# *Differential Invariants of Surfaces*

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Iowa, May, 2007

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## via Moving Frames

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# Examples of Differential Invariants

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## Euclidean Group on $\mathbb{R}^3$

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\Longrightarrow \quad \text{group of rigid motions}$$

$$z \quad \longmapsto \quad R\,z + b \qquad\qquad R \in \mathrm{SO}(3)$$

- Induced action on curves and surfaces.

# Euclidean Curves $C \subset \mathbb{R}^3$

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$$I = H(\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \ \ldots \ )$$

Thus, $\kappa$ and $\tau$ *generate* the differential invariants of space curves under the Euclidean group.

# Euclidean Surfaces $S \subset \mathbb{R}^3$

- $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — mean curvature: order $= 2$
- $K = \kappa_1 \kappa_2$ — Gauss curvature: order $= 2$

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**Theorem.** Every Euclidean differential invariant of a non-umbilic surface $S \subset \mathbb{R}^3$ can be written

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Thus, $H, K$ *generate* the differential invariants of (generic) Euclidean surfaces.

# Equi-affine Group on $\mathbb{R}^3$

$$G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3 \qquad \text{—} \qquad \text{volume preserving}$$

$$z \quad \longmapsto \quad A\,z + b, \qquad \det A = 1$$

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Curves in $\mathbb{R}^3$:

- $\kappa$ — equi-affine curvature: order $= 4$
- $\tau$ — equi-affine torsion: order $= 5$
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — diff. w.r.t. equi-affine arc length

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Surfaces in $\mathbb{R}^3$:

- $P$ — Pick invariant: order $= 3$
- $Q_0, Q_1, \ldots, Q_4$ — fourth order invariants
- $\mathcal{D}_1 P, \mathcal{D}_2 P, \mathcal{D}_1 Q_\nu, \ldots$ diff. w.r.t. the equi-affine frame

# *General Problem*

Find a minimal system of
generating differential invariants.

# Curves

**Theorem.** Let $G$ be an ordinary$^\star$ Lie group acting on the $m$-dimensional manifold $M$. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the $G$-invariant arc length element $ds$.

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$\Longrightarrow \quad m = 3 \quad$ — curvature $\kappa$ & torsion $\tau$

# Equi-affine Surfaces

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$$Q_\nu = \Phi_\nu(P, \mathcal{D}_1 P, \mathcal{D}_2 P, \ \ldots \ )$$

# Euclidean Surfaces

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# Euclidean Surfaces

**Theorem.**

The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

$$K = \Phi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots \ )$$

# Euclidean Proof

Commutation relation:

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1 \mathcal{D}_2 - \mathcal{D}_2 \mathcal{D}_1 = Z_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2,$$

Commutator invariants:

$$Z_1 = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad Z_2 = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

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Codazzi relation:

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Z_1) Z_1 - (\mathcal{D}_2 + Z_2) Z_2$$

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$\Longrightarrow$ Gauss' Theorema Egregium

(*Guggenheimer*)

To determine the commutator invariants:

$$\mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H = Z_2\,\mathcal{D}_1 H - Z_1\,\mathcal{D}_2 H$$
$$\mathcal{D}_1\mathcal{D}_2\mathcal{D}_J H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_J H = Z_2\,\mathcal{D}_1\mathcal{D}_J H - Z_1\,\mathcal{D}_2\mathcal{D}_J H \qquad (*)$$

Nondegenerate surface:

$$\det \begin{pmatrix} \mathcal{D}_1 H & \mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_J H & \mathcal{D}_2\mathcal{D}_J H \end{pmatrix} \neq 0,$$

Solve $(*)$ for $Z_1, Z_2$ in terms of derivatives of $H$.

*Q.E.D.*

# General (Moving) Framework

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$M$ — $m$-dimensional manifold

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — $n^{\text{th}}$ order jet space for

$p$-dimensional submanifolds $S \subset M$

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$G$ — transformation group acting on $M$

$G^{(n)}$ — prolonged action

on the submanifold jet space $\mathrm{J}^n$

# Differential Invariants

Differential invariant $\qquad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

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Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p \qquad\qquad p = \dim S$$

---

$\mathcal{I}(G)$ — the algebra (sheaf) of differential invariants

# The Basis Theorem

**Theorem.** The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \; \ldots \; , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \; \ldots \; , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Pohjanpelto–O*

# Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

# Applications

- Equivalence and signatures of submanifolds
- Characterization of moduli spaces
- Invariant differential equations:

$$H( \ldots \mathcal{D}_J I_\kappa \ldots ) = 0$$

- Group splitting of PDEs and explicit solutions
- Invariant variational problems:

$$\int L( \ldots \mathcal{D}_J I_\kappa \ldots ) \boldsymbol{\omega}$$

# Equivariant Moving Frames

**Definition.** An $n^{\text{th}}$ order *moving frame* is a $G$-equivariant map

$$\rho^{(n)} : V^n \subset \mathrm{J}^n \longrightarrow G$$

- *É. Cartan*
- *Griffiths, Jensen, Green*
- *Fels–O*

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Equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} g \cdot \rho(z^{(n)}) & \text{left moving frame} \\ \rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

---

Note: $\qquad \rho_{left}(z^{(n)}) = \rho_{right}(z^{(n)})^{-1}$

**Theorem.** A moving frame exists in a neighborhood of a jet $z^{(n)} \in \mathrm{J}^n$ if and only if $G$ acts freely and regularly near $z^{(n)}$.

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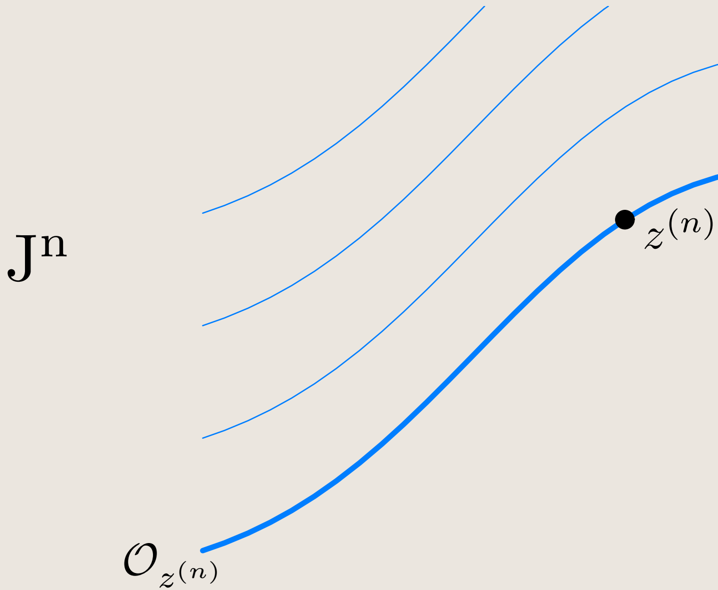
**Theorem.** If $G$ acts locally effectively on subsets, then for $n \gg 0$, the (prolonged) action of $G$ is locally free on an open subset of $\mathrm{J}^n$.

$\Longrightarrow$ *Ovsiannikov, O*

# Geometric Construction




Normalization = choice of cross-section to the group orbits

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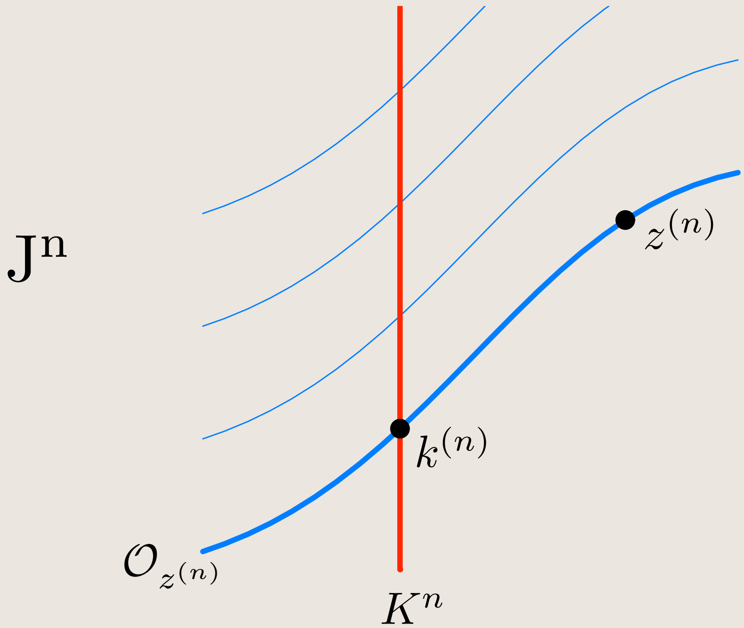


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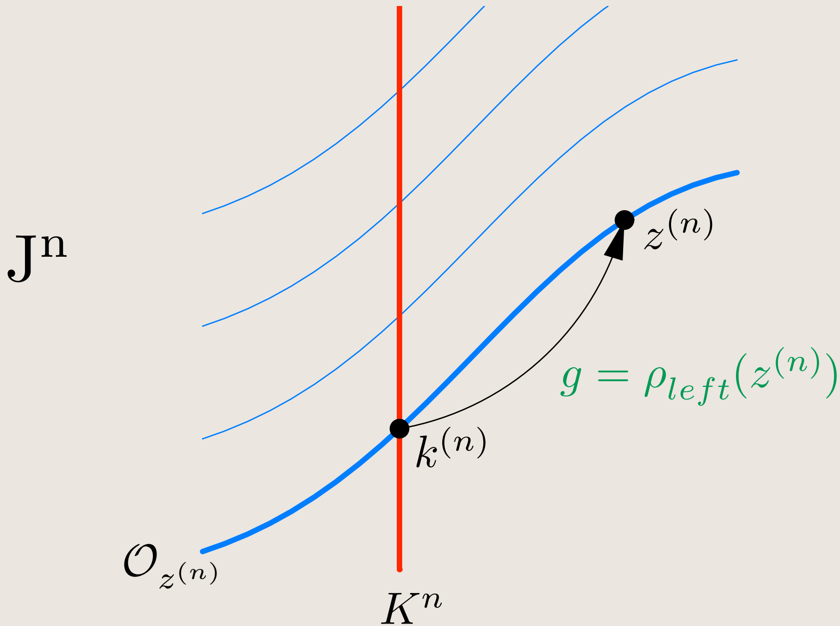


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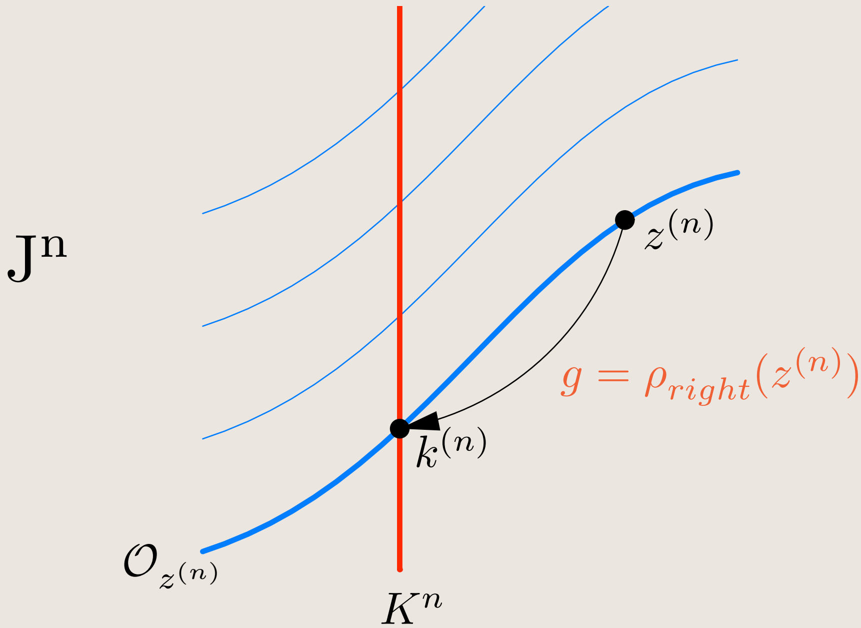


Normalization = choice of cross-section to the group orbits

# The Normalization Construction

**1.** Write out the explicit formulas for the prolonged group action:

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$$

$$\Longrightarrow \quad \textit{Implicit differentiation}$$

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$\Longrightarrow$ *Implicit differentiation*

**2.** From the components of $w^{(n)}$, choose $r = \dim G$ *normalization equations* to define the cross-section:

$$w_1(g, z^{(n)}) = c_1 \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r$$

**3.** Solve the normalization equations for the group parameters $g = (g_1, \ldots, g_r)$:

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The solution is the right moving frame.

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**4.** Substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants of order $\leq n$:

$$I_k(x, u^{(n)}) = w_k(\rho(z^{(n)}), z^{(n)})), \qquad k = r+1, \ \ldots \ , \dim \mathrm{J}^n$$

# Invariantization

The process of replacing group parameters in transformation rules by their moving frame formulae is known as invariantization:

$$\iota : \begin{cases} \text{Functions} & \longrightarrow \quad \text{Invariants} \\ \text{Forms} & \longrightarrow \quad \text{Invariant Forms} \\ \begin{array}{c}\text{Differential} \\ \text{Operators}\end{array} & \longrightarrow \quad \begin{array}{c}\text{Invariant Differential} \\ \text{Operators}\end{array} \\ \quad \vdots & \qquad\qquad \vdots \end{cases}$$

- Invariantization defines an (exterior) algebra morphism.
- Invariantization does not affect invariants: $\iota(I) = I$

# The Fundamental Differential Invariants

Invariantized jet coordinate functions:

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)$$

- The constant differential invariants, as dictated by the moving frame normalizations, are known as the phantom invariants.
- The remaining non-constant differential invariants are the basic invariants and form a complete system of functionally independent differential invariants for the prolonged group action.

**Invariantization of general differential functions:**

$$\iota\,[\,F(\;\ldots\;x^i\;\;\ldots\;u^\alpha_J\;\;\ldots\;)\,] = F(\;\ldots\;H^i\;\;\ldots\;I^\alpha_J\;\;\ldots\;)$$

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## The Replacement Theorem:

If $J$ is a differential invariant, then $\iota(J) = J$.

$$J(\,\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\,) = J(\,\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\,)$$

**Invariantization of general differential functions:**

$$\iota\,[\,F(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )\,] = F(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

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Key fact: Invariantization and differentiation do not commute:

$$\iota(D_i F)\ \neq\ \mathcal{D}_i\,\iota(F)$$

# Infinitesimal Generators

Infinitesimal generators of action of $G$ on $M$:

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha_\kappa(x,u)\,\frac{\partial}{\partial u^\alpha} \qquad \kappa = 1,\ldots,r$$

Prolonged infinitesimal generators on $\mathrm{J}^n$:

$$\mathbf{v}^{(n)}_\kappa = \mathbf{v}_\kappa + \sum_{\alpha=1}^{q} \sum_{j=\#J=1}^{n} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\ \frac{\partial}{\partial u^\alpha_J}$$

Prolongation formula:

$$\varphi^\alpha_{J,\kappa} = D_K\left(\varphi^\alpha_\kappa - \sum_{i=1}^{p} u^\alpha_i\,\xi^i_\kappa\right) + \sum_{i=1}^{p} u^\alpha_{J,i}\,\xi^i_\kappa$$

$D_1,\ldots,D_p$ — total derivatives

# Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa \, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

$\omega^i = \iota(dx^i)$ — invariant coframe

$\mathcal{D}_i = \iota(D_{x^i})$ — dual invariant differential operators

$R_j^\kappa$ — Maurer–Cartan invariants

$\mathbf{v}_1, \ \ldots \ \mathbf{v}_r \in \mathfrak{g}$ — infinitesimal generators

$\mu^1, \ \ldots \ \mu^r \in \mathfrak{g}^*$ — dual Maurer–Cartan forms

# The Maurer–Cartan Invariants

Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R_j^\kappa \, \omega^j$$

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*Remark*: When $G \subset \mathrm{GL}(N)$, the Maurer–Cartan invariants $R_j^\kappa$ are the entries of the Frenet matrices

$$\mathcal{D}_i \, \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$$

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$$\mathcal{D}_i \, \rho(x, u^{(n)}) \cdot \rho(x, u^{(n)})^{-1}$$

**Theorem.** (*E. Hubert*) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants serve to generate the differential invariant algebra $\mathcal{I}(G)$.

# Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa\, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants $R_j^\kappa$!

♡ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}(G)$!

# The Universal Recurrence Formula

Let $\Omega$ be any differential form on $\mathrm{J}^n$.

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(\Omega)]$$

$\Longrightarrow$ *The invariant variational bicomplex*

Commutator invariants:

$$d\omega^i = d[\iota(dx^i)] = \iota(d^2x^i) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(dx^i)]$$

$$= - \sum_{j<k} Y^i_{jk}\,\omega^j \wedge \omega^k + \cdots$$

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} Y^i_{jk}\,\mathcal{D}_i$$

# The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

**Theorem.** If $G$ acts transitively on $M$, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

# Equi-affine Surfaces

$$M = \mathbb{R}^3 \qquad G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3 \qquad \dim G = 11.$$

$$g \cdot z = A\,z + b, \qquad \det A = 1, \qquad z = \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbb{R}^3.$$

Surfaces $S \subset M = \mathbb{R}^3$:

$$u = f(x, y)$$

# Hyperbolic case

$$u_{xx}u_{yy} - u_{xy}^2 < 0$$

Cross-section:

$$x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = -1,$$

$$u_{xyy} = u_{xxx}, \qquad u_{xxy} = u_{yyy} = 0.$$

Power series normal form:

$$u(x,y) = \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}c\,(x^3 + 3\,xy^2) + \cdots$$

$$\Longrightarrow \quad \textit{Nonsingular}: c \neq 0.$$

Invariantization — differential invariants: $\quad I_{jk} = \iota(u_{jk})$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = \iota(u_{xxy}) = \iota(u_{yyy}) = 0,$$

$$\iota(u_{xx}) = 1, \qquad \iota(u_{yy}) = -1, \qquad \iota(u_{xxx}) - \iota(u_{xyy}) = 0.$$

Pick invariant:

$$P = \iota(u_{xxx}) = \iota(u_{xyy}).$$

Basic differential invariants of order 4:

$$Q_0 = \iota(u_{xxxx}), \quad Q_1 = \iota(u_{xxxy}), \quad Q_2 = \iota(u_{xxyy}),$$

$$Q_3 = \iota(u_{xyyy}), \quad Q_4 = \iota(u_{yyyy}),$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y).$$

- Since the moving frame has order 3, one can generate all higher order differential invariants from the basic differential invariants of order $\leq 4$.
- This is a consequence of a general theorem, that follows directly from the recurrence formulae.

- Thus, to prove that the Pick invariant generates $\mathcal{I}(G)$, it suffices to generate $Q_0, \ldots, Q_4$ from $P$ by invariant differentiation.

Infinitesimal generators:

$$\mathbf{v}_1 = x\,\partial_x - u\,\partial_u, \qquad \mathbf{v}_2 = y\,\partial_y - u\,\partial_u,$$

$$\mathbf{v}_3 = y\,\partial_x, \qquad \mathbf{v}_4 = u\,\partial_x, \qquad \mathbf{v}_5 = x\,\partial_y,$$

$$\mathbf{v}_6 = u\,\partial_y, \quad \mathbf{v}_7 = x\,\partial_u, \quad \mathbf{v}_8 = y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u,$$

- The translations will be ignored, as they play no role in the higher order recurrence formulae.

# Recurrence formulae

$$\mathcal{D}_i \iota(u_{jk}) = \iota(D_i u_{jk}) + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(x, y, u^{(j+k)}) R_i^\kappa, \qquad j + k \geq 1$$

$$\mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0, 0, I^{(j+k)}) R_1^\kappa$$

$$\mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0, 0, I^{(j+k)}) R_2^\kappa$$

$\varphi_\kappa^{jk}(0, 0, I^{(j+k)}) = \iota[\,\varphi_\kappa^{jk}(x, y, u^{(j+k)})\,]$ — invariantized prolonged infinitesimal generator coefficients

$R_i^\kappa$ — Maurer–Cartan invariants

Phantom recurrence formulae:

$$0 = \mathcal{D}_1 I_{10} = 1 + R_1^7, \qquad 0 = \mathcal{D}_2 I_{10} = R_2^7,$$

$$0 = \mathcal{D}_1 I_{01} = R_1^8, \qquad 0 = \mathcal{D}_2 I_{01} = -1 + R_2^8,$$

$$0 = \mathcal{D}_1 I_{20} = I_{30} - 3R_1^1 - R_1^2, \qquad 0 = \mathcal{D}_2 I_{20} = -3R_2^1 - R_2^2,$$

$$0 = \mathcal{D}_1 I_{11} = -R_1^3 + R_1^5, \qquad 0 = \mathcal{D}_2 I_{11} = I_{30} - R_2^3 + R_2^5,$$

$$0 = \mathcal{D}_1 I_{02} = I_{12} + R_1^1 + 3R_1^2, \qquad 0 = \mathcal{D}_2 I_{02} = R_2^1 + 3R_2^2,$$

$$0 = \mathcal{D}_1 I_{21} = I_{31} - I_{30} R_1^3 - 2I_{30} R_1^5 + R_1^6,$$

$$0 = \mathcal{D}_2 I_{21} = I_{22} - I_{30} R_2^3 - 2I_{30} R_2^5 + R_2^6,$$

$$0 = \mathcal{D}_1 I_{03} = I_{13} - 3I_{30} R_2^3 - 3R_2^6, \qquad 0 = \mathcal{D}_2 I_{03} = I_{04} - 3I_{30} R_2^3 - 3R_2^6.$$

Maurer–Cartan invariants:

$$R_1 = \left( \tfrac{1}{2}P, -\tfrac{1}{2}P, \frac{3Q_1 + Q_3}{12P}, \tfrac{1}{4}Q_0 - \tfrac{1}{4}Q_2 - \tfrac{1}{2}P^2, \frac{3Q_1 + Q_3}{12P}, -\tfrac{1}{4}Q_1 + \tfrac{1}{4}Q_3, -1, 0 \right)$$

$$R_2 = \left( 0, 0, \frac{3Q_2 + Q_4}{12P} + \tfrac{1}{2}P, \tfrac{1}{4}Q_1 - \tfrac{1}{4}Q_3, \frac{3Q_2 + Q_4}{12P} - \tfrac{1}{2}P, -\tfrac{1}{4}Q_2 + \tfrac{1}{4}Q_4 - \tfrac{1}{2}P^2, 0, 1 \right)$$

Fourth order invariants:

$$P_1 = \mathcal{D}_1 P = \tfrac{1}{4}Q_0 + \tfrac{3}{4}Q_2, \qquad P_2 = \mathcal{D}_2 P = \tfrac{1}{4}Q_1 + \tfrac{3}{4}Q_3.$$

Commutator:

$$\mathcal{D}_3 = [\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1 = Y_1\mathcal{D}_1 + Y_2\mathcal{D}_2,$$

Commutator invariants:

$$Y_1 = R_2^1 - R_1^3 = -\frac{3Q_1 + Q_3}{12P}, \qquad Y_2 = R_2^5 - R_1^2 = \frac{3Q_2 + Q_4}{12P}.$$

Another fourth order invariant:

$$P_3 = \mathcal{D}_3 P = \mathcal{D}_1 \mathcal{D}_2 P - \mathcal{D}_2 \mathcal{D}_1 P = Y_1 P_1 + Y_2 P_2. \qquad (*)$$

Nondegeneracy condition: If

$$\det \begin{pmatrix} P_1 & P_2 \\ \mathcal{D}_1 P_j & \mathcal{D}_2 P_j \end{pmatrix} \neq 0 \qquad \text{for} \qquad j = 1, 2, \text{ or } 3,$$

we can solve $(*)$ and

$$\mathcal{D}_3 P_j = Y_1 \mathcal{D}_1 P_j + Y_2 \mathcal{D}_2 P_j$$

for the fourth order commutator invariants:

$$Y_1 = -\frac{3 Q_1 + Q_3}{12 P}, \qquad Y_2 = \frac{3 Q_2 + Q_4}{12 P}.$$

So far, we have constructed four combinations of the fourth order differential invariants

$$S_1 = Q_0 + 3\,Q_2, \qquad S_2 = Q_1 + 3\,Q_3,$$

$$S_3 = 3\,Q_1 + Q_3, \qquad S_4 = 3\,Q_2 + Q_4.$$

as rational functions of the invariant derivatives of the Pick invariant. To obtain the final fourth order differential invariant:

$$12\,P(\,\mathcal{D}_1 S_4 - \mathcal{D}_2 S_3\,) = 48\,P^2 Q_0 - 30\,P^2 S_1 + 18\,P^2 S_4 \\ - 3\,S_2 S_3 - S_3^2 + 3\,S_1 S_4 + S_4^2.$$

⋆ ⋆ ⋆ This completes the proof ⋆ ⋆ ⋆

# The Final Message

The equivariant moving frame methods (Fels-Kogan-O) are completely constructive. They can be applied to arbitrary finite-dimensional transformation groups, as well as to (eventually locally freely acting) infinite-dimensional Lie pseudo-groups (Pohjanpelto–O).