quad \ldots \quad w_r(g, x, u^{(n)}) = c_r$$

Solution:

$$g = \rho^{(n)}(z^{(n)}) = \rho^{(n)}(x, u^{(n)}) \qquad \Longrightarrow \quad \text{moving frame}$$

The Fundamental Differential Invariants

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)})$$

$$\boxed{\begin{aligned} H^i(x,u^{(n)}) &= y^i(\rho^{(n)}(x,u^{(n)}),x,u) \\ I^\alpha_K(x,u^{(k)}) &= v^\alpha_K(\rho^{(n)}(x,u^{(n)}),x,u^{(k)}) \end{aligned}}$$

Phantom differential invariants

$$w_1 = c_1 \ \ldots \ w_r = c_r \qquad \Longrightarrow \quad \text{normalizations}$$

Theorem. Every n^{th} order differential invariant can be locally uniquely written as a function of the non-phantom fundamental differential invariants in $I^{(n)}$.

Invariant Differentiation

Contact-invariant coframe

$$dy^i \quad \longmapsto \quad \omega^i = \sum_{j=1}^{p} P^i_j(\rho^{(n)}(z^{(n)}), z^{(n)}) \; dx^i$$

$\Longrightarrow$ arc length element

Invariant differential operators:

$$D_{y^j} \quad \longmapsto \quad \mathcal{D}_j = \sum_{i=1}^{p} Q^i_j(\rho^{(n)}(z^{(n)}), z^{(n)}) \, D_{x^i}$$

$\Longrightarrow$ arc length derivative

Duality:

$$dF = \sum_{i=1}^{p} \mathcal{D}_i F \cdot \omega^i$$

Theorem. The higher order differential invariants are obtained by invariant differentiation with respect to $\mathcal{D}_1, \ldots, \mathcal{D}_p$.

Euclidean Curves $\qquad G = \mathrm{SE}(2)$

Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

Prolong to J^3 via implicit differentiation

$$\left.\begin{aligned} y &= \cos\theta\,(x-a) + \sin\theta\,(u-b) \\ v &= -\sin\theta\,(x-a) + \cos\theta\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-b)$$

$$v_y = \frac{-\sin\theta + u_x \cos\theta}{\cos\theta + u_x \sin\theta}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x \sin\theta)^3}$$

$$v_{yyy} = \frac{(\cos\theta + u_x \sin\theta)u_{xxx} - 3u_{xx}^2 \sin\theta}{(\cos\theta + u_x \sin\theta)^5}$$

$$\vdots$$

Normalization $\qquad r = \dim G = 3$

$$y = 0, \qquad v = 0, \qquad v_y = 0$$

Left moving frame $\qquad \rho\colon \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x, \qquad b = u, \qquad \theta = \tan^{-1} u_x$$

Differential invariants

$$\begin{aligned} v_{yy} &\longmapsto \kappa &&= \frac{u_{xx}}{(1+u_x^2)^{3/2}} \\ v_{yyy} &\longmapsto \frac{d\kappa}{ds} &&= \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3} \\ v_{yyyy} &\longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 &&= \cdots \end{aligned}$$

Invariant one-form — arc length

$$dy = (\cos\theta + u_x \sin\theta)\, dx \qquad \longmapsto \qquad ds = \sqrt{1+u_x^2}\; dx$$

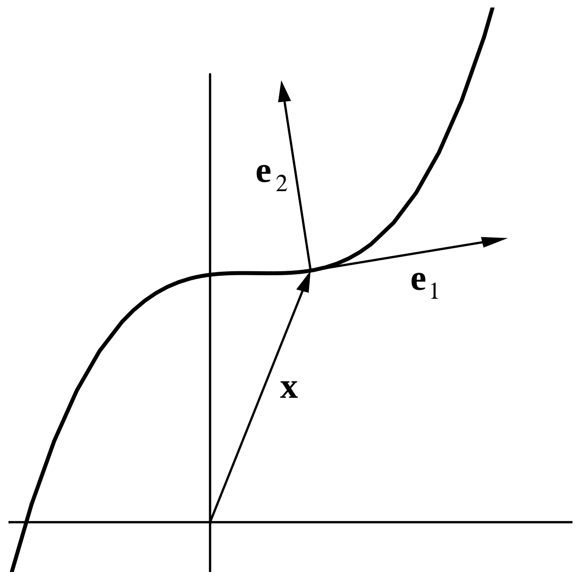
Invariant differential operator

$$\frac{d}{dy} = \frac{1}{\cos\theta + u_x\sin\theta}\,\frac{d}{dx} \qquad \longmapsto \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

Euclidean Curves



Moving frame $\qquad \rho : \ (x, u, u_x) \ \longmapsto \ (R, \mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\mathbf{e}_1, \mathbf{e}_2) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{e}_1 = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix} \qquad \mathbf{e}_2 = \mathbf{e}_1^{\perp} = \begin{pmatrix} -y_s \\ x_s \end{pmatrix}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \kappa\, \mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = -\,\kappa\, \mathbf{e}_1$$

The Replacement Theorem

Any differential invariant has the form

$$I = F(x, u^{(n)}) = F(y, w^{(n)}) = F(I^{(n)})$$

$\Longrightarrow$ T.Y. Thomas

$$\kappa = \frac{v_{yy}}{(1+v_y^2)^2} = \frac{u_{xx}}{(1+u_x^2)^2}$$

$$\iota(x) = \iota(u) = (u_x) = 0$$

$$\iota(u_{xx}) = \kappa$$

Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\,z + b \qquad\qquad A \in \mathrm{SL}(2), \qquad b \in \mathbb{R}^2$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - u_x \beta)\ dx \qquad\qquad D_y = \frac{1}{\delta - u_x \beta}\ D_x$$

$$\left.\begin{aligned} y &= \delta(x-a) - \beta(u-b) \\ v &= -\,\gamma(x-a) + \alpha(u-b) \end{aligned}\right\} \qquad w = A^{-1}(z-b)$$

$$v_y = -\,\frac{\gamma - \alpha u_x}{\delta - \beta u_x} \qquad\qquad v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta u_x) u_{xxx} + 3\beta u_{xx}^2}{(\delta - \beta u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta u_x)^2 + 10 u_{xx} u_{xxx} \beta(\delta - \beta u_x) + 15 u_{xx}^3 \beta^2}{(\alpha + \beta u_x)^7}$$

$$\vdots$$

Nondegeneracy $\qquad u_{xx} = 0$

$\Longrightarrow$ Straight lines are totally singular

(three-dimensional equi-affine symmetry group)

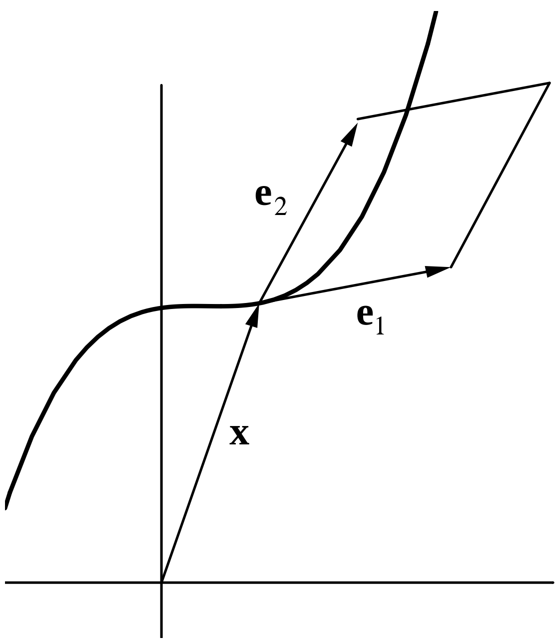
Normalization $\qquad r = \dim G = 5$

$$y = 0, \quad v = 0, \quad v_y = 0, \quad v_{yy} = 1, \quad v_{yyy} = 0.$$

Left Moving frame $\quad \rho \colon \mathrm{J}^3 \longrightarrow \mathrm{SA}(2)$

$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3} u_{xx}^{-5/3} u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3} u_{xx}^{-5/3} u_{xxx} \end{pmatrix} \qquad \mathbf{b} = z = \begin{pmatrix} x \\ u \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{dz}{ds} & \dfrac{d^2 z}{ds^2} \end{pmatrix}$$



Frenet frame

$$\mathbf{e}_1 = \frac{dz}{ds} \qquad\qquad \mathbf{e}_2 = \frac{d^2 z}{ds^2}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{dz}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = \kappa\, \mathbf{e}_1$$

Equi-affine arc length

$$dy \quad \longmapsto \quad ds = \sqrt[3]{u_{xx}}\; dx = \sqrt[3]{\dot z \wedge \ddot z}\; dt$$

Invariant differential operator

$$D_y \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}}\, D_x = \frac{1}{\sqrt[3]{\dot z \wedge \ddot z}}\, D_t$$

Equi-affine curvature

$$v_{4y} \quad \longmapsto \quad \kappa = \frac{5u_{xx}u_{xxxx} - 3u_{xxx}^2}{9u_{xx}^{8/3}} = z_s \wedge z_{ss}$$

$$v_{5y} \quad \longmapsto \quad \frac{d\kappa}{ds} \qquad\qquad v_{6y} \quad \longmapsto \quad \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

Equivalence & Signature

Cartan's main idea: The equivalence and symmetry properties of submanifolds will be found by restricting the differential invariants to the submanifold $J(x) = I(\,\mathrm{j}_n N|_x\,)$.

Equivalent submanifolds should have the same invariants.

However, unless an invariant $J(x)$ is constant, it carries little information by itself, since the equivalence map will typically drastically change the dependence of the invariant on the parameter x.

$\Longrightarrow$ Constant curvature submanifolds

However, a functional dependency or *syzygy* among the invariants *is* intrinsic:

$$J_k(x) = \Phi(J_1(x), \ldots, J_{k-1}(x))$$

The Signature Map

Equivalence and symmetry properties of submanifolds are governed by the functional dependencies — "syzygies" — among the differential invariants.

$$J_k(x) = \Phi(J_1(x), \ldots, J_{k-1}(x))$$

The syzygies are encoded by the *signature map*

$$\Sigma : N \quad \longrightarrow \quad \mathcal{S}$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\Sigma(x) = (J_1(x), \ldots, J_m(x))$$
$$= (I_1 \mid N, \ \ldots \ , I_m \mid N)$$

The image $\mathcal{S} = \operatorname{Im} \Sigma$ is the signature subset (or submanifold) of N.

Geometrically, the signature

$$\mathcal{S} \subset \mathcal{K}$$

is the image of $\mathrm{j}_n N$ in the cross-section $\mathcal{K} \subset \mathrm{J}^n$, where $n \gg 0$ is sufficiently large.

$$\Sigma : \; N \quad \longrightarrow \quad \mathrm{j}_n N \quad \longrightarrow \quad \mathcal{S} \subset \mathcal{K}$$

Theorem. Two submanifolds are equivalent:

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical:

$$\mathcal{S} = \overline{\mathcal{S}}$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the first two differential invariants κ and κ_s

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Theorem. Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry

Signature map

$$\Sigma \; : \; N \quad \longrightarrow \quad \mathcal{S}$$

Theorem. Let $\mathcal{S}$ denote the signature of the submanifold N. Then the dimension of its symmetry group $G_N = \{ \, g \mid g \cdot N \subset N \, \}$ equals

$$\dim G_N \; = \; \dim N - \dim \mathcal{S}$$

Corollary. For a regular submanifold $N \subset M$,

$$0 \; \leq \; \dim G_N \; \leq \; \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point

$$\dim \mathcal{S} = 0$$

- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ In Euclidean geometry, these are the circles, straight lines, helices, spheres & planes.

$\Longrightarrow$ In equi-affine plane geometry, these are the conic sections.

$\Longrightarrow$ In projective geometry, these are the W *curves* of Lie and Klein.

Discrete Symmetries

Definition. The *index* of a submanifold N equals the number of points in N which map to a generic point of its signature $\mathcal{S}$:

$$\iota_N = \min\left\{ \#\, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

Theorem. The cardinality of the symmetry group of N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

Classical Invariant Theory

$$M = \mathbb{R}^2 \setminus \{ u = 0 \} \qquad G = \mathrm{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \alpha\delta - \beta\gamma \neq 0 \right\}$$

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}\,,\ \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

$$\sigma = \gamma\, x + \delta \qquad \Delta = \alpha\delta - \beta\gamma$$

Prolongation:

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}$$

$$v = \sigma^{-n} u$$

$$v_y = \frac{\sigma u_x - n\gamma u}{\Delta\ \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2 u_{xx} - 2(n-1)\gamma\sigma u_x + n(n-1)\gamma^2 u}{\Delta^2\ \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

Normalization:

$$y = 0 \qquad v = 1 \qquad v_y = 0 \qquad v_{yy} = \frac{1}{n(n-1)}$$

Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\, u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\, u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\, x u^{(1-n)/n}$$

$$H = n(n-1)uu_{xx} - (n-1)^2 u_x^2 \qquad \text{— Hessian}$$

Nonsingular form: $H \neq 0$

Note: $H \equiv 0$ if and only if $Q(x) = (ax+b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} \approx \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3(n-2)}{n^3(n-1)} \approx \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} = n(n-1)QQ'' - (n-1)^2 Q'^2 \sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = (Q,\, H)^{(1)} = (2n-4)Q'H - nQH' \sim Q_x H_y - Q_y H_x$$

$$U = (Q,\, T)^{(1)} = (3n-6)Q'T - nQT' \sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \qquad \deg H = 2n-4 \qquad \deg T = 3n-6 \qquad \deg U = 4n-8$$

Signatures of Binary Forms

Signature curve of a nonsingular binary form $Q(x)$:

$$\mathcal{S}_Q = \left\{ (J(x)^2, K(x)) = \left(\frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $H(x) \neq 0$ and $(J'(x), K'(x)) \neq 0$.

Signature map

$$\Sigma \colon N_Q \longrightarrow \mathcal{S}_Q \qquad \Sigma(x) = (J(x)^2, K(x))$$

Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

Maximally Symmetric Binary Forms

Theorem. If $u = Q(x)$ is a polynomial, then the following are equivalent:

- $Q(x)$ admits a one-parameter symmetry group
- T^2 is a constant multiple of H^3
- $Q(x) \simeq x^k$ is complex-equivalent to a monomial
- the signature curve degenerates to a single point
- all the (absolute) differential invariants of Q are constant
- the graph of Q coincides with the orbit of a one-parameter subgroup

$\Longrightarrow$ diagonalizable

Symmetries of Binary Forms

Theorem. The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree n is:

- A two-parameter group if and only if $H \equiv 0$ if and only if Q is equivalent to a constant.

  $\Longrightarrow$ totally singular

- A one-parameter group if and only if $H \not\equiv 0$ and T^2 is a constant multiple of H^3 if and only if Q is complex-equivalent to a monomial x^k, with $k \neq 0, n$.

  $\Longrightarrow$ maximally symmetric

- In all other cases, a finite group whose cardinality equals the index

$$\iota_Q = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

of the signature curve, and is bounded by

$$\iota_Q \quad \leq \quad \begin{cases} 6n - 12 & \qquad U = cH^2 \\ 4n - 8 & \qquad \text{otherwise} \end{cases}$$

The Variational Bicomplex

$\Longrightarrow$ *Dedecker, Vinogradov, Tsujishita, I. Anderson*

Infinite jet space

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

Horizontal one-forms

$$dx^1, \ldots, dx^p$$

Contact (vertical) one-forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i} \, dx^i$$

Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum \ A^\alpha_J \, \theta^\alpha_J$$

Bigrading of the differential forms on J^∞

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s} \qquad \begin{array}{l} r = \# \quad \text{of} \quad dx^i \\ s = \# \quad \text{of} \quad \theta^\alpha_J \end{array}$$

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

Variational Bicomplex:

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

$$d_H^2 = 0 \qquad d_H\, d_V + d_V\, d_H = 0 \qquad d_V^2 = 0$$

$F(x, u^{(n)})$ — differential function

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total differential}$$

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u^\alpha_J}\, \theta^\alpha_J \qquad \text{— “variation”}$$

The Simplest Example. $\quad M = \mathbb{R}^2, \quad x, u \in \mathbb{R}$

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x\, dx$$

$$\theta_x = du_x - u_{xx}\, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx}\, dx$$

$$\vdots$$

Differential $\quad F = F(x, u, u_x, u_{xx}, \ \ldots\ )$

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial u}\, du + \frac{\partial F}{\partial u_x}\, du_x + \frac{\partial F}{\partial u_{xx}}\, du_{xx} + \ \cdots \\ &= (D_x F)\, dx \ + \ \frac{\partial F}{\partial u}\theta + \frac{\partial F}{\partial u_x}\theta_x + \frac{\partial F}{\partial u_{xx}}\theta_{xx} + \ \cdots \\ &= \ d_H F \qquad + \qquad\quad d_V F \end{aligned}$$

Total derivative

$$D_x F = \frac{\partial F}{\partial u}\, u_x + \frac{\partial F}{\partial u_x}\, u_{xx} + \frac{\partial F}{\partial u_{xx}}\, u_{xxx} + \ \cdots$$

Lagrangian form

$$\lambda = L(x, u^{(n)})\, dx \ \in \ \Omega^{1,1}$$

Vertical derivative — variation

$$d\lambda = d_V \lambda = d_V L \wedge dx$$

$$= \left(\frac{\partial L}{\partial u}\,\theta + \frac{\partial L}{\partial u_x}\,\theta_x + \frac{\partial L}{\partial u_{xx}}\,\theta_{xx} + \cdots \right) \wedge \, dx \ \in \ \Omega^{1,1}$$

Integration by parts

$$d_H (A\,\theta) = (D_x A)\, dx \wedge \theta - A\ \theta_x \wedge dx$$

$$= -[\,(D_x A)\,\theta + A\,\theta_x\,] \wedge dx$$

so

$$A\ \theta_x \wedge dx \ \sim \ -(D_x A)\,\theta \wedge dx \quad \text{mod im } d_H$$

Variational derivative — compute modulo im d_H :

$$d\lambda \ \sim \ \delta\lambda = \left(\frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge \, dx$$

$$= \mathbf{E}(L)\ \theta \wedge \, dx$$

$\Longrightarrow$ Euler-Lagrange source form.

Variational Derivative

p = # independent variables

Variation:

$$d_V : \Omega^{p,0} \longrightarrow \Omega^{p,1}$$

Integration by Parts:

$$\pi : \Omega^{p,1} \longrightarrow \mathcal{F}^1 = \Omega^{p,1} \,/\, d_H\, \Omega^{p-1,1}$$

$\Longrightarrow$ source forms

Variational derivative or Euler operator:

$$\delta = \pi \circ d_V : \qquad \Omega^{p,0} \longrightarrow \mathcal{F}^1$$

$$\lambda = L\,d\mathbf{x} \longrightarrow \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\; \theta^\alpha \wedge d\mathbf{x}$$

Variational Problems $\longrightarrow$ Source Forms

The Variational Bicomplex

$$
\begin{array}{cccccccccccc}
 & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & \nearrow{\scriptstyle \delta} & \\
 & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & & \\
\nearrow & & & & & & & & & & & \\
\mathbb{R} & & & & & & & & & & &
\end{array}
$$

$$
\begin{aligned}
p &= \#\ \text{independent variables} \\
q &= \#\ \text{dependent variables}
\end{aligned}
$$

Invariantization of the Bicomplex

$$\iota : \quad \begin{array}{lcl} \text{Functions} & \longrightarrow & \text{Invariants} \\ \text{Forms} & \longrightarrow & \text{Invariant Forms} \end{array}$$

- Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i) = \omega^i + \sigma^i$$

- Invariant contact forms

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

Invariant Variational Complex

$\Longrightarrow$ *I. Kogan, PJO*

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$\begin{array}{llll} d_{\mathcal{H}}: & \widehat{\Omega}^{r,s} & \longrightarrow & \widehat{\Omega}^{r+1,s} \\ d_{\mathcal{V}}: & \widehat{\Omega}^{r,s} & \longrightarrow & \widehat{\Omega}^{r,s+1} \\ d_{\mathcal{W}}: & \widehat{\Omega}^{r,s} & \longrightarrow & \widehat{\Omega}^{r-1,s+2} \end{array}$$

$$d_{\mathcal{H}}^2 = 0 \qquad d_{\mathcal{H}}\, d_{\mathcal{V}} + d_{\mathcal{V}}\, d_{\mathcal{H}} = 0$$

$$d_{\mathcal{V}}^2 + d_{\mathcal{H}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{H}} = 0$$

$$d_{\mathcal{V}}\, d_{\mathcal{W}} + d_{\mathcal{W}}\, d_{\mathcal{V}} = 0 \qquad d_{\mathcal{W}}^2 = 0$$

The Key Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{k=1}^{p} \nu^\kappa \wedge\ \iota[\mathbf{v}_\kappa(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$

$$\nu^\kappa = \sigma^* \mu^\kappa = \gamma^\kappa + \varepsilon^\kappa \qquad \kappa = 1, \ldots, r$$

$$\gamma^\kappa \in \widehat{\Omega}^{1,0} \qquad \varepsilon^\kappa \in \widehat{\Omega}^{0,1}$$

— pull back of the dual basis Maurer–Cartan forms via the moving frame section

$$\sigma^* : \mathrm{J}^\infty \to \mathcal{B}^\infty$$

⋆ ⋆ ⋆ All recurrence formulae, syzygies, commutation formulae, etc. are found by applying the key formula for various forms and functions Ω

Euclidean Curves

Lifted invariants

$$y = w^*(x) = x\cos\phi - u\sin\phi + a$$

$$v = w^*(u) = x\cos\phi + u\sin\phi + b$$

$$v_y = w^*(u_x) = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}$$

$$v_{yy} = w^*(u_{xx}) = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

$$v_{yyy} = w^*(u_{xx}) = \frac{(\cos\phi\ - u_x\sin\phi\,)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi\ - u_x\sin\phi\,)^5}$$

$$dy = (\cos\phi - u_x\sin\phi)\,dx - (\sin\phi)\,\theta + da - v\,d\phi$$

$$d_J y = \pi_J(dy) = (\cos\phi - u_x\sin\phi)\,dx - (\sin\phi)\,\theta$$

$$D_y = \frac{1}{\cos\phi - u_x\sin\phi}\,D_x \qquad\qquad \theta = du - u_x\,dx$$

Normalization

$$y = 0 \qquad\qquad v = 0 \qquad\qquad v_y = 0$$

Right moving frame $\qquad \rho\colon \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$\phi = -\ \tan^{-1} u_x \qquad\qquad a = -\,\frac{x + uu_x}{\sqrt{1+u_x^2}} \qquad\qquad b = \frac{xu_x - u}{\sqrt{1+u_x^2}}$$

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa$$

$$\iota(u_{xxx}) = I_3 = \kappa_s$$

$$\iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

Invariant horizontal one-form

$$\iota(dx) = \sigma^*(\, d_J\, y) = \varpi \quad = \quad \omega \quad + \quad \eta$$

$$= \sqrt{1+u_x^2}\, dx \quad + \quad \frac{u_x}{\sqrt{1+u_x^2}}\, \theta$$

Invariant contact forms

$$\iota(\theta) = \vartheta = \frac{\theta}{\sqrt{1+u_x^2}}$$

$$\iota(\theta_x) = \vartheta_1 = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx} \theta}{(1+u_x^2)^2}$$

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x \qquad\qquad \mathbf{v}_2 = \partial_u$$

$$\mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

$$d_{\mathcal{H}} I = D_s I \cdot \varpi$$

Horizontal recurrence formula

$$d_{\mathcal{H}}\,\iota(F) = \iota(d_H F) + \iota(\mathbf{v}_1(F))\,\gamma^1 + \iota(\mathbf{v}_2(F))\,\gamma^2 + \iota(\mathbf{v}_3(F))\,\gamma^3$$

Use phantom invariants

$$0 = d_{\mathcal{H}}\,H = \iota(d_H x) + \sum \iota(\mathbf{v}_\kappa(x))\,\gamma^\kappa = \varpi + \gamma^1,$$

$$0 = d_{\mathcal{H}}\,I_0 = \iota(d_H u) + \sum \iota(\mathbf{v}_\kappa(u))\,\gamma^\kappa = \gamma^2,$$

$$0 = d_{\mathcal{H}}\,I_1 = \iota(d_H u_x) + \sum \iota(\mathbf{v}_\kappa(u_x))\,\gamma^\kappa = \kappa\,\varpi + \gamma^3,$$

to solve for

$$\gamma^1 = -\varpi \qquad \gamma^2 = 0 \qquad \gamma^3 = -\kappa\,\varpi$$

$$\gamma^1 = -\varpi \qquad \gamma^2 = 0 \qquad \gamma^3 = -\kappa\,\varpi$$

Recurrence formulae

$$\kappa_s \varpi = d_{\mathcal{H}}\,\kappa = d_{\mathcal{H}}\,(I_2) = \iota(d_H u_{xx}) + \iota(\mathbf{v}_3(u_{xx}))\,\gamma^3$$

$$= \iota(u_{xxx}\,dx) - \iota(3u_x u_{xx}))\,\kappa\,\varpi = I_3\,\varpi$$

$$\kappa_{ss}\varpi = d_{\mathcal{H}}\,(I_3) = \iota(d_H u_{xxx}) + \iota(\mathbf{v}_3(u_{xxx})\,\gamma^3$$

$$= \iota(u_{xxxx}\,dx) - \iota(4u_x u_{xxx} + 3u_{xx}^2)\,\kappa\,\varpi = I_4 - 3I_2^3\,\varpi$$

$$\kappa = I_2 \qquad I_2 = \kappa$$

$$\kappa_s = I_3 \qquad I_3 = \kappa_s$$

$$\kappa_{ss} = I_4 - 3I_2^3 \qquad I_4 = \kappa_{ss} + 3\kappa^3$$

$$\kappa_{sss} = I_5 - 19I_2^2 I_3 \qquad I_4 = \kappa_{sss} + 19\kappa^2\kappa_s$$

Vertical recurrence formula

$$d_{\mathcal{V}}\,\iota(F) = \iota(\,d_V\,F) + \iota(\mathbf{v}_1(F))\,\varepsilon^1 + \iota(\mathbf{v}_2(F))\,\varepsilon^2 + \iota(\mathbf{v}_3(F))\,\varepsilon^3$$

Use phantom invariants

$$0 = \,d_{\mathcal{V}}\,H = \varepsilon^1$$

$$0 = \,d_{\mathcal{V}}\,I_0 = \vartheta + \varepsilon^2$$

$$0 = \,d_{\mathcal{V}}\,I_1 = \vartheta_1 + \varepsilon^3$$

to solve for

$$\varepsilon^1 = 0 \qquad \varepsilon^2 = -\,\vartheta = -\,\iota(\theta) \qquad \varepsilon^3 = -\,\vartheta_1 = -\,\iota(\theta_1)$$

Recurrence formulae

$$d_{\mathcal{V}}\,I_2 = \,d_{\mathcal{V}}\,\kappa = \iota(\theta_2) + \iota(\mathbf{v}_3(u_{xx}))\,\varepsilon^3 = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta,$$

$d_{\mathcal{H}}\,\vartheta$:

$$\mathcal{D}\vartheta = \vartheta_1 \qquad \mathcal{D}\vartheta_1 = \vartheta_2 - \kappa^2\,\vartheta$$

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Example

$(x^1, x^2, u) \in M = \mathbb{R}^3$ $\qquad\qquad$ $G = \mathrm{GL}(2)$

$$(x^1, x^2, u) \longmapsto (\alpha x^1 + \beta x^2, \gamma x^1 + \delta x^2, \lambda u)$$

$$\lambda = \alpha\delta - \beta\gamma$$

$\Longrightarrow$ Classical invariant theory

Prolongation (lifted differential invariants):

$$y^1 = \lambda^{-1}(\delta x^1 - \beta x^2) \qquad y^2 = \lambda^{-1}(-\gamma x^1 + \alpha x^2)$$

$$v = \lambda^{-1} u$$

$$v_1 = \frac{\alpha u_1 + \gamma u_2}{\lambda} \qquad v_2 = \frac{\beta u_1 + \delta u_2}{\lambda}$$

$$v_{11} = \frac{\alpha^2 u_{11} + 2\alpha\gamma u_{12} + \gamma^2 u_{22}}{\lambda}$$

$$v_{12} = \frac{\alpha\beta u_{11} + (\alpha\delta + \beta\gamma) u_{12} + \gamma\delta u_{22}}{\lambda}$$

$$v_{22} = \frac{\beta^2 u_{11} + 2\beta\delta u_{12} + \delta^2 u_{22}}{\lambda}$$

Normalization

$$y^1 = 1 \qquad y^2 = 0 \qquad v_1 = 1 \qquad v_2 = 0$$

Nondegeneracy

$$x^1 \frac{\partial u}{\partial x^1} + x^2 \frac{\partial u}{\partial x^2} \neq 0$$

First order moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^1 & -u_2 \\ x^2 & u_1 \end{pmatrix}$$

Normalized differential invariants

$$J^1 = 1 \qquad\qquad J^2 = 0$$

$$I = \frac{u}{x^1 u_1 + x^2 u_2}$$

$$I_1 = 1 \qquad\qquad I_2 = 0$$

$$I_{11} = \frac{(x^1)^2 u_{11} + 2x^1 x^2 u_{12} + (x^2)^2 u_{22}}{x^1 u_1 + x^2 u_2}$$

$$I_{12} = \frac{-x^1 u_2 u_{11} + (x^1 u_1 - x^2 u_2) u_{12} + x^2 u_1 u_{22}}{x^1 u_1 + x^2 u_2}$$

$$I_{22} = \frac{(u_2)^2 u_{11} - 2u_1 u_2 u_{12} + (u_1)^2 u_{22}}{x^1 u_1 + x^2 u_2}$$

Phantom differential invariants

$$I_1 \qquad\qquad I_2$$

Generating differential invariants

$$I \qquad\quad I_{11} \qquad\quad I_{12} \qquad\quad I_{22}$$

Invariant differential operators

$\mathcal{D}_1 = x^1 D_1 + x^2 D_2$ — scaling process

$\mathcal{D}_2 = -u_2 D_1 + u_1 D_2$ — Jacobian process

Recurrence formulae

$$\mathcal{D}_1 J^1 = \delta_1^1 - 1 = 0 \qquad \mathcal{D}_2 J^1 = \delta_2^1 - 0 = 0$$

$$\mathcal{D}_1 J^2 = \delta_1^2 - 0 = 0 \qquad \mathcal{D}_2 J^2 = \delta_2^2 - 1 = 0$$

$$\mathcal{D}_1 I = I_1 - I(1 + I_{11}) = -I(1 + I_{11}) \qquad \mathcal{D}_2 I = I_2 - I\, I_{12} = -I\, I_{12}$$

$$\mathcal{D}_1 I_1 = I_{11} - I_{11} = 0 \qquad \mathcal{D}_2 I_1 = I_{12} - I_{12} = 0$$

$$\mathcal{D}_1 I_2 = I_{12} - I_{12} = 0 \qquad \mathcal{D}_2 I_2 = I_{22} - I_{22} = 0$$

$$\mathcal{D}_1 I_{11} = I_{111} + (1 - I_{11}) I_{11} \qquad \mathcal{D}_2 I_{11} = I_{112} + (2 - I_{11}) I_{12}$$

$$\mathcal{D}_1 I_{12} = I_{112} - I_{11} I_{12} \qquad \mathcal{D}_2 I_{12} = I_{122} + (1 - I_{11}) I_{22}$$

$$\mathcal{D}_1 I_{22} = I_{122} + (I_{11} - 1) I_{22} - 2 I_{12}^2 \qquad \mathcal{D}_2 I_{22} = I_{222} - I_{12} I_{22}$$

$\Longrightarrow$ Use I to generate I_{11} and I_{12}

Syzygies

$$\begin{aligned}
\mathcal{D}_1 I_{12} - \mathcal{D}_2 I_{11} &= -2 I_{12} \\
\mathcal{D}_1 I_{22} - \mathcal{D}_2 I_{12} &= 2(I_{11} - 1) I_{22} - 2 I_{12}^2 \\
(\mathcal{D}_1)^2 I_{22} - (\mathcal{D}_2)^2 I_{11} &= \\
&= 2 I_{22} \mathcal{D}_1 I_{11} + (5 I_{12} - 2) \mathcal{D}_1 I_{12} + (3 I_{11} - 5) \mathcal{D}_1 I_{22} - \\
&\quad - (2 I_{11} - 5)(I_{11} - 1) I_{12} + 4 (I_{11} - 1) I_{12}^2
\end{aligned}$$

Commutation formulae

$$[\mathcal{D}_1, \mathcal{D}_2] = -I_{12} \mathcal{D}_1 + (I_{11} - 1) \mathcal{D}_2$$

Invariant Variational Problems

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\;\ldots\; \mathcal{D}_K I^\alpha \;\ldots\;)\, \boldsymbol{\omega}$$

$I_1, \ldots, I_\ell$ — fundamental differential invariants

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — contact-invariant volume form

Invariant Euler-Lagrange equations

$$\mathbf{E}(L) = F(\;\ldots\; \mathcal{D}_K I^\alpha \;\ldots\;) = 0$$

Problem.

Construct F directly from P.

$\Longrightarrow$ P. Griffiths, I. Anderson

Example. Planar Euclidean group $G = \mathrm{SE}(2)$

Invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) = F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\, \kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1 + u_x^2)^{5/2}}$$

Euler-Lagrange equation

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\, \kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler operator

$$\mathcal{E} = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial}{\partial \kappa_n} \qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian operator

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\, (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(P) + \kappa\, \mathcal{H}(P).$$

Elastica

$$P = \tfrac{1}{2}\kappa^2 \qquad \mathcal{E}(P) = \kappa \qquad \mathcal{H}(P) = -P = -\tfrac{1}{2}\kappa^2$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0$$

Euler-Lagrange Equations

Integration by Parts:

$$\pi : \Omega^{p,1} \longrightarrow \mathcal{F}^1 = \Omega^{p,1} \,/\, d_H\, \Omega^{p-1,1}$$

$$\Longrightarrow \quad \text{Source forms}$$

Variational derivative or Euler operator:

$$\delta = \pi \circ d_V : \Omega^{p,0} \quad \longrightarrow \quad \mathcal{F}^1$$

$$\text{Variational Problems} \quad \longrightarrow \quad \text{Source Forms}$$

$$\delta : \lambda = L\, d\mathbf{x} \quad \longrightarrow \quad \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\ \theta^\alpha \wedge\ d\mathbf{x}$$

Hamiltonian

$$\mathbf{H}(L) = \sum_{\alpha=1}^{m} \sum_{i>j\geq 0} u^\alpha_{i-j}(-D_x)^j \, \frac{\partial L}{\partial u^\alpha_i} - L$$

The Simplest Example. $\quad M = \mathbb{R}^2 \quad x, u \in \mathbb{R}$

Lagrangian form

$$\lambda = L(x, u^{(n)})\, dx$$

Vertical derivative

$$\begin{aligned} d\lambda &= d_V \lambda \\ &= \left(\frac{\partial L}{\partial u}\, \theta + \frac{\partial L}{\partial u_x}\, \theta_x + \frac{\partial L}{\partial u_{xx}}\, \theta_{xx} + \cdots \right) \wedge\, dx \in \Omega^{1,1} \end{aligned}$$

Integration by parts

$$\begin{aligned} d_H\, (A\, \theta) &= (D_x A)\, dx \wedge \theta - A\, \theta_x \wedge dx \\ &= -[\, (D_x A)\, \theta + A\, \theta_x\,] \wedge dx \end{aligned}$$

Variational derivative

$$\begin{aligned} \delta\lambda &= \left(\frac{\partial L}{\partial u} - D_x\, \frac{\partial L}{\partial u_x} + D_x^2\, \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge\, dx \\ &= \mathbf{E}(L)\, \theta \wedge\, dx \in \mathcal{F}^1 \end{aligned}$$

Plane Curves

Invariant Lagrangian

$$\int P(\kappa, \kappa_s, \ldots)\, \varpi$$

κ — fundamental differential invariant (curvature)

$\varpi = \omega + \eta$ — fully invariant horizontal form

$\omega = ds$ — contact-invariant arc length

Invariant integration by parts

$$d_{\mathcal{V}}\,(P\,\varpi) = \mathcal{E}(P)\; d_{\mathcal{V}}\,\kappa \wedge \varpi - \mathcal{H}(P)\; d_{\mathcal{V}}\,\varpi$$

Vertical differentiation formulae

$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta)$ $\qquad$ $\mathcal{A}$ — Eulerian operator

$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$ $\qquad$ $\mathcal{B}$ — Hamiltonian operator

$\Longrightarrow$ The explicit formulae follow from our fundamental recurrence formula, based on the infinitesimal generators of the action.

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = 0}$$

General Framework

Fundamental differential invariants

$$I^1, \ldots, I^\ell$$

Invariant horizontal coframe

$$\varpi^1, \ldots, \varpi^p$$

Dual invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

Invariant volume form

$$\boldsymbol{\varpi} = \varpi^1 \wedge \cdots \wedge \varpi^p$$

Differentiated invariants

$$I^\alpha_{,K} = \mathcal{D}^K J^\alpha = \mathcal{D}_{k_1} \cdots \mathcal{D}_{k_n} J^\alpha$$

$\Longrightarrow$ order is important!

Eulerian operator

$$d_{\mathcal{V}}\, I^\alpha = \sum_{\beta=1}^{q} \mathcal{A}_\beta^\alpha(\vartheta^\beta) \qquad \mathcal{A} = (\, \mathcal{A}_\beta^\alpha \,)$$

$\Longrightarrow$ $m \times q$ matrix of invariant differential operators

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Hamiltonian operator complex

$$d_{\mathcal{V}}\, \varpi^j = \sum_{\beta=1}^{q} \mathcal{B}_{i,\beta}^j(\vartheta^\beta) \wedge \varpi^i \qquad \mathcal{B}_i^j = (\, \mathcal{B}_{i,\beta}^j \,)$$

$\Longrightarrow$ p^2 row vectors of invariant differential operators

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$$\boldsymbol{\varpi}_{(i)} = (-1)^{i-1}\, \varpi^1 \wedge \cdots \wedge \varpi^{i-1} \wedge \varpi^{i+1} \wedge \cdots \wedge \varpi^p$$

Twist invariants

$$d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(i)} = Z_i\, \boldsymbol{\varpi}$$

Twisted adjoint

$$\mathcal{D}_i^\dagger = -(\mathcal{D}_i + Z_i)$$

Invariant variational problem

$$\int P(I^{(n)})\,\boldsymbol{\varpi}$$

Invariant Eulerian

$$\mathcal{E}_\alpha(P) = \sum_K \mathcal{D}_K^\dagger \frac{\partial P}{\partial I^\alpha_{,K}}$$

Invariant Hamiltonian tensor

$$\mathcal{H}^i_j(P) = -\,P\,\delta^i_j + \sum_{\alpha=1}^{q} \sum_{J,K} I^\alpha_{,J,j}\,\mathcal{D}_K^\dagger \frac{\partial P}{\partial I^\alpha_{,J,i,K}}\,,$$

Invariant Euler-Lagrange equations

$$\mathcal{A}^\dagger\,\mathcal{E}(P) - \sum_{i,j=1}^{p} (\mathcal{B}^j_i)^\dagger\,\mathcal{H}^i_j(P) = 0.$$

Euclidean Surfaces

$S \subset M = \mathbb{R}^3$ coordinates $z = (x, y, u)$

Group: $G = \mathrm{E}(3)$

$$z \longmapsto R\,z + a, \qquad R \in \mathrm{O}(3)$$

Normalization — coordinate cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Left moving frame

$$a = z \qquad R = (\ \mathbf{t}_1\ \mathbf{t}_2\ \mathbf{n}\ )$$

- $\mathbf{t}_1, \mathbf{t}_2 \in TS$ — Frenet frame
- $\mathbf{n}$ — unit normal

Fundamental differential invariants

$$\kappa^1 = \iota(u_{xx}) \qquad \kappa^2 = \iota(u_{yy})$$

$\Longrightarrow$ principal curvatures

Frenet coframe

$$\varpi^1 = \iota(dx^1) = \omega^1 + \eta^1 \qquad \varpi^2 = \iota(dx^2) = \omega^2 + \eta^2$$

Invariant differential operators

$$\mathcal{D}_1 \qquad \mathcal{D}_2$$

$\Longrightarrow$ Frenet differentiation

Fundamental Syzygy:

Use the recurrence formula to compare

$$\iota(u_{xxyy}) \qquad \text{with} \qquad \begin{aligned} \kappa^1_{,22} &= \mathcal{D}_2^2 \iota(u_{xx}) \\ \kappa^2_{,11} &= \mathcal{D}_1^2 \iota(u_{yy}) \end{aligned}$$

$$\kappa^1_{,22} - \kappa^2_{,11} + \frac{\kappa^1_{,1}\kappa^2_{,1} + \kappa^1_{,2}\kappa^2_{,2} - 2(\kappa^2_{,1})^2 - 2(\kappa^1_{,2})^2}{\kappa^1 - \kappa^2} - \kappa^1\kappa^2(\kappa^1 - \kappa^2) = 0$$

$\Longrightarrow$ Codazzi equations

Twisted adjoints

$$\mathcal{D}_1^\dagger = -\,(\mathcal{D}_1 + Z_1) \qquad Z_1 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}$$

$$\mathcal{D}_2^\dagger = -(\mathcal{D}_2 + Z_2) \qquad Z_2 = \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}$$

Gauss curvature — Codazzi equations:

$$K = \kappa^1\kappa^2 = \mathcal{D}_1^\dagger(Z_1) + \mathcal{D}_2^\dagger(Z_2)$$
$$= -\,(\mathcal{D}_1 + Z_1)Z_1 - (\mathcal{D}_2 + Z_2)Z_2$$

K is an invariant divergence

$\Longrightarrow$ Gauss–Bonnet Theorem!

Invariant contact form

$$\vartheta = \iota(\theta) = \iota(du - u_x\, dx - u_y\, dy)$$

Invariant vertical derivatives

$$d_{\mathcal{V}}\, \kappa^1 = \iota(\theta_{xx}) = (\, \mathcal{D}_1^2 + Z_2\, \mathcal{D}_2 + (\kappa^1)^2 \,)\, \vartheta$$

$$d_{\mathcal{V}}\, \kappa^2 = \iota(\theta_{yy}) = (\, \mathcal{D}_2^2 + Z_1\, \mathcal{D}_1 + (\kappa^2)^2 \,)\, \vartheta$$

Eulerian operator

$$\mathcal{A} = \begin{pmatrix} \mathcal{D}_1^2 + Z_2\, \mathcal{D}_2 + (\kappa^1)^2 \\ \mathcal{D}_2^2 + Z_1\, \mathcal{D}_1 + (\kappa^2)^2 \end{pmatrix}$$

$$d_{\mathcal{V}}\, \varpi^1 = -\,\kappa^1\, \vartheta \wedge \varpi^1 + \frac{1}{\kappa^1 - \kappa^2}(\, \mathcal{D}_1 \mathcal{D}_2 - Z_2 \mathcal{D}_1\,)\vartheta \wedge \varpi^2,$$

$$d_{\mathcal{V}}\, \varpi^2 = \frac{1}{\kappa^2 - \kappa^1}(\, \mathcal{D}_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2\,)\vartheta \wedge \varpi^1 - \kappa^2\, \vartheta \wedge \varpi^2,$$

Hamiltonian operator complex

$$\begin{aligned} \mathcal{B}_1^1 &= -\kappa^1, \\ \mathcal{B}_2^2 &= -\kappa^2, \end{aligned} \qquad \mathcal{B}_2^1 = \frac{1}{\kappa^1 - \kappa^2}(\, \mathcal{D}_1 \mathcal{D}_2 - Z_2 \mathcal{D}_1\,) = -\,\mathcal{B}_1^2$$

Euclidean-invariant variational problem

$$\int P(\kappa^{(n)})\,\omega^1 \wedge \omega^2 = \int P(\kappa^{(n)})\; dA$$

Euler-Lagrange equations

$$\mathbf{E}(L) = \mathcal{A}^\dagger\,\mathcal{E}(P) - \mathcal{B}^\dagger\,\mathcal{H}(P) = 0,$$

Special case: $P(\kappa^1, \kappa^2)$

$$\mathbf{E}(L) = [\,(\mathcal{D}_1^\dagger)^2 + \mathcal{D}_2^\dagger \cdot Z_2 + (\kappa^1)^2\,]\,\frac{\partial \widetilde{L}}{\partial \kappa^1} +$$

$$+ [\,(\mathcal{D}_2^\dagger)^2 + \mathcal{D}_1^\dagger \cdot Z_1 + (\kappa^2)^2\,]\,\frac{\partial \widetilde{L}}{\partial \kappa^2} - (\kappa^1 + \kappa^2)\widetilde{L}.$$

Minimal surfaces: $P = 1$

$$-(\kappa^1 + \kappa^2) = -2H = 0$$

Minimizing mean curvature: $P = H = \frac{1}{2}(\kappa^1 + \kappa^2)$

$$\tfrac{1}{2}\left[\,(\kappa^1)^2 + (\kappa^2)^2 - (\kappa^1 + \kappa^2)^2\,\right] = -\,\kappa^1\,\kappa^2 = -K = 0.$$

Willmore surfaces: $P = \frac{1}{2}(\kappa^1)^2 + \frac{1}{2}(\kappa^2)^2$

$$\Delta(\kappa^1 + \kappa^2) + \tfrac{1}{2}(\kappa^1 + \kappa^2)(\kappa^1 - \kappa^2)^2 = 2\,\Delta H + 4(H^2 - K)H = 0$$

Laplace–Beltrami operator

$$\Delta = (\mathcal{D}_1 + Z_1)\mathcal{D}_1 + (\mathcal{D}_2 + Z_2)\mathcal{D}_2 = -\,\mathcal{D}_1^\dagger \cdot \mathcal{D}_1 - \mathcal{D}_2^\dagger \cdot \mathcal{D}_2$$