Invariant Variational Problems
&
Integrable Curve Flows

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Variational Problems

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$u^\alpha_J = \partial_J u^\alpha$ — derivatives

Variational problem:

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x}$$

$L(x, u^{(n)})$ — Lagrangian

Variational derivative — Euler-Lagrange equations

$$\mathbf{E}(L) = 0$$

Components:

$$\mathbf{E}_\alpha(L) = \sum_J (-D)^J \, \frac{\partial L}{\partial u^\alpha_J}$$

Invariant Variational Problems

G — transformation group

G–invariant variational problem

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}}$$

$\Longrightarrow$ *Lie*

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

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Invariant Euler-Lagrange equations

$$\mathbf{E}(L) = F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0$$

Main Problem:

Construct F directly from P.

(*P. Griffiths, I. Anderson*)

Example. Planar Euclidean group $\quad G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \qquad \text{— curvature}$$

$$ds = \sqrt{1+u_x^2}\,dx \qquad \text{— arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx} \qquad \text{— arc length derivative}$$

Invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\,ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) = F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[u] = \int ds = \int \sqrt{1+u_x^2}\ dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\,ds = \int \frac{u_{xx}^2\,dx}{(1+u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General Euclidean–invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\, (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(P) + \kappa\, \mathcal{H}(P).$$

Elastica : $\qquad P = \frac{1}{2}\kappa^2$

$$\mathcal{E}(P) = \kappa \qquad\qquad \mathcal{H}(P) = -\,P = -\,\tfrac{1}{2}\kappa^2$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0$$

Moving Frames

$\Longrightarrow$ Mark Fels and PJO

G — r-dimensional Lie group acting on M

$J^n = J^n(M, p)$ — n^{th} order jet bundle for p-dimensional submanifolds $N = \{ u = f(x) \} \subset M$

$z^{(n)} = (x, u^{(n)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$ — coordinates on J^n

Definition.

An n^{th} order *moving frame* is a G-equivariant map

$$\rho = \rho^{(n)} : \ V \subset J^n \longrightarrow G$$

Equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} g \cdot \rho(z^{(n)}) & \text{left moving frame} \\ \rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

Note $\qquad \rho_{left}(z^{(n)}) = \rho_{right}(z^{(n)})^{-1}$

Theorem.

A moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if G acts freely and regularly near $z^{(n)}$.

Theorem.

If G acts locally effectively on subsets, then for $n \gg 0$, the (prolonged) action of G is locally free on an open subset of J^n.

$\Longrightarrow$ Ovsiannikov, PJO

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:

  $$g \cdot z = z \text{ if and only if } g = e.$$

- locally free — the orbits have the same dimension as G.

- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once ($\not\approx$ irrational flow on the torus)

- effective — the only group element $g \in G$ which fixes *every* point $z \in M$ is the identity:

  $$g \cdot z = z \text{ for all } z \in M \text{ if and only if } g = e.$$

The Normalization Construction

1. Write out the explicit formulas for the prolonged group action:

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$$

$\Longrightarrow$ *Implicit differentiation*

2. From the components of $w^{(n)}$, choose $r = \dim G$ *normalization equations*:

$$w_1(g, z^{(n)}) = c_1 \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r$$

3. Solve the normalization equations for the group parameters $g = (g_1, \ldots, g_r)$:

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The solution is the right moving frame.

4. Substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants of order $\leq n$:

$$I_k(x, u^{(n)}) = w_k(\rho(z^{(n)}), z^{(n)}))$$

$$k = r + 1, \ \ldots \ , \dim \mathrm{J}^n$$

Euclidean Plane Curves $\qquad G = \mathrm{SE}(2)$

Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

Prolong to J^3 via implicit differentiation

$$\left.\begin{aligned} y &= x\cos\phi - u\sin\phi + a \\ v &= x\cos\phi + u\sin\phi + b \end{aligned}\right\} \qquad w = R\,z + c$$

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi - u_x\sin\phi)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi - u_x\sin\phi)^5}$$

$$\vdots$$

Normalization: $\qquad r = \dim G = 3$

$$y = 0 \qquad\qquad v = 0 \qquad\qquad v_y = 0$$

Right moving frame $\qquad \rho\colon \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$\phi = -\tan^{-1} u_x \qquad a = -\frac{x + uu_x}{\sqrt{1+u_x^2}} \qquad b = \frac{xu_x - u}{\sqrt{1+u_x^2}}$$

Differential invariants

$$
\begin{aligned}
v_{yy} &\longmapsto \kappa &&= \frac{u_{xx}}{(1+u_x^2)^{3/2}} \\
v_{yyy} &\longmapsto \frac{d\kappa}{ds} &&= \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3} \\
v_{yyyy} &\longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3 &&= \cdots
\end{aligned}
$$

Invariant one-form — arc length

$$
dy = (\cos\phi - u_x \sin\phi)\, dx \qquad \longmapsto \qquad ds = \sqrt{1+u_x^2}\;\, dx
$$

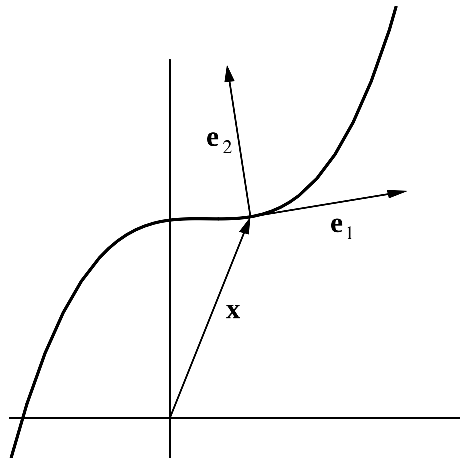
Invariant differential operator

$$
\frac{d}{dy} = \frac{1}{\cos\phi - u_x \sin\phi}\;\frac{d}{dx} \qquad \longmapsto \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\;\frac{d}{dx}
$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$
\kappa \qquad \frac{d\kappa}{ds} \qquad \frac{d^2\kappa}{ds^2} \qquad \cdots
$$

Euclidean Curves



Left moving frame $\quad \widetilde{\rho}(x, u^{(1)}) = \rho(x, u^{(1)})^{-1}$

$$\tilde{a} = x \qquad \tilde{b} = u \qquad \widetilde{\theta} = \tan^{-1} u_x$$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{t} & \mathbf{n} \end{pmatrix} \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix} \qquad \mathbf{n} = \mathbf{t}^{\perp} = \begin{pmatrix} -y_s \\ x_s \end{pmatrix}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \kappa\, \mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = -\,\kappa\, \mathbf{e}_1$$

Invariantization

The process of replacing group parameters in transformation rules by their moving frame formulae is known as *invariantization*:

$$\iota : \quad \begin{array}{ccc} \text{Functions} & \longrightarrow & \text{Invariants} \\ \text{Forms} & \longrightarrow & \text{Invariant Forms} \\ \text{Differential Operators} & \longrightarrow & \text{Invariant Differential Operators} \end{array}$$

Fundamental differential invariants

$$\boxed{H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)}$$

$\Longrightarrow$ The constant differential invariants, coming from the moving frame normalizations, are known as the *phantom invariants*

Invariantization:

$$\iota\,[\,F(\;\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\;)\,] = F(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;)$$

Replacement Theorem:

If J is a differential invariant, then $\iota(J) = J$.

$$J(\;\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\;) = J(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;)$$

Euclidean Curves

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa \qquad \iota(u_{xxx}) = I_3 = \kappa_s \qquad \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

In general:

$$\iota(\, F(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, \ldots\,)) = F(0, 0, 0, \kappa, \kappa_s, \kappa_{ss} + 3\kappa^3, \ldots\ )$$

Invariant one-form

$$dy = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\, \theta$$

$$\begin{aligned} \varpi = \iota(dx) \;&=\; \omega \;+\; \eta \\ &=\; \sqrt{1+u_x^2}\; dx \;+\; \frac{u_x}{\sqrt{1+u_x^2}}\,\theta \end{aligned}$$

$$\Longrightarrow \quad \theta = du - u_x\, dx$$

Invariant contact forms

$$\iota(\theta) = \vartheta = \frac{\theta}{\sqrt{1+u_x^2}} \qquad \iota(\theta_x) = \vartheta_1 = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}$$

The Variational Bicomplex

$\Longrightarrow$ *Vinogradov, Tsujishita, I. Anderson*

Infinite jet space

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ )$$

Horizontal one-forms

$$dx^1, \ldots, dx^p$$

Contact (vertical) one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha\, dx^i$$

Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum A_J^\alpha\, \theta_J^\alpha$$

Bigrading of the differential forms on J^∞

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s} \qquad \begin{array}{l} r = \# \quad \text{of} \quad dx^i \\ s = \# \quad \text{of} \quad \theta^\alpha_J \end{array}$$

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

Variational Bicomplex:

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

$F(x, u^{(n)})$ — differential function

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total differential}$$

$$d_V F = \sum_{\alpha,J} \frac{\partial F}{\partial u^\alpha_J}\, \theta^\alpha_J \qquad \text{— “variation”}$$

The Simplest Example. $M = \mathbb{R}^2 \quad x, u \in \mathbb{R}$

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x\, dx$$

$$\theta_x = du_x - u_{xx}\, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx}\, dx$$

$$\vdots$$

Differential $\quad F = F(x, u, u_x, u_{xx}, \ \ldots\ )$

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial u}\, du + \frac{\partial F}{\partial u_x}\, du_x + \frac{\partial F}{\partial u_{xx}}\, du_{xx} + \cdots \\ &= (D_x F)\, dx \ + \ \frac{\partial F}{\partial u}\,\theta + \frac{\partial F}{\partial u_x}\,\theta_x + \frac{\partial F}{\partial u_{xx}}\,\theta_{xx} + \cdots \\ &= \ d_H F \qquad + \qquad\quad d_V F \end{aligned}$$

Total derivative

$$D_x F = \frac{\partial F}{\partial u}\, u_x + \frac{\partial F}{\partial u_x}\, u_{xx} + \frac{\partial F}{\partial u_{xx}}\, u_{xxx} + \cdots$$

Lagrangian form

$$\lambda = L(x, u^{(n)})\, dx \ \in \ \Omega^{1,1}$$

Vertical derivative — variation

$$\begin{aligned} d\lambda &= d_V\, \lambda = d_V\, L \wedge dx \\ &= \left(\frac{\partial L}{\partial u}\, \theta + \frac{\partial L}{\partial u_x}\, \theta_x + \frac{\partial L}{\partial u_{xx}}\, \theta_{xx} + \cdots \right) \wedge\, dx \ \in \ \Omega^{1,1} \end{aligned}$$

Integration by parts

$$\begin{aligned} d_H\, (A\, \theta) &= (D_x A)\, dx \wedge \theta - A\, \theta_x \wedge dx \\ &= -[\, (D_x A)\, \theta + A\, \theta_x\,] \wedge dx \end{aligned}$$

so

$$A\ \theta_x \wedge dx \ \sim \ (D_x A)\, \theta \wedge dx \quad \text{mod im } d_H$$

Variational derivative — compute modulo im d_H :

$$\begin{aligned} d\lambda \ \sim \ \delta\lambda &= \left(\frac{\partial L}{\partial u} - D_x\, \frac{\partial L}{\partial u_x} + D_x^2\, \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge\, dx \\ &= \mathbf{E}(L)\ \theta \wedge\, dx \end{aligned}$$

$\Longrightarrow$ Euler-Lagrange source form.

Variational Derivative

Variation:

$$d_V : \ \Omega^{p,0} \quad \longrightarrow \quad \Omega^{p,1}$$

Integration by Parts:

$$\pi : \ \Omega^{p,1} \quad \longrightarrow \quad \mathcal{F}^1 = \Omega^{p,1} \,/\, d_H\, \Omega^{p-1,1}$$

$$\Longrightarrow \quad \text{source forms}$$

Variational derivative or Euler operator:

$$\delta = \pi \circ d_V : \qquad \Omega^{p,0} \qquad \longrightarrow \qquad \mathcal{F}^1$$

$$\lambda = L\, d\mathbf{x} \qquad \longrightarrow \qquad \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\ \theta^\alpha \wedge\, d\mathbf{x}$$

Variational Problems $\quad\longrightarrow\quad$ Source Forms

Invariant Variational Complex

- Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i)$$

- Invariant contact forms

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$\begin{aligned} d_{\mathcal{H}} :& \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r+1,s} \\ d_{\mathcal{V}} :& \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r,s+1} \\ d_{\mathcal{W}} :& \quad \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r-1,s+2} \end{aligned}$$

The Key Recurrence Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \mu^\kappa \wedge\ \iota[\mathbf{v}_\kappa(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$

— basis for infinitesimal generators $\mathfrak{g}$

$\mu_1, \ldots, \mu_r$

— invariantized dual Maurer–Cartan forms

$$\mu_k = \gamma_k + \varepsilon_k \in \Omega^{1,0} \oplus \Omega^{0,1}$$

⋆ ⋆ ⋆ **All identities, commutation formulae, etc., in the variational bicomplex can be found by applying the key formula with Ω replaced by the basic functions and differential forms!**

Euclidean Curves

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x \qquad\qquad \mathbf{v}_2 = \partial_u$$

$$\mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + 3u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

Horizontal recurrence formula

$$d_{\mathcal{H}}\,\iota(F) = \iota(d_H F) + \iota(\mathbf{v}_1(F))\,\gamma^1 + \iota(\mathbf{v}_2(F))\,\gamma^2 + \iota(\mathbf{v}_3(F))\,\gamma^3$$

$$d_{\mathcal{H}}\,I = \mathcal{D}I \cdot \varpi \qquad\qquad \iota(d_H F) = \iota(D_x F)\,\varpi$$

$$\Longrightarrow \quad \mathcal{D} = d/ds$$

Use phantom invariants

$$0 = \; d_{\mathcal{H}}\,H = \iota(d_H x) + \sum\, \iota(\mathbf{v}_\kappa(x))\,\gamma^\kappa = \varpi + \gamma^1$$

$$0 = \; d_{\mathcal{H}}\,I_0 = \iota(d_H u) + \sum\, \iota(\mathbf{v}_\kappa(u))\,\gamma^\kappa = \gamma^2$$

$$0 = \; d_{\mathcal{H}}\,I_1 = \iota(d_H u_x) + \sum\, \iota(\mathbf{v}_\kappa(u_x))\,\gamma^\kappa = \kappa\,\varpi + \gamma^3,$$

to solve for

$$\gamma^1 = -\,\varpi \qquad\qquad \gamma^2 = 0 \qquad\qquad \gamma^3 = -\,\kappa\,\varpi$$

$$\gamma^1 = -\varpi \qquad \gamma^2 = 0 \qquad \gamma^3 = -\kappa\,\varpi$$

Horizontal recurrence formulae

$$\begin{aligned}
\kappa_s\varpi = d_{\mathcal{H}}\,\kappa = d_{\mathcal{H}}\,(I_2) &= \iota(d_H u_{xx}) + \iota(\mathbf{v}_3(u_{xx}))\,\gamma^3 \\
&= \iota(u_{xxx}\,dx) - \iota(3u_x u_{xx}))\,\kappa\,\varpi = I_3\,\varpi
\end{aligned}$$

$$\begin{aligned}
\kappa_{ss}\varpi = d_{\mathcal{H}}\,(I_3) &= \iota(d_H u_{xxx}) + \iota(\mathbf{v}_3(u_{xxx})\,\gamma^3 \\
&= \iota(u_{xxxx}\,dx) - \iota(4u_x u_{xxx} + 3u_{xx}^2)\,\kappa\,\varpi = I_4 - 3I_2^3\,\varpi \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
\kappa &= I_2 & I_2 &= \kappa \\
\kappa_s &= I_3 & I_3 &= \kappa_s \\
\kappa_{ss} &= I_4 - 3I_2^3 & I_4 &= \kappa_{ss} + 3\kappa^3 \\
\kappa_{sss} &= I_5 - 19I_2^2 I_3 & I_5 &= \kappa_{sss} + 19\kappa^2\kappa_s \\
&\vdots & &\vdots
\end{aligned}$$

Vertical recurrence formula

$$d_{\mathcal{V}}\,\iota(F) = \iota(\,d_V\,F) + \iota(\mathbf{v}_1(F))\,\varepsilon^1 + \iota(\mathbf{v}_2(F))\,\varepsilon^2 + \iota(\mathbf{v}_3(F))\,\varepsilon^3$$

Use phantom invariants

$$0 = \,d_{\mathcal{V}}\,H = \varepsilon^1 \qquad 0 = \,d_{\mathcal{V}}\,I_0 = \vartheta + \varepsilon^2 \qquad 0 = \,d_{\mathcal{V}}\,I_1 = \vartheta_1 + \varepsilon^3$$

to solve for

$$\varepsilon^1 = 0 \qquad \varepsilon^2 = -\,\vartheta = -\,\iota(\theta) \qquad \varepsilon^3 = -\,\vartheta_1 = -\,\iota(\theta_1)$$

$$d_{\mathcal{V}}\,\kappa = \,d_{\mathcal{V}}\,I_2 = \iota(\theta_2) + \iota(\mathbf{v}_3(u_{xx}))\,\varepsilon^3 = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

$$\vdots$$

Key recurrence formulae:

$$\boxed{d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta}$$

$$\boxed{d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi}$$

Plane Curves

Invariant Lagrangian:

$$\widetilde{\lambda} = L(x, u^{(n)})\, dx = P(\kappa, \kappa_s, \ldots)\, \varpi$$

Euler–Lagrange form:

$$d_{\mathcal{V}}\, \widetilde{\lambda} \;\sim\; \mathbf{E}(L)\, \vartheta \wedge \varpi$$

Invariant Integration by Parts Formula

$$F\, d_{\mathcal{V}}\, (\mathcal{D}H) \wedge \varpi \quad\sim\quad -(\mathcal{D}F)\, d_{\mathcal{V}}\, H \wedge \varpi - (F \cdot \mathcal{D}H)\, d_{\mathcal{V}}\, \varpi$$

$$\begin{aligned} d_{\mathcal{V}}\, \widetilde{\lambda} &= d_{\mathcal{V}}\, P \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &= \sum_n \frac{\partial P}{\partial \kappa_n}\; d_{\mathcal{V}}\, \kappa_n \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &\sim\; \mathcal{E}(P)\; d_{\mathcal{V}}\, \kappa \wedge \varpi + \mathcal{H}(P)\; d_{\mathcal{V}}\, \varpi \end{aligned}$$

Vertical differentiation formulae

$d_{\mathcal{V}}\, \kappa = \mathcal{A}(\vartheta)$ $\qquad$ $\mathcal{A}$ — "Eulerian operator"

$d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi$ $\qquad$ $\mathcal{B}$ — "Hamiltonian operator"

$$\begin{aligned} d_{\mathcal{V}}\, \widetilde{\lambda} &\sim\; \mathcal{E}(P)\; \mathcal{A}(\vartheta) \wedge \varpi + \mathcal{H}(P)\; \mathcal{B}(\vartheta) \wedge \varpi \\ &\sim\; \Big[\, \mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) \Big]\; \vartheta \wedge \varpi \end{aligned}$$

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) = 0}$$

Euclidean Plane Curves

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

Eulerian operator

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{A}^* = \mathcal{D}^2 + \kappa^2$$

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Hamiltonian operator

$$\mathcal{B} = -\,\kappa \qquad\qquad \mathcal{B}^* = -\,\kappa$$

Euclidean–invariant Euler-Lagrange formula

$$\begin{aligned}\mathbf{E}(L) &= \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P)\\ &= (\mathcal{D}^2 + \kappa^2)\;\mathcal{E}(P) + \kappa\,\mathcal{H}(P).\end{aligned}$$

Invariant Plane Curve Flows

G — Lie group acting on $\mathbb{R}^2$

ϖ — invariant horizontal form (arc length)

ϑ — invariant contact form

$C(t)$ — parametrized family of plane curves

G-invariant curve flow:

$$\frac{dC}{dt} = \mathcal{F}[C]$$

Infinitesimal generator:

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

- I, J — differential invariants
- $\mathbf{t}$ — "unit tangent"
- $\mathbf{n}$ — "unit normal"

$\mathbf{t},\ \mathbf{n}$ — basis of the invariant vector fields dual to the invariant one-forms:

$$\langle\, \mathbf{t}\,;\varpi\,\rangle = 1, \quad \langle\, \mathbf{n}\,;\varpi\,\rangle = 0,$$

$$\langle\, \mathbf{t}\,;\vartheta\,\rangle = 0, \quad \langle\, \mathbf{n}\,;\vartheta\,\rangle = 1.$$

$\mathcal{D}$ — invariant arc length derivative

$\mathcal{B}$ — invariant Hamiltonian operator

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

Theorem. The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}I = 0$$

Proof: Cartan's formula for Lie derivatives:

$$\begin{aligned} \mathbf{v}(\varpi) &= d(\mathbf{v} \,\lrcorner\, \varpi) + \mathbf{v} \,\lrcorner\, d\varpi \\ &= dI + \mathbf{v} \,\lrcorner\, (\mathcal{B}(\vartheta) \wedge \varpi) \\ &\equiv [\,\mathcal{D}I + \mathcal{B}(J)\,]\,\varpi \quad \bmod \vartheta \end{aligned}$$

Corollary.

$$[\,\mathbf{v},\,\mathcal{D}\,] = 0$$

- Every curve flow is equivalent, modulo reparametrization, to an arc length-preserving flow.

Evolution of Differential Invariants

κ — curvature (generating differential invariant)

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta)$$

$\mathcal{A}$ — invariant Eulerian operator

Theorem.

Under such an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad (*)$$

where

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B}$$

In surprisingly many situations, $(*)$ is a well-known integrable evolution equation, and $\mathcal{R}$ is its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang

$\Longrightarrow$ Qu, Chou

$\Longrightarrow$ and many more ...

Why????

Examples

Euclidean plane curves:

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\, \kappa = (\mathcal{D}^2 + \kappa^2)\, \vartheta, \qquad d_{\mathcal{V}}\, \varpi = -\, \kappa\, \vartheta \wedge \varpi$$

$$\Longrightarrow \qquad \mathcal{A} = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{B} = -\, \kappa$$

$$\begin{aligned} \mathcal{R} &= \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \\ &= \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa \end{aligned}$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\, \kappa^2 \kappa_s$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\, \kappa = \mathcal{A}(\vartheta), \qquad d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\, \kappa\, \mathcal{D}^2 + \tfrac{5}{3}\, \kappa_s \mathcal{D} + \tfrac{1}{3}\, \kappa_{ss} + \tfrac{4}{9}\, \kappa^2,$$

$$\mathcal{B} = \tfrac{1}{3}\, \mathcal{D}^2 - \tfrac{2}{9}\, \kappa$$

$$\begin{aligned} \mathcal{R} &= \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \\ &= \mathcal{D}^4 + \tfrac{5}{3}\, \kappa\, \mathcal{D}^2 + \tfrac{4}{3}\, \kappa_s \mathcal{D} + \tfrac{1}{3}\, \kappa_{ss} \\ &\qquad\qquad + \tfrac{4}{9}\, \kappa^2 + \tfrac{2}{9}\, \kappa_s \mathcal{D}^{-1} \cdot \kappa \end{aligned}$$

$$\begin{aligned} \kappa_t &= \mathcal{R}(\kappa_s) \\ &= \kappa_{5s} + 2\, \kappa\, \kappa_{ss} + \tfrac{4}{3}\, \kappa_s^2 + + \tfrac{5}{9}\, \kappa^2\, \kappa_s \end{aligned}$$

$\Longrightarrow$ Sawada–Kotera equation

Euclidean space curves in $\mathbb{R}^3$:

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\begin{pmatrix} d_{\mathcal{V}}\,\kappa \\ d_{\mathcal{V}}\,\tau \end{pmatrix} = \mathcal{A} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix}, \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \wedge \varpi$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \dfrac{2\tau}{\kappa} D_s^2 + \dfrac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \dfrac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \dfrac{1}{\kappa} D_s^3 - \dfrac{\kappa_s}{\kappa^2} D_s^2 + \dfrac{\kappa^2 - \tau^2}{\kappa} D_s + \dfrac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} \kappa & 0 \end{pmatrix}$$

Recursion operator:

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}$$

$\Longrightarrow$ vortex filament flow

$\Longrightarrow$ nonlinear Schrödinger equation (Hasimoto)