

Invariant Variational Problems
&
Integrable Curve Flows

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Basic Notation

$x = (x^1, \dots, x^p)$ — independent variables

$u = (u^1, \dots, u^q)$ — dependent variables

$u_J^\alpha = \partial_J u^\alpha$ — partial derivatives

$F(x, u^{(n)}) = F(\dots x^k \dots u_J^\alpha \dots)$
— differential function

$D_k F = \frac{\partial F}{\partial x^k} + \sum_{\alpha, J} u_{J,k}^\alpha \frac{\partial F}{\partial u_J^\alpha}$
— total derivative of F with respect to x^k

Variational Problems

$$\mathcal{I}[u] = \int L(x, u^{(n)}) dx$$

$L(x, u^{(n)})$ — Lagrangian

Variational derivative — Euler-Lagrange equations

$$\mathbf{E}(L) = 0$$

Components:

$$\mathbf{E}_\alpha(L) = \sum_J (-D)^J \frac{\partial L}{\partial u_J^\alpha}$$

Invariant Variational Problems

G — transformation group

G -invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)}) d\mathbf{x} = \int P(\dots \mathcal{D}_K I^\alpha \dots) \omega$$

\implies *Lie*

I^1, \dots, I^ℓ — fundamental differential invariants

$\mathcal{D}_1, \dots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\omega = \omega^1 \wedge \dots \wedge \omega^p$ — invariant volume form

Invariant Euler-Lagrange equations

$$\mathbf{E}(L) = F(\dots \mathcal{D}_K I^\alpha \dots) = 0$$

Main Problem:

Construct F directly from P .

(*P. Griffiths, I. Anderson*)

Planar Euclidean group $G = \text{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \quad \text{— curvature}$$

$$ds = \sqrt{1 + u_x^2} dx \quad \text{— arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}} \frac{d}{dx} \quad \text{— arc length derivative}$$

Invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \dots) ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) = F(\kappa, \kappa_s, \kappa_{ss}, \dots) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[u] = \int ds = \int \sqrt{1 + u_x^2} dx$$

$$\mathbf{E}(L) = -\kappa = 0$$

The Elastica (Euler):

$$\mathcal{I}[u] = \int \frac{1}{2} \kappa^2 ds = \int \frac{u_{xx}^2 dx}{(1 + u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \frac{1}{2} \kappa^3 = 0$$

\implies elliptic functions

General Euclidean-invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \dots) ds$$

Invariantized Euler-Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \quad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j} (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2) \mathcal{E}(P) + \kappa \mathcal{H}(P).$$

Elastica : $P = \frac{1}{2} \kappa^2$

$$\mathcal{E}(P) = \kappa \quad \mathcal{H}(P) = -P = -\frac{1}{2} \kappa^2$$

$$\mathbf{E}(L) = \kappa_{ss} + \frac{1}{2} \kappa^3 = 0$$

Moving Frames

⇒ Mark Fels and PJO

G — r -dimensional Lie group acting on M

$J^n = J^n(M, p)$ — n^{th} order jet bundle for
 p -dimensional submanifolds $N = \{u = f(x)\} \subset M$

$z^{(n)} = (x, u^{(n)}) = (\dots x^i \dots u_J^\alpha \dots)$
— coordinates on J^n

Definition.

An n^{th} order *moving frame* is a G -equivariant map

$$\rho = \rho^{(n)} : V \subset J^n \longrightarrow G$$

Equivariance:

$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} g \cdot \rho(z^{(n)}) & \text{left moving frame} \\ \rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

Note $\rho_{left}(z^{(n)}) = \rho_{right}(z^{(n)})^{-1}$

Theorem.

A moving frame exists in a neighborhood of a point $z^{(n)} \in \mathbf{J}^n$ if and only if G acts freely and regularly near $z^{(n)}$.

Theorem.

If G acts locally effectively on subsets, then for $n \gg 0$, the (prolonged) action of G is locally free on an open subset of \mathbf{J}^n .

\implies Ovsianikov, PJO

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:

$$g \cdot z = z \text{ if and only if } g = e.$$
- locally free — the orbits have the same dimension as G .
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once (\neq irrational flow on the torus)
- effective — the only group element $g \in G$ which fixes *every* point $z \in M$ is the identity:

$$g \cdot z = z \text{ for all } z \in M \text{ if and only if } g = e.$$

The Normalization Construction

1. Write out the explicit formulas for the prolonged group action:

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$$

\implies *Implicit differentiation*

2. From the components of $w^{(n)}$, choose $r = \dim G$ *normalization equations*:

$$w_1(g, z^{(n)}) = c_1 \quad \dots \quad w_r(g, z^{(n)}) = c_r$$

3. Solve the normalization equations for the group parameters

$$g = (g_1, \dots, g_r):$$

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The solution is the right moving frame.

4. Substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants of order $\leq n$:

$$I_k(x, u^{(n)}) = w_k(\rho(z^{(n)}), z^{(n)})$$

$$k = r + 1, \dots, \dim \mathbf{J}^n$$

Euclidean plane curves

$$G = \text{SE}(2)$$

Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

Prolong to \mathbb{J}^3 via implicit differentiation

$$\left. \begin{aligned} y &= x \cos \phi - u \sin \phi + a \\ v &= x \cos \phi + u \sin \phi + b \end{aligned} \right\} w = Rz + c$$

$$v_y = \frac{\sin \phi + u_x \cos \phi}{\cos \phi - u_x \sin \phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos \phi - u_x \sin \phi)^3}$$

$$v_{yyy} = \frac{(\cos \phi - u_x \sin \phi)u_{xxx} - 3u_{xx}^2 \sin \phi}{(\cos \phi - u_x \sin \phi)^5}$$

⋮

Normalization: $r = \dim G = 3$

$$y = 0$$

$$v = 0$$

$$v_y = 0$$

Right moving frame

$$\rho: J^1 \longrightarrow \text{SE}(2)$$

$$\phi = -\tan^{-1} u_x$$

$$a = -\frac{x + uu_x}{\sqrt{1 + u_x^2}}$$

$$b = \frac{xu_x - u}{\sqrt{1 + u_x^2}}$$

Differential invariants

$$\begin{aligned}v_{yy} &\longmapsto \kappa &= \frac{u_{xx}}{(1+u_x^2)^{3/2}} \\v_{yyy} &\longmapsto \frac{d\kappa}{ds} &= \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3} \\v_{yyyy} &\longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3 &= \dots\end{aligned}$$

Invariant one-form — arc length

$$dy = (\cos \phi - u_x \sin \phi) dx \quad \longmapsto \quad ds = \sqrt{1+u_x^2} \, dx$$

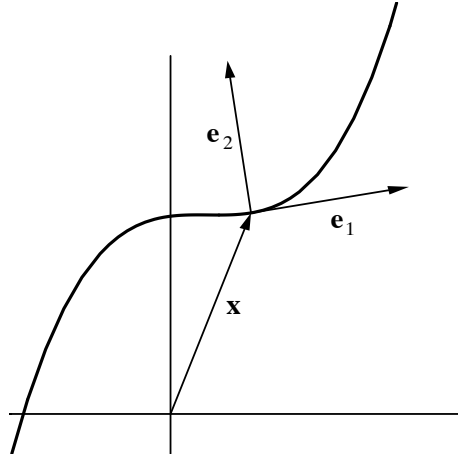
Invariant differential operator

$$\frac{d}{dy} = \frac{1}{\cos \phi - u_x \sin \phi} \frac{d}{dx} \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}} \frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa \qquad \frac{d\kappa}{ds} \qquad \frac{d^2\kappa}{ds^2} \qquad \dots$$

Euclidean Curves



Left moving frame $\tilde{\rho}(x, u^{(1)}) = \rho(x, u^{(1)})^{-1}$

$$\tilde{a} = x \qquad \tilde{b} = u \qquad \tilde{\theta} = \tan^{-1} u_x$$

$$R = \frac{1}{\sqrt{1 + u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\mathbf{t} \quad \mathbf{n}) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix} \quad \mathbf{n} = \mathbf{t}^\perp = \begin{pmatrix} -y_s \\ x_s \end{pmatrix}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1 \quad \frac{d\mathbf{e}_1}{ds} = \kappa \mathbf{e}_2 \quad \frac{d\mathbf{e}_2}{ds} = -\kappa \mathbf{e}_1$$

Invariantization

The process of replacing group parameters in transformation rules by their moving frame formulae is known as *invariantization*:

	Functions	→	Invariants
ι :	Forms	→	Invariant Forms
	Differential	→	Invariant Differential
	Operators	→	Operators

Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \quad I_K^\alpha(x, u^{(l)}) = \iota(u_K^\alpha)$$

\implies The constant differential invariants, coming from the moving frame normalizations, are known as the *phantom invariants*. The remaining non-constant differential invariants are the *basic invariants* and form a complete system of functionally independent differential invariants for the prolonged group action.

Invariantization:

$$\iota [F(\dots x^i \dots u_J^\alpha \dots)] = F(\dots H^i \dots I_J^\alpha \dots)$$

Replacement Theorem:

If J is a differential invariant, then $\iota(J) = J$.

$$J(\dots x^i \dots u_J^\alpha \dots) = J(\dots H^i \dots I_J^\alpha \dots)$$

The Infinite Jet Bundle

Jet bundles

$$M = J^0 \longleftarrow J^1 \longleftarrow J^2 \longleftarrow \dots$$

Inverse limit

$$J^\infty = \lim_{n \rightarrow \infty} J^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\dots x^i \dots u_J^\alpha \dots)$$

\implies Taylor series

Differential Forms

Coframe — basis for the cotangent space T^*J^∞ :

- Horizontal one-forms

$$dx^1, \dots, dx^p$$

- Contact (vertical) one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha dx^i$$

Intrinsic definition of contact form

$$\theta \mid j_\infty N = 0 \quad \iff \quad \theta = \sum A_J^\alpha \theta_J^\alpha$$

The Variational Bicomplex

\implies *Dedecker, Vinogradov, Tsujishita, I. Anderson, ...*

Bigrading of the differential forms on J^∞ :

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

$r = \#$ horizontal forms

$s = \#$ contact forms

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

Vertical and Horizontal Differentials

$F(x, u^{(n)})$ — differential function

$$d_H F = \sum_{i=1}^p (D_i F) dx^i \quad \text{— total differential}$$

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u_J^\alpha} \theta_J^\alpha \quad \text{— variation}$$

$$d_H (dx^i) = d_V (dx^i) = 0,$$

$$d_H (\theta_J^\alpha) = \sum_{i=1}^p dx^i \wedge \theta_{J,i}^\alpha, \quad d_V (\theta_J^\alpha) = 0.$$

The Simplest Example

$$(x, u) \in M = \mathbb{R}^2$$

x — independent variable

u — dependent variable

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x dx$$

$$\theta_x = du_x - u_{xx} dx$$

$$\theta_{xx} = du_{xx} - u_{xxx} dx$$

\vdots

$$\theta = du - u_x dx, \quad \theta_x = du_x - u_{xx} dx, \quad \theta_{xx} = du_{xx} - u_{xxx} dx$$

Differential:

$$\begin{aligned} dF &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial u} du + \frac{\partial F}{\partial u_x} du_x + \frac{\partial F}{\partial u_{xx}} du_{xx} + \dots \\ &= (D_x F) dx + \frac{\partial F}{\partial u} \theta + \frac{\partial F}{\partial u_x} \theta_x + \frac{\partial F}{\partial u_{xx}} \theta_{xx} + \dots \\ &= d_H F + d_V F \end{aligned}$$

Total derivative:

$$D_x F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} u_x + \frac{\partial F}{\partial u_x} u_{xx} + \frac{\partial F}{\partial u_{xx}} u_{xxx} + \dots$$

The Variational Bicomplex

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & \delta \uparrow \\
 \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 d_V \uparrow & & d_V \uparrow & & & & d_V \uparrow & & d_V \uparrow & & \delta \uparrow \\
 \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 d_V \uparrow & & d_V \uparrow & & & & d_V \uparrow & & d_V \uparrow & & \delta \uparrow \\
 \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 d_V \uparrow & & d_V \uparrow & & & & d_V \uparrow & & d_V \uparrow & & \delta \uparrow \\
 \mathbb{R} \rightarrow \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & & \mathcal{E} \nearrow
 \end{array}$$

- conservation laws
Lagrangians
PDEs (Euler–Lagrange)
Helmholtz conditions

Variational Derivative

$$\delta = \pi \circ d_V$$

d_V — first variation
 π — integration by parts

$$\begin{array}{ccccc}
 \Omega^{p,0} & \xrightarrow{d_V} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 \lambda = L d\mathbf{x} & \longrightarrow & \sum_{\alpha,J} \frac{\partial L}{\partial u_J^\alpha} \theta_J^\alpha \wedge d\mathbf{x} & \longrightarrow & \sum_{\alpha=1}^q \mathbf{E}_\alpha(L) \theta^\alpha \wedge d\mathbf{x} \\
 \text{Variational} & \longrightarrow & \text{First} & \longrightarrow & \text{Euler-Lagrange} \\
 \text{problem} & & \text{variation} & & \text{Form}
 \end{array}$$

The Simplest Example: $x, u \in \mathbb{R}$

Lagrangian form

$$\lambda = L(x, u^{(n)}) dx \in \Omega^{1,0}$$

Vertical derivative — variation

$$\begin{aligned} d\lambda &= d_V \lambda = d_V L \wedge dx \\ &= \left(\frac{\partial L}{\partial u} \theta + \frac{\partial L}{\partial u_x} \theta_x + \frac{\partial L}{\partial u_{xx}} \theta_{xx} + \dots \right) \wedge dx \in \Omega^{1,1} \end{aligned}$$

Integration by parts

$$\begin{aligned} d_H (A \theta) &= - (D_x A) dx \wedge \theta - A \theta_x \wedge dx \\ &= - [(D_x A) \theta + A \theta_x] \wedge dx \end{aligned}$$

so

$$A \theta_x \wedge dx \sim (D_x A) \theta \wedge dx \quad \text{mod im } d_H$$

Lagrangian form: $\lambda = L(x, u^{(n)}) dx \in \Omega^{1,0}$

Vertical derivative — variation:

$$\begin{aligned} d\lambda &= d_V \lambda = d_V L \wedge dx \\ &= \left(\frac{\partial L}{\partial u} \theta + \frac{\partial L}{\partial u_x} \theta_x + \frac{\partial L}{\partial u_{xx}} \theta_{xx} + \dots \right) \wedge dx \in \Omega^{1,1} \end{aligned}$$

Integration by parts — compute modulo $\text{im } d_H$:

$$\begin{aligned} d\lambda \sim \delta\lambda &= \left(\frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \dots \right) \theta \wedge dx \\ &= \mathbf{E}(L) \theta \wedge dx \end{aligned}$$

\implies Euler-Lagrange source form.

Differential Invariants and Invariant Differential Forms

ι — invariantization associated with moving frame ρ .

- Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \quad I_K^\alpha(x, u^{(l)}) = \iota(u_K^\alpha)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i)$$

- Invariant contact forms

$$\vartheta_J^\alpha = \iota(\theta_J^\alpha)$$

The Invariant “Quasi–Tricomplex”

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$d_{\mathcal{H}} : \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r+1,s}$$

$$d_{\mathcal{V}} : \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r,s+1}$$

$$d_{\mathcal{W}} : \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r-1,s+2}$$

The Universal Recurrence Formula

$$d\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^r \mu^\kappa \wedge \iota[\mathbf{v}_\kappa(\Omega)]$$

$\mathbf{v}_1, \dots, \mathbf{v}_r$ — basis for infinitesimal generators \mathfrak{g}

μ_1, \dots, μ_r — invariantized dual Maurer–Cartan forms

★ ★ ★ All identities, commutation formulae, etc., in the invariant variational bicomplex are consequences of this universal formula, by letting Ω range over the basic functions and differential forms!

Euclidean plane curves

Fundamental normalized differential invariants

$$\left. \begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned} \right\} \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa \quad \iota(u_{xxx}) = I_3 = \kappa_s \quad \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

In general:

$$\iota(F(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, \dots)) = F(0, 0, 0, \kappa, \kappa_s, \kappa_{ss} + 3\kappa^3, \dots)$$

Invariant one-form

$$dy = (\cos \phi - u_x \sin \phi) dx - (\sin \phi) \theta$$

$$\begin{aligned}\varpi = \iota(dx) &= \omega + \eta \\ &= \sqrt{1 + u_x^2} dx + \frac{u_x}{\sqrt{1 + u_x^2}} \theta\end{aligned}$$

$$\implies \theta = du - u_x dx$$

Invariant contact forms

$$\iota(\theta) = \vartheta = \frac{\theta}{\sqrt{1 + u_x^2}} \quad \iota(\theta_x) = \vartheta_1 = \frac{(1 + u_x^2) \theta_x - u_x u_{xx} \theta}{(1 + u_x^2)^2}$$

Prolonged infinitesimal generators

$$\begin{aligned}\mathbf{v}_1 &= \partial_x & \mathbf{v}_2 &= \partial_u \\ \mathbf{v}_3 &= -u \partial_x + x \partial_u + (1 + u_x^2) \partial_{u_x} + 3u_x u_{xx} \partial_{u_{xx}} + \dots\end{aligned}$$

Horizontal recurrence formula

$$\begin{aligned}d_{\mathcal{H}} \iota(F) &= \iota(d_H F) + \iota(\mathbf{v}_1(F)) \gamma^1 + \iota(\mathbf{v}_2(F)) \gamma^2 + \iota(\mathbf{v}_3(F)) \gamma^3 \\ d_{\mathcal{H}} I &= \mathcal{D}I \cdot \varpi & \iota(d_H F) &= \iota(D_x F) \varpi\end{aligned}$$

Use phantom invariants

$$\begin{aligned}0 &= d_{\mathcal{H}} H = \iota(d_H x) + \sum \iota(\mathbf{v}_\kappa(x)) \gamma^\kappa = \varpi + \gamma^1 \\ 0 &= d_{\mathcal{H}} I_0 = \iota(d_H u) + \sum \iota(\mathbf{v}_\kappa(u)) \gamma^\kappa = \gamma^2 \\ 0 &= d_{\mathcal{H}} I_1 = \iota(d_H u_x) + \sum \iota(\mathbf{v}_\kappa(u_x)) \gamma^\kappa = \kappa \varpi + \gamma^3,\end{aligned}$$

to solve for

$$\gamma^1 = -\varpi \quad \gamma^2 = 0 \quad \gamma^3 = -\kappa \varpi$$

$$\gamma^1 = -\varpi \quad \gamma^2 = 0 \quad \gamma^3 = -\kappa \varpi$$

Horizontal recurrence formulae

$$\begin{aligned} \kappa_s \varpi &= d_{\mathcal{H}} \kappa = d_{\mathcal{H}} (I_2) = \iota(d_H u_{xx}) + \iota(\mathbf{v}_3(u_{xx})) \gamma^3 \\ &= \iota(u_{xxx} dx) - \iota(3u_x u_{xx}) \kappa \varpi = I_3 \varpi \end{aligned}$$

$$\begin{aligned} \kappa_{ss} \varpi &= d_{\mathcal{H}} (I_3) = \iota(d_H u_{xxx}) + \iota(\mathbf{v}_3(u_{xxx})) \gamma^3 \\ &= \iota(u_{xxxx} dx) - \iota(4u_x u_{xxx} + 3u_{xx}^2) \kappa \varpi = (I_4 - 3I_2^3) \varpi \end{aligned}$$

$$\kappa = I_2 \quad \kappa_s = I_3 \quad \kappa_{ss} = I_4 - 3I_2^3 \quad \kappa_{sss} = I_5 - 19I_2^2 I_3$$

$$I_2 = \kappa \quad I_3 = \kappa_s \quad I_4 = \kappa_{ss} + 3\kappa^3 \quad I_5 = \kappa_{sss} + 19\kappa^2 \kappa_s$$

Vertical recurrence formula

$$d_{\mathcal{V}} \iota(F) = \iota(d_{\mathcal{V}} F) + \iota(\mathbf{v}_1(F)) \varepsilon^1 + \iota(\mathbf{v}_2(F)) \varepsilon^2 + \iota(\mathbf{v}_3(F)) \varepsilon^3$$

Use phantom invariants

$$0 = d_{\mathcal{V}} H = \varepsilon^1 \quad 0 = d_{\mathcal{V}} I_0 = \vartheta + \varepsilon^2 \quad 0 = d_{\mathcal{V}} I_1 = \vartheta_1 + \varepsilon^3$$

to solve for

$$\varepsilon^1 = 0 \quad \varepsilon^2 = -\vartheta = -\iota(\theta) \quad \varepsilon^3 = -\vartheta_1 = -\iota(\theta_1)$$

$$d_{\mathcal{V}} \kappa = d_{\mathcal{V}} I_2 = \iota(\theta_2) + \iota(\mathbf{v}_3(u_{xx})) \varepsilon^3 = \vartheta_2 = (\mathcal{D}^2 + \kappa^2) \vartheta$$

Key recurrence formulae:

$$d_{\mathcal{V}} \kappa = (\mathcal{D}^2 + \kappa^2) \vartheta$$

$$d_{\mathcal{V}} \varpi = -\kappa \vartheta \wedge \varpi$$

Plane Curves

Invariant Lagrangian:

$$\tilde{\lambda} = L(x, u^{(n)}) dx = P(\kappa, \kappa_s, \dots) \varpi$$

Euler–Lagrange form:

$$d_{\mathcal{V}} \tilde{\lambda} \sim \mathbf{E}(L) \vartheta \wedge \varpi$$

Invariant Integration by Parts Formula

$$F d_{\mathcal{V}} (\mathcal{D}H) \wedge \varpi \sim -(\mathcal{D}F) d_{\mathcal{V}} H \wedge \varpi - (F \cdot \mathcal{D}H) d_{\mathcal{V}} \varpi$$

$$\begin{aligned} d_{\mathcal{V}} \tilde{\lambda} &= d_{\mathcal{V}} P \wedge \varpi + P d_{\mathcal{V}} \varpi \\ &= \sum_n \frac{\partial P}{\partial \kappa_n} d_{\mathcal{V}} \kappa_n \wedge \varpi + P d_{\mathcal{V}} \varpi \\ &\sim \mathcal{E}(P) d_{\mathcal{V}} \kappa \wedge \varpi + \mathcal{H}(P) d_{\mathcal{V}} \varpi \end{aligned}$$

Vertical differentiation formulae

$$d_{\mathcal{V}} \kappa = \mathcal{A}(\vartheta) \quad \mathcal{A} \text{ — “Eulerian operator”}$$

$$d_{\mathcal{V}} \varpi = \mathcal{B}(\vartheta) \wedge \varpi \quad \mathcal{B} \text{ — “Hamiltonian operator”}$$

$$\begin{aligned} d_{\mathcal{V}} \tilde{\lambda} &\sim \mathcal{E}(P) \mathcal{A}(\vartheta) \wedge \varpi + \mathcal{H}(P) \mathcal{B}(\vartheta) \wedge \varpi \\ &\sim \left[\mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) \right] \vartheta \wedge \varpi \end{aligned}$$

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) = 0}$$

Euclidean Plane Curves

$$d_{\mathcal{V}} \kappa = (\mathcal{D}^2 + \kappa^2) \vartheta$$

Eulerian operator

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad \mathcal{A}^* = \mathcal{D}^2 + \kappa^2$$

$$d_{\mathcal{V}} \varpi = -\kappa \vartheta \wedge \varpi$$

Hamiltonian operator

$$\mathcal{B} = -\kappa \qquad \mathcal{B}^* = -\kappa$$

Euclidean-invariant Euler-Lagrange formula

$$\mathbf{E}(L) = \mathcal{A}^* \mathcal{E}(P) - \mathcal{B}^* \mathcal{H}(P) = (\mathcal{D}^2 + \kappa^2) \mathcal{E}(P) + \kappa \mathcal{H}(P).$$

Invariant Plane Curve Flows

G — Lie group acting on \mathbb{R}^2

$C(t)$ — parametrized family of plane curves

G -invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I \mathbf{t} + J \mathbf{n}$$

- I, J — differential invariants
- \mathbf{t} — “unit tangent”
- \mathbf{n} — “unit normal”

\mathbf{t} , \mathbf{n} — basis of the invariant vector fields dual to the invariant one-forms:

$$\langle \mathbf{t}; \varpi \rangle = 1, \quad \langle \mathbf{n}; \varpi \rangle = 0,$$

$$\langle \mathbf{t}; \vartheta \rangle = 0, \quad \langle \mathbf{n}; \vartheta \rangle = 1.$$

$$C_t = \mathbf{V} = I \mathbf{t} + J \mathbf{n}$$

- The tangential component $I \mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set I to be anything we like without affecting the curve evolution.
- There are two principal choices of tangential component:

Normal Curve Flows

$$C_t = J \mathbf{n}$$

Examples — Euclidean Curve Flows

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa \mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3} \mathbf{n}$ — equi-affine invariant curve shortening flow:
$$C_t = \mathbf{n}_{\text{equi-affine}} ;$$
- $C_t = \kappa_s \mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss} \mathbf{n}$ — thermal grooving of metals.

Intrinsic Curve Flows

Theorem. The curve flow generated by

$$\mathbf{v} = I \mathbf{t} + J \mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}I = 0.$$

\mathcal{D} — invariant arc length derivative

$$d_{\mathcal{V}} \varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

\mathcal{B} — invariant Hamiltonian operator

Normal Evolution of Differential Invariants

Theorem. Under a normal flow $C_t = J \mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \quad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

Invariant Fréchet derivatives:

$$d_{\mathcal{V}} \kappa = \mathcal{A}_\kappa(\vartheta), \quad d_{\mathcal{V}} \kappa_s = \mathcal{A}_{\kappa_s}(\vartheta).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant Eulerian operator or
invariant Fréchet derivative of curvature;

$\mathcal{A}_{\kappa_s} = \mathcal{D} \mathcal{A}_\kappa + \kappa \kappa_s$ — invariant Fréchet derivative of κ_s .

Euclidean Signature Evolution

Normal Euclidean-invariant curve flow:

$$C_t = J \mathbf{n}$$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2) J,$$

$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2 \mathcal{D} + 3\kappa \kappa_s) J.$$

Euclidean signature curve:

$$\kappa_s = \Phi(t, \kappa).$$

Evolution of the Euclidean signature curve

$$\kappa_s = \Phi(t, \kappa).$$

Grassfire flow:

$$\frac{\partial \Phi}{\partial t} = 3\kappa \Phi - \kappa^2 \frac{\partial \Phi}{\partial \kappa}.$$

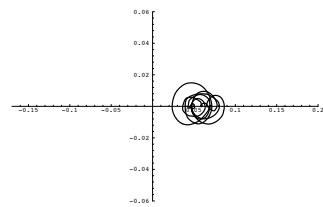
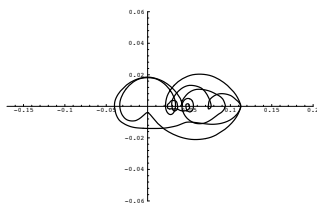
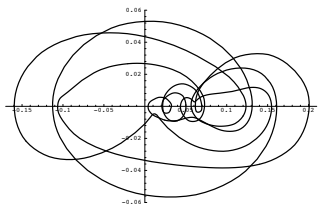
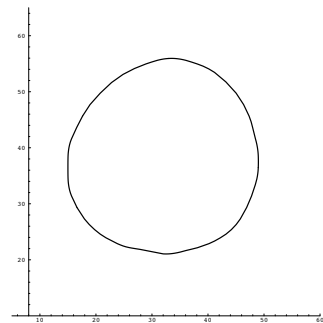
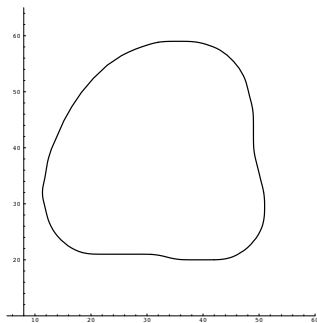
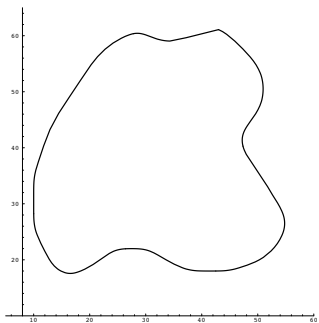
Curve shortening flow:

$$\frac{\partial \Phi}{\partial t} = \Phi^2 \Phi_{\kappa\kappa} - \kappa^3 \Phi_{\kappa} + 4\kappa^2 \Phi.$$

Modified Korteweg-deVries flow:

$$\frac{\partial \Phi}{\partial t} = \Phi^3 \Phi_{\kappa\kappa\kappa} + 3\Phi^2 \Phi_{\kappa} \Phi_{\kappa\kappa} + 3\kappa \Phi^2.$$

Smoothed Ventricle Signature



Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \quad (*)$$

where

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B}.$$

In surprisingly many situations, (*) is a well-known integrable evolution equation, and \mathcal{R} is its recursion operator!

\implies Hasimoto

\implies Langer, Singer, Perline

\implies Marí-Beffa, Sanders, Wang

\implies Qu, Chou, and many more ...

Euclidean plane curves

$$G = \text{SE}(2) = \text{SO}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}} \kappa = (\mathcal{D}^2 + \kappa^2) \vartheta, \quad d_{\mathcal{V}} \varpi = -\kappa \vartheta \wedge \varpi$$

$$\implies \mathcal{A} = \mathcal{D}^2 + \kappa^2, \quad \mathcal{B} = -\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \frac{3}{2} \kappa^2 \kappa_s$$

\implies modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \text{SA}(2) = \text{SL}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}} \kappa = \mathcal{A}(\vartheta), \quad d_{\mathcal{V}} \varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$$\mathcal{A} = \mathcal{D}^4 + \frac{5}{3} \kappa \mathcal{D}^2 + \frac{5}{3} \kappa_s \mathcal{D} + \frac{1}{3} \kappa_{ss} + \frac{4}{9} \kappa^2,$$

$$\mathcal{B} = \frac{1}{3} \mathcal{D}^2 - \frac{2}{9} \kappa,$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B}$$

$$= \mathcal{D}^4 + \frac{5}{3} \kappa \mathcal{D}^2 + \frac{4}{3} \kappa_s \mathcal{D} + \frac{1}{3} \kappa_{ss} + \frac{4}{9} \kappa^2 + \frac{2}{9} \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + 2 \kappa \kappa_{ss} + \frac{4}{3} \kappa_s^2 + \frac{5}{9} \kappa^2 \kappa_s$$

\implies Sawada–Kotera equation

Euclidean space curves

$$G = \text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3$$

$$\begin{pmatrix} d_{\mathcal{V}} \kappa \\ d_{\mathcal{V}} \tau \end{pmatrix} = \mathcal{A} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \quad d_{\mathcal{V}} \varpi = \mathcal{B} \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \wedge \varpi$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) \\ \frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} \\ -2\tau D_s - \tau_s \\ \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = (\kappa \quad 0)$$

Recursion operator:

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$
$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}$$

\implies vortex filament flow

\implies nonlinear Schrödinger equation (Hasimoto)