Invariant Variational Problems

&

Moving Frames

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$\Longrightarrow$ Joint work with Irina Kogan.

Variational Problems

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$u^\alpha_J = \partial_J u^\alpha$ — derivatives

Variational problem:

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x}$$

$L(x, u^{(n)})$ — Lagrangian

Variational derivative — Euler-Lagrange equations

$$\mathbf{E}(L) = 0$$

Components:

$$\mathbf{E}_\alpha(L) = \sum_J (-D)^J \frac{\partial L}{\partial u^\alpha_J}$$

Invariant Variational Problems

G — transformation group

G – invariant variational problem (*Lie*):

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

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Invariant Euler-Lagrange equations

$$\mathbf{E}(L) = F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0$$

Main Problem:

Construct F directly from P.

(*P. Griffiths, I. Anderson*)

Example. Planar Euclidean group $G = \mathrm{SE}(2)$

κ — curvature

ds — arc length & derivative — $\mathcal{D} = \dfrac{d}{ds}$

Invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) = F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Minimal curves (geodesics):

$$\mathcal{I}[u] = \int ds = \int \sqrt{1 + u_x^2}\ dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1 + u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General SE(2) – invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler operator

$$\mathcal{E} = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian operator

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j} \ (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\ \mathcal{E}(P) + \kappa\, \mathcal{H}(P).$$

Elastica : $\qquad P = \frac{1}{2}\,\kappa^2$

$$\mathcal{E}(P) = \kappa \qquad\qquad \mathcal{H}(P) = -\,P = -\,\tfrac{1}{2}\,\kappa^2$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

Basic Issues

- Classification of differential invariants

$$(I^1, \ldots, I^N) \qquad\qquad N = ???$$

- Invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

- Commutation formulae

$$[\,\mathcal{D}_i,\, \mathcal{D}_j\,] = \sum_{i=1}^{p} A_{ij}^k\, \mathcal{D}_k$$

- Classification of syzygies

$$F(\ \ldots\ \mathcal{D}^J I^\alpha\ \ \ldots\ ) = 0$$

- Invariant variational problems

$$\int P\, \boldsymbol{\varpi} \quad \longmapsto \quad \mathbf{E}(L) = 0$$

Moving Frames

$\Longrightarrow$ Mark Fels and PJO

G — r-dimensional Lie group acting on M

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — n^{th} order jet bundle for p-dimensional submanifolds $N \subset M$

$(x, u^{(n)})$ — coordinates on J^n

Definition.

An n^{th} order *moving frame* is a G-equivariant map

$$\rho = \rho^{(n)} : \mathrm{J}^n \longrightarrow G$$

Equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} g \cdot \rho(z^{(n)}) & \text{left moving frame} \\ \rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

Note $\rho_{left}(z^{(n)}) = \rho_{right}(z^{(n)})^{-1}$

Theorem.

A moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if G acts freely and regularly near $z^{(n)}$.

Theorem.

If G acts locally effectively on subsets, then for $n \gg 0$, the prolonged group action $G^{(n)}$ is locally free on an open subset of J^n.

$\Longrightarrow$ Ovsiannikov, PJO

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:
  $g \cdot z = z$ iff $g = e$.

- locally free — the orbits have the same dimension as G.

- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once ($\not\approx$ irrational flow on the torus)

- effective — the only group element $g \in G$ which fixes *every* point $z \in M$ is the identity:
  $g \cdot z = z$ for all $z \in M$ iff $g = e$.

Normalization

Prolonged action:

$$\boxed{w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}}$$

Normalization Equations: $\qquad r = \dim G$

$$w_1(g, z^{(n)}) = c_1 \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r$$

The solution is the right moving frame:

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The nonconstant components of

$$I^{(n)}(z) = w^{(n)}(\rho(z^{(n)}), z^{(n)}) = \rho(z^{(n)}) \cdot z^{(n)}$$

are the fundamental differential invariants
of order $\leq n$

Euclidean Plane Curves $\qquad G = \mathrm{SE}(2)$

Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

Prolong to J^3 via implicit differentiation

$$\left.\begin{aligned} y &= x\cos\phi - u\sin\phi + a \\ v &= x\cos\phi + u\sin\phi + b \end{aligned}\right\} \quad w = R\,z + c$$

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi - u_x\sin\phi)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi - u_x\sin\phi)^5}$$

$$\vdots$$

Normalization: $\qquad r = \dim G = 3$

$$y = 0 \qquad v = 0 \qquad v_y = 0$$

Right moving frame $\qquad \rho\colon \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$\phi = -\tan^{-1} u_x \qquad a = -\frac{x + u u_x}{\sqrt{1+u_x^2}} \qquad b = \frac{x u_x - u}{\sqrt{1+u_x^2}}$$

Differential invariants

$$\begin{aligned} v_{yy} &\longmapsto \kappa &&= \frac{u_{xx}}{(1+u_x^2)^{3/2}} \\ v_{yyy} &\longmapsto \frac{d\kappa}{ds} &&= \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3} \\ v_{yyyy} &\longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3 &&= \cdots \end{aligned}$$

Invariant one-form — arc length

$$dy = (\cos\phi - u_x \sin\phi)\, dx \quad \longmapsto \quad ds = \sqrt{1+u_x^2}\; dx$$

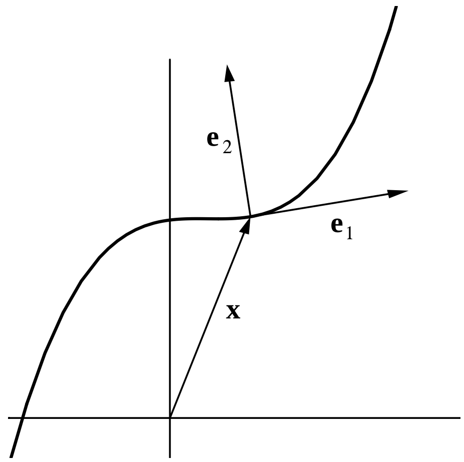
Invariant differential operator

$$\frac{d}{dy} = \frac{1}{\cos\phi - u_x \sin\phi}\,\frac{d}{dx} \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa \qquad \frac{d\kappa}{ds} \qquad \frac{d^2\kappa}{ds^2} \qquad \cdots$$

Euclidean Curves



Left moving frame $\widetilde{\rho}(x, u^{(1)}) = \rho(x, u^{(1)})^{-1}$

$$\tilde{a} = x \qquad \tilde{b} = u \qquad \tilde{\theta} = \tan^{-1} u_x$$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\mathbf{e}_1, \mathbf{e}_2) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{e}_1 = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix} \qquad \mathbf{e}_2 = \mathbf{e}_1^{\perp} = \begin{pmatrix} -y_s \\ x_s \end{pmatrix}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \kappa\,\mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = -\,\kappa\,\mathbf{e}_1$$

The Variational Bicomplex

$\Longrightarrow$ *Vinogradov, Tsujishita, I. Anderson*

Infinite jet space

$$\mathrm{J}^\infty = \lim_{n\to\infty} \mathrm{J}^n$$

is the inverse limit

$$M = \mathrm{J}^0 \longleftarrow \mathrm{J}^1 \longleftarrow \mathrm{J}^2 \longleftarrow \cdots$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\; \ldots \; x^i \; \ldots \; u^\alpha_J \; \ldots \;)$$

Coframe — basis for the cotangent space $T^*\mathrm{J}^\infty$:

Horizontal one-forms

$$dx^1, \ldots, dx^p$$

Contact (vertical) one-forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, dx^i$$

Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum \; A^\alpha_J \, \theta^\alpha_J$$

Bigrading of the differential forms on J^∞

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s} \qquad \begin{array}{l} r = \# \quad \text{of} \quad dx^i \\ s = \# \quad \text{of} \quad \theta^\alpha_J \end{array}$$

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

Bicomplex

$$\begin{array}{lcl} d_H : \Omega^{r,s} & \longrightarrow & \Omega^{r+1,s} \\ d_V : \Omega^{r,s} & \longrightarrow & \Omega^{r,s+1} \end{array}$$

$F(x, u^{(n)})$ — differential function

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total differential}$$

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u^\alpha_J}\, \theta^\alpha_J \qquad \text{— variation}$$

The Simplest Example. $M = \mathbb{R}^2 \quad x, u \in \mathbb{R}$

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x\, dx$$

$$\theta_x = du_x - u_{xx}\, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx}\, dx$$

$$\vdots$$

Differential $\qquad F = F(x, u, u_x, u_{xx}, \ \ldots\ )$

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial u}\, du + \frac{\partial F}{\partial u_x}\, du_x + \frac{\partial F}{\partial u_{xx}}\, du_{xx} + \cdots \\ &= (D_x F)\, dx \ + \ \frac{\partial F}{\partial u}\,\theta + \frac{\partial F}{\partial u_x}\,\theta_x + \frac{\partial F}{\partial u_{xx}}\,\theta_{xx} + \cdots \\ &= \ d_H F \qquad + \qquad d_V F \end{aligned}$$

Total derivative

$$D_x F = \frac{\partial F}{\partial u}\, u_x + \frac{\partial F}{\partial u_x}\, u_{xx} + \frac{\partial F}{\partial u_{xx}}\, u_{xxx} + \cdots$$

Lagrangian form

$$\lambda = L(x, u^{(n)})\, dx \;\in\; \Omega^{1,1}$$

Vertical derivative — variation

$$\begin{aligned} d\lambda &= d_V\, \lambda \\ &= \left(\frac{\partial L}{\partial u}\,\theta + \frac{\partial L}{\partial u_x}\,\theta_x + \frac{\partial L}{\partial u_{xx}}\,\theta_{xx} + \cdots \right) \wedge\, dx \;\in\; \Omega^{1,1} \end{aligned}$$

Integration by parts

$$\begin{aligned} d_H\,(A\,\theta) &= (D_x A)\, dx \wedge \theta - A\,\theta_x \wedge dx \\ &= -[\,(D_x A)\,\theta + A\,\theta_x\,] \wedge dx \end{aligned}$$

so

$$A\;\theta_x \wedge dx \;\sim\; (D_x A)\,\theta \wedge dx \quad \text{mod im } d_H$$

Variational derivative

$$\begin{aligned} d\lambda \;\sim\; \delta\lambda &= \left(\frac{\partial L}{\partial u} - D_x\,\frac{\partial L}{\partial u_x} + D_x^2\,\frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge\, dx \\ &= \mathbf{E}(L)\;\theta \wedge\, dx \end{aligned}$$

$\Longrightarrow$ Euler-Lagrange source form.

Variational Derivative

Variation:

$$d_V : \ \Omega^{p,0} \quad \longrightarrow \quad \Omega^{p,1}$$

Integration by Parts:

$$\pi : \ \Omega^{p,1} \quad \longrightarrow \quad \mathcal{F}^1 = \Omega^{p,1} \, / \, d_H \, \Omega^{p-1,1}$$

$\Longrightarrow$ source forms

Variational derivative or Euler operator:

$$\delta = \pi \circ d_V : \qquad \Omega^{p,0} \qquad \longrightarrow \qquad \mathcal{F}^1$$

$$\lambda = L \, d\mathbf{x} \quad \longrightarrow \quad \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L) \ \theta^\alpha \wedge \ d\mathbf{x}$$

Variational Problems $\longrightarrow$ Source Forms

Invariantization

$$\iota : \begin{array}{lcl} \text{Functions} & \longrightarrow & \text{Invariants} \\ \text{Forms} & \longrightarrow & \text{Invariant Forms} \end{array}$$

Fundamental differential invariants

$$I^{(n)}(x, u^{(n)}) = \iota(x, u^{(n)}) = \rho(x, u^{(n)}) \cdot (x, u^{(n)})$$

$$\boxed{H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)}$$

$\Longrightarrow$ The constant differential invariants, coming from the moving frame normalizations, are known as the *phantom invariants*

Invariantization

$$\iota(F(x, u^{(n)})) \;=\; F[\, I^{(n)}(x, u^{(n)}) \,]$$

Replacement Theorem:

If J is a differential invariant, then $\iota(J) = J$.

$$J(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) = J(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

Invariantization of Differential Forms

Prolonged group action

$$\begin{aligned} w^{(n)} : G \times \mathrm{J}^n &\longrightarrow \mathrm{J}^n \\ (g, z^{(n)}) &\longmapsto g^{(n)} \cdot z^{(n)} \end{aligned}$$

Moving frame section

$$\begin{aligned} \sigma^{(n)} : \mathrm{J}^n &\longrightarrow G \times \mathrm{J}^n \\ z^{(n)} &\longmapsto (\rho(z^{(n)}), z^{(n)}) \end{aligned}$$

Composition = fundamental differential invariants

$$w^{(n)} \circ \sigma^{(n)}(z^{(n)}) = \rho(z^{(n)}) \cdot z^{(n)} = I^{(n)}(z^{(n)})$$

Invariantization of differential functions:

$$\iota(F) \;=\; \sigma^* \circ w^* \,(F) \;=\; F \circ I^{(\infty)}$$

Invariantization of differential forms:

$$\iota(\Omega) = \sigma^*(\pi_J(w^*\,\Omega)).$$

Jet projection: $\quad \pi_J \;:\; \mu^\kappa \;\longmapsto\; 0$

μ^κ — Maurer–Cartan forms

Invariant Variational Complex

- Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i)$$

- Invariant contact forms

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$\begin{array}{llll} d_{\mathcal{H}} : & \widehat{\Omega}^{r,s} & \longrightarrow & \widehat{\Omega}^{r+1,s} \\ d_{\mathcal{V}} : & \widehat{\Omega}^{r,s} & \longrightarrow & \widehat{\Omega}^{r,s+1} \\ d_{\mathcal{W}} : & \widehat{\Omega}^{r,s} & \longrightarrow & \widehat{\Omega}^{r-1,s+2} \end{array}$$

Euclidean Curves

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa \qquad \iota(u_{xxx}) = I_3 = \kappa_s \qquad \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

Invariant horizontal one-form

$$dy = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\, \theta + da - v\, d\phi$$

$$d_J y = \pi_J(dy) = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\, \theta$$

$$\begin{aligned} \iota(dx) = \sigma^*(\, d_J\, y) = \varpi \ &= \qquad \omega \qquad + \qquad \eta \\ &= \sqrt{1+u_x^2}\; dx \ + \ \frac{u_x}{\sqrt{1+u_x^2}}\,\theta \end{aligned}$$

Invariant contact forms

$$\iota(\theta) = \vartheta = \frac{\theta}{\sqrt{1+u_x^2}} \qquad \iota(\theta_x) = \vartheta_1 = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx} \theta}{(1+u_x^2)^2}$$

The Key Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^\kappa \wedge\ \iota[\mathbf{v}_\kappa(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for infinitesimal generators $\mathfrak{g}$

Invariantized dual Maurer–Cartan forms:

$$\nu^\kappa = \sigma^* \mu^\kappa = \gamma^\kappa + \varepsilon^\kappa \qquad\qquad \kappa = 1, \ldots, r$$

$$\gamma^\kappa \in \widehat{\Omega}^{1,0} \qquad \varepsilon^\kappa \in \widehat{\Omega}^{0,1}$$

Duality

$$d_G\,\Theta = \sum_{\kappa=1}^{r} \mu^\kappa \wedge \hat{\mathbf{v}}_\kappa(\Theta)$$

⋆ ⋆ ⋆ All recurrence formulae, syzygies, commutation formulae, etc. are found by applying the key formula with Ω replaced by the basic forms and functions!

Euclidean Curves

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x \qquad \mathbf{v}_2 = \partial_u$$

$$\mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

Horizontal recurrence formula

$$d_{\mathcal{H}}\,\iota(F) = \iota(d_H F) + \iota(\mathbf{v}_1(F))\,\gamma^1 + \iota(\mathbf{v}_2(F))\,\gamma^2 + \iota(\mathbf{v}_3(F))\,\gamma^3$$

$$d_{\mathcal{H}}\,I = \mathcal{D}I \cdot \varpi \qquad \iota(d_H F) = \iota(D_x F)\,\varpi$$

$$\Longrightarrow \quad \mathcal{D} = d/ds$$

Use phantom invariants

$$0 = \ d_{\mathcal{H}}\,H = \iota(d_H x) + \sum\ \iota(\mathbf{v}_\kappa(x))\,\gamma^\kappa = \varpi + \gamma^1$$

$$0 = \ d_{\mathcal{H}}\,I_0 = \iota(d_H u) + \sum\ \iota(\mathbf{v}_\kappa(u))\,\gamma^\kappa = \gamma^2$$

$$0 = \ d_{\mathcal{H}}\,I_1 = \iota(d_H u_x) + \sum\ \iota(\mathbf{v}_\kappa(u_x))\,\gamma^\kappa = \kappa\,\varpi + \gamma^3,$$

to solve for

$$\gamma^1 = -\,\varpi \qquad \gamma^2 = 0 \qquad \gamma^3 = -\,\kappa\,\varpi$$

$$\gamma^1 = -\varpi \qquad \gamma^2 = 0 \qquad \gamma^3 = -\kappa\,\varpi$$

Horizontal recurrence formulae

$$\begin{aligned}\kappa_s\varpi = d_{\mathcal{H}}\,\kappa = d_{\mathcal{H}}\,(I_2) &= \iota(d_H u_{xx}) + \iota(\mathbf{v}_3(u_{xx}))\,\gamma^3 \\ &= \iota(u_{xxx}\,dx) - \iota(3u_x u_{xx}))\,\kappa\,\varpi = I_3\,\varpi\end{aligned}$$

$$\begin{aligned}\kappa_{ss}\varpi = d_{\mathcal{H}}\,(I_3) &= \iota(d_H u_{xxx}) + \iota(\mathbf{v}_3(u_{xxx})\,\gamma^3 \\ &= \iota(u_{xxxx}\,dx) - \iota(4u_x u_{xxx} + 3u_{xx}^2)\,\kappa\,\varpi = I_4 - 3I_2^3\,\varpi \\ &\vdots\end{aligned}$$

$$\begin{aligned}\kappa &= I_2 & I_2 &= \kappa \\ \kappa_s &= I_3 & I_3 &= \kappa_s \\ \kappa_{ss} &= I_4 - 3I_2^3 & I_4 &= \kappa_{ss} + 3\kappa^3 \\ \kappa_{sss} &= I_5 - 19I_2^2 I_3 & I_4 &= \kappa_{sss} + 19\kappa^2\kappa_s \\ &\vdots & &\vdots\end{aligned}$$

Vertical recurrence formula

$$d_{\mathcal{V}}\,\iota(F) = \iota(\,d_V\,F) + \iota(\mathbf{v}_1(F))\,\varepsilon^1 + \iota(\mathbf{v}_2(F))\,\varepsilon^2 + \iota(\mathbf{v}_3(F))\,\varepsilon^3$$

Use phantom invariants

$$0 = \,d_{\mathcal{V}}\,H = \varepsilon^1 \qquad 0 = \,d_{\mathcal{V}}\,I_0 = \vartheta + \varepsilon^2 \qquad 0 = \,d_{\mathcal{V}}\,I_1 = \vartheta_1 + \varepsilon^3$$

to solve for

$$\varepsilon^1 = 0 \qquad \varepsilon^2 = -\,\vartheta = -\,\iota(\theta) \qquad \varepsilon^3 = -\,\vartheta_1 = -\,\iota(\theta_1)$$

$$d_{\mathcal{V}}\,\kappa = \,d_{\mathcal{V}}\,I_2 = \iota(\theta_2) + \iota(\mathbf{v}_3(u_{xx}))\,\varepsilon^3 = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

$$\vdots$$

Key recurrence formulae:

$$\boxed{d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta}$$

$$\boxed{d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi}$$

Plane Curves

Invariant Lagrangian: $\widetilde{\lambda} = P(\kappa, \kappa_s, \ldots)\, \varpi$

Invariant vertical differential

$$d_{\mathcal{V}}\, \widetilde{\lambda} = d_{\mathcal{V}}\, P \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi = \sum_n \frac{\partial P}{\partial \kappa_n}\; d_{\mathcal{V}}\, \kappa_n \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi$$

Invariant Integration by Parts

$$F\, d_{\mathcal{V}}\, (\mathcal{D}H) \wedge \varpi \;\sim\; -(\mathcal{D}F)\, d_{\mathcal{V}}\, H \wedge \varpi - (F \cdot \mathcal{D}H)\, d_{\mathcal{V}}\, \varpi$$

$$d_{\mathcal{V}}\, \widetilde{\lambda} = \mathcal{E}(P)\; d_{\mathcal{V}}\, \kappa \wedge \varpi + \mathcal{H}(P)\; d_{\mathcal{V}}\, \varpi$$

Vertical differentiation formulae

$d_{\mathcal{V}}\, \kappa = \mathcal{A}(\vartheta)$ $\qquad$ $\mathcal{A}$ — "Eulerian operator"

$d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi$ $\qquad$ $\mathcal{B}$ — "Hamiltonian operator"

$$\begin{aligned} d_{\mathcal{V}}\, \widetilde{\lambda} &= \mathcal{E}(P)\, \mathcal{A}(\vartheta) \wedge \varpi + \mathcal{H}(P)\, \mathcal{B}(\vartheta) \wedge \varpi \\ &= \Big[\, \mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) \,\Big]\; \vartheta \wedge \varpi \end{aligned}$$

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) = 0}$$

Euclidean Plane Curves

$$d_{\mathcal{V}}\, \kappa = (\mathcal{D}^2 + \kappa^2)\, \vartheta$$

Eulerian operator

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{A}^* = \mathcal{D}^2 + \kappa^2$$

$$d_{\mathcal{V}}\, \varpi = -\, \kappa\, \vartheta \wedge \varpi$$

Hamiltonian operator

$$\mathcal{B} = -\, \kappa \qquad\qquad \mathcal{B}^* = -\, \kappa$$

Euclidean–invariant Euler-Lagrange formula

$$\begin{aligned} \mathbf{E}(L) &= \mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) \\ &= (\mathcal{D}^2 + \kappa^2)\; \mathcal{E}(P) + \kappa\, \mathcal{H}(P). \end{aligned}$$

Recurrence Formulae:

$$\boxed{\begin{aligned} \mathcal{D}_j H^i &= \delta^i_j + M^i_j \\ \mathcal{D}_j I^\alpha_K &= I^\alpha_{K,j} + M^\alpha_{K,j} \end{aligned}}$$

$M^i_j, M^\alpha_{K,j}$ — correction terms

- The correction terms can be computed directly from the infinitesimal generators!

Commutation Formulae:

$$\boxed{[\mathcal{D}_i, \mathcal{D}_j] = \sum_{i=1}^{p} A^k_{ij}\, \mathcal{D}_k}$$

Generating Invariants

Theorem. A generating system of differential invariants consists of

- all non-phantom differential invariants H^i and I^α coming from the un-normalized zero$^{\text{th}}$ order lifted invariants y^i, v^α, and
- all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where I^α_J is a phantom differential invariant.

In other words, every other differential invariant can, locally, be written as a function of the generating invariants and their invariant derivatives, $\mathcal{D}_K H^i$, $\mathcal{D}_K I^\alpha_{J,i}$.

$\Longrightarrow$ Not necessarily a minimal set!

Syzygies

Theorem. All syzygies among the differentiated invariants are differential consequences of the following three fundamental types:

$$\boxed{\mathcal{D}_j H^i = \delta^i_j + M^i_j}$$

— H^i non-phantom

$$\boxed{\mathcal{D}_J I^\alpha_K = c_\nu + M^\alpha_{K,J}}$$

— I^α_K generating

— $I^\alpha_{J,K} = w_\nu = c_\nu$ phantom

$$\boxed{\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}}$$

— I^α_{LK}, I^α_{LJ} generating, $K \cap J = \varnothing$

$\Longrightarrow$ Not necessarily a minimal system!

General Framework

Fundamental differential invariants

$$I^1, \ldots, I^\ell$$

Invariant horizontal coframe

$$\varpi^1, \ldots, \varpi^p$$

Dual invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

Invariant volume form

$$\boldsymbol{\varpi} = \varpi^1 \wedge \cdots \wedge \varpi^p$$

Differentiated invariants

$$I^\alpha_{,K} = \mathcal{D}^K I^\alpha = \mathcal{D}_{k_1} \cdots \mathcal{D}_{k_n} I^\alpha$$

$\Longrightarrow$ order is important!

Eulerian operator

$$d_{\mathcal{V}}\, I^\alpha = \sum_{\beta=1}^{q} \mathcal{A}^\alpha_\beta(\vartheta^\beta) \qquad \mathcal{A} = (\, \mathcal{A}^\alpha_\beta \,)$$

$\Longrightarrow$ $m \times q$ matrix of invariant differential operators

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Hamiltonian operator complex

$$d_{\mathcal{V}}\, \varpi^j = \sum_{\beta=1}^{q} \mathcal{B}^j_{i,\beta}(\vartheta^\beta) \wedge \varpi^i \qquad \mathcal{B}^j_i = (\, \mathcal{B}^j_{i,\beta} \,)$$

$\Longrightarrow$ p^2 row vectors of invariant differential operators

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$$\boldsymbol{\varpi}_{(i)} = (-1)^{i-1}\, \varpi^1 \wedge \cdots \wedge \varpi^{i-1} \wedge \varpi^{i+1} \wedge \cdots \wedge \varpi^p$$

Twist invariants

$$d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(i)} = Z_i\, \boldsymbol{\varpi}$$

Twisted adjoint

$$\mathcal{D}_i^\dagger = -\,(\mathcal{D}_i + Z_i)$$

Invariant variational problem

$$\int P(I^{(n)})\,\boldsymbol{\varpi}$$

Invariant Eulerian

$$\mathcal{E}_\alpha(P) = \sum_K \mathcal{D}_K^\dagger \frac{\partial P}{\partial I^\alpha_{,K}}$$

Invariant Hamiltonian tensor

$$\mathcal{H}^i_j(P) = -\,P\,\delta^i_j + \sum_{\alpha=1}^{q} \sum_{J,K} I^\alpha_{,J,j}\,\mathcal{D}_K^\dagger \frac{\partial P}{\partial I^\alpha_{,J,i,K}}\,,$$

Invariant Euler-Lagrange equations

$$\boxed{\mathcal{A}^\dagger\,\mathcal{E}(P) - \sum_{i,j=1}^{p} (\mathcal{B}^j_i)^\dagger\,\mathcal{H}^i_j(P) = 0}$$

Euclidean Surfaces

$S \subset M = \mathbb{R}^3$ coordinates $z = (x, y, u)$

Group: $G = \mathrm{SE}(3)$

$$z \longmapsto R\,z + a, \qquad R \in \mathrm{SO}(3)$$

Normalization — coordinate cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0$$

Left moving frame

$$a = z \qquad R = (\ \mathbf{t}_1\ \mathbf{t}_2\ \mathbf{n}\ )$$

- $\mathbf{t}_1, \mathbf{t}_2 \in TS$ — Frenet frame
- $\mathbf{n}$ — unit normal

Fundamental differential invariants

$$\kappa^1 = \iota(u_{xx}) \qquad \kappa^2 = \iota(u_{yy})$$

$\Longrightarrow$ principal curvatures

Frenet coframe

$$\varpi^1 = \iota(dx^1) = \omega^1 + \eta^1 \qquad \varpi^2 = \iota(dx^2) = \omega^2 + \eta^2$$

Invariant differential operators

$$\mathcal{D}_1 \qquad \mathcal{D}_2$$

$\Longrightarrow$ Frenet differentiation

Fundamental Syzygy:

Use the recurrence formula to compare

$$\iota(u_{xxyy}) \qquad \text{with} \qquad \begin{aligned} \kappa^1_{,22} &= \mathcal{D}_2^2\, \iota(u_{xx}) \\ \kappa^2_{,11} &= \mathcal{D}_1^2\, \iota(u_{yy}) \end{aligned}$$

$$\kappa^1_{,22} - \kappa^2_{,11} + \frac{\kappa^1_{,1}\kappa^2_{,1} + \kappa^1_{,2}\kappa^2_{,2} - 2(\kappa^2_{,1})^2 - 2(\kappa^1_{,2})^2}{\kappa^1 - \kappa^2} - \kappa^1\kappa^2(\kappa^1 - \kappa^2) = 0$$

$\Longrightarrow$ Codazzi equations

Twist invariants

$$d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(1)} = d_{\mathcal{H}}\,\varpi^2 = Z_1\,\varpi^1 \wedge \varpi^2$$

$$d_{\mathcal{H}}\,\boldsymbol{\varpi}_{(2)} = -\,d_{\mathcal{H}}\,\varpi^1 = Z_2\,\varpi^1 \wedge \varpi^2$$

$$Z_1 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2} \qquad\qquad Z_2 = \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}$$

Twisted adjoints

$$\mathcal{D}_1^\dagger = -\,(\mathcal{D}_1 + Z_1) \qquad\qquad \mathcal{D}_2^\dagger = -\,(\mathcal{D}_2 + Z_2)$$

Gauss curvature — Codazzi equations:

$$K = \kappa^1\kappa^2 = \mathcal{D}_1^\dagger(Z_1) + \mathcal{D}_2^\dagger(Z_2)$$

$$= -\,(\mathcal{D}_1 + Z_1)Z_1 - (\mathcal{D}_2 + Z_2)Z_2$$

K is an invariant divergence

$\Longrightarrow$ Gauss–Bonnet Theorem!

Invariant contact form

$$\vartheta = \iota(\theta) = \iota(du - u_x\, dx - u_y\, dy)$$

Invariant vertical derivatives

$$d_{\mathcal{V}}\, \kappa^1 = \iota(\theta_{xx}) = (\, \mathcal{D}_1^2 + Z_2\, \mathcal{D}_2 + (\kappa^1)^2\,)\, \vartheta$$

$$d_{\mathcal{V}}\, \kappa^2 = \iota(\theta_{yy}) = (\, \mathcal{D}_2^2 + Z_1\, \mathcal{D}_1 + (\kappa^2)^2\,)\, \vartheta$$

Eulerian operator

$$\mathcal{A} = \begin{pmatrix} \mathcal{D}_1^2 + Z_2\, \mathcal{D}_2 + (\kappa^1)^2 \\ \mathcal{D}_2^2 + Z_1\, \mathcal{D}_1 + (\kappa^2)^2 \end{pmatrix}$$

$$d_{\mathcal{V}}\, \varpi^1 = \kappa^1\, \vartheta \wedge \varpi^1 - \frac{1}{\kappa^1 - \kappa^2}(\, \mathcal{D}_1 \mathcal{D}_2 - Z_2 \mathcal{D}_1\,)\, \vartheta \wedge \varpi^2$$

$$d_{\mathcal{V}}\, \varpi^2 = \frac{1}{\kappa^1 - \kappa^2}(\, \mathcal{D}_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2\,)\, \vartheta \wedge \varpi^1 + \kappa^2\, \vartheta \wedge \varpi^2$$

Hamiltonian operator complex

$$\mathcal{B}_1^1 = \kappa^1 \qquad \mathcal{B}_2^1 = \frac{1}{\kappa^1 - \kappa^2}(\, \mathcal{D}_1 \mathcal{D}_2 - Z_2 \mathcal{D}_1\,) = -\, \mathcal{B}_1^2$$

$$\mathcal{B}_2^2 = \kappa^2$$

Euclidean-invariant variational problem

$$\int P(\kappa^{(n)})\,\omega^1 \wedge \omega^2 = \int P(\kappa^{(n)})\; dS$$

Euler-Lagrange equations

$$\mathbf{E}(L) = \mathcal{A}^{\dagger}\,\mathcal{E}(P) - \mathcal{B}^{\dagger}\,\mathcal{H}(P) = 0,$$

Special case: $P(\kappa^1, \kappa^2)$

$$\mathbf{E}(L) = [\,(\mathcal{D}_1^{\dagger})^2 - \mathcal{D}_2^{\dagger} \cdot Z_2 + (\kappa^1)^2\,]\,\frac{\partial P}{\partial \kappa^1} +$$

$$+ [\,(\mathcal{D}_2^{\dagger})^2 - \mathcal{D}_1^{\dagger} \cdot Z_1 + (\kappa^2)^2\,]\,\frac{\partial P}{\partial \kappa^2} + (\kappa^1 + \kappa^2)\,P$$

Minimal surfaces: $P = 1$

$$\kappa^1 + \kappa^2 = 2H = 0$$

Minimizing mean curvature: $P = H = \frac{1}{2}(\kappa^1 + \kappa^2)$

$$\tfrac{1}{2}\,\Big[\,(\kappa^1)^2 + (\kappa^2)^2 + \kappa^1 + \kappa^2\,\Big] = 2H^2 + H - K = 0.$$

Willmore surfaces: $P = \frac{1}{2}(\kappa^1)^2 + \frac{1}{2}(\kappa^2)^2$

$$\Delta(\kappa^1 + \kappa^2) + \tfrac{1}{2}(\kappa^1 + \kappa^2)(\kappa^1 - \kappa^2)^2 = 2\,\Delta H + 4(H^2 - K)H = 0$$

Laplace–Beltrami operator

$$\Delta = (\mathcal{D}_1 + Z_1)\mathcal{D}_1 + (\mathcal{D}_2 + Z_2)\mathcal{D}_2 = -\,\mathcal{D}_1^{\dagger} \cdot \mathcal{D}_1 - \mathcal{D}_2^{\dagger} \cdot \mathcal{D}_2$$