Moving Frames in Classical Invariant Theory

Peter J. Olver

University of Minnesota

http://www.math.umn.edu/ ∼ olver

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Classical Invariant Theory

Peter J. Olver

London Mathematical Society
Student Texts **44**

Binary Forms

$\Longrightarrow \quad \mathbb{R}$ or $\mathbb{C}$.

Homogeneous version:

$$Q(x, y) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k y^{n-k}$$

Inhomogeneous (projective) version:

$$Q(p) = Q(p, 1) = \sum_{k=0}^{n} \binom{n}{k} \, a_k \, p^k$$

Note:

$$Q(x, y) = y^n \, Q\left(\frac{x}{y}\right)$$

Equivalence of Binary Forms

Transformation group: $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$

Equivalence:

$$\overline{Q} = g \cdot Q$$

Symmetry = Self-equivalence:

$$Q = g \cdot Q$$

$\Longrightarrow$ Galois theory???

Homogeneous transformation rule:

$$\bar{x} = \alpha\, x + \beta\, y, \qquad \bar{y} = \gamma\, x + \delta\, y,$$

$$Q(x, y) = \overline{Q}(\alpha\, x + \beta\, y, \gamma\, x + \delta\, y)$$

Inhomogeneous transformation rule:

$$\bar{p} = \frac{\alpha\, p + \beta}{\gamma\, p + \delta}, \qquad Q(p) = (\gamma\, p + \delta)^n\, \overline{Q}\left(\frac{\alpha\, p + \beta}{\gamma\, p + \delta}\right)$$

- multiplier representation of GL(2)
- section of line bundle over $\mathbb{CP}^1$
- modular forms

Invariants

Definition. An invariant of a binary form Q of degree n is a function $I(\mathbf{a}) = I(a_0, \ldots, a_n)$ depending on its coefficients which satisfies

$$I(\mathbf{a}) = (\alpha\delta - \beta\gamma)^k I(\overline{\mathbf{a}})$$

for some integer k under the action of GL(2).

- $k = \mathrm{wt}\, I$ — *weight*
- Strictly speaking, I is only an invariant of the action of SL(2) and a *relative invariant* of GL(2)

♡ The vanishing of an invariant, $I = 0$, has intrinsic meaning.

Examples of Invariants

Binary quadratic:

$$Q(x, y) = a_2 x^2 + 2\,a_1 x\,y + a_0 y^2,$$

Discriminant:

$$\Delta = a_0 a_2 - a_1^2$$

Since

$$\overline{\Delta} = (\alpha\delta - \beta\gamma)^2 \Delta$$

the discriminant is an invariant of weight 2

$\Longrightarrow$ *Boole, Cayley, ...*

Binary cubic:

$$Q(x,y) = a_3 x^3 + 3\,a_2 x^2 y + 3\,a_1 x\,y^2 + a_0 y^3.$$

Discriminant:

$$\Delta = a_0^2 a_3^2 - 6\,a_0 a_1 a_2 a_3 + 4\,a_0 a_2^3 - 3\,a_1^2 a_2^2 + 4\,a_1^3 a_3$$

$$\Longrightarrow \quad \text{weight } 6$$

$$\Delta = 0 \qquad \Longleftrightarrow \qquad \text{multiple root}$$

Binary quartic:

$$Q(\mathbf{x}) = a_4 x^4 + 4\,a_3 x^3 y + 6\,a_2 x^2 y^2 + 4\,a_1 x\,y^3 + a_0 y^4.$$

Invariants:

$$i = a_0 a_4 - 4\,a_1 a_3 + 3\,a_2^2 \qquad \text{weight } 4$$

$$j = \det \begin{vmatrix} a_4 & a_3 & a_2 \\ a_3 & a_2 & a_1 \\ a_2 & a_1 & a_0 \end{vmatrix} \qquad \text{weight } 6$$

Discriminant: $\Delta = i^3 - 27\,j^2.$

Absolute rational invariant: $r = \dfrac{j^2}{i^3}\,.$

Covariants

Definition. A covariant of *weight* k is a function

$$J(\mathbf{a}, \mathbf{x}) = J(a_0, \ldots, a_n, x, y)$$

which satisfies

$$J(\mathbf{a}, \overline{\mathbf{x}}) = (\alpha\,\delta - \beta\,\gamma)^k\,\bar{J}(\bar{\mathbf{a}}, \mathbf{x}).$$

under the action of GL(2).

Hessian Covariant

$$H = Q_{xx}Q_{yy} - Q_{xy}^2 \qquad \text{weight } 2$$

Projective version:

$$H = n(n-1)Q\,Q_{pp} - (n-1)^2 Q_p^2$$

Theorem.

$H \equiv 0$ if and only if $Q(x,y) = (a\,x + b\,y)^n$.

Jacobian Covariants

If K, L are covariants, so is

$$J = \frac{\partial(K, L)}{\partial(x, y)} = K_x L_y - K_y L_x \qquad \text{weight } j + k + 1$$

Examples:

$$T = Q_x H_y - Q_y H_x \qquad \text{weight } 3$$

$$= -Q_y Q_{yy} Q_{xxx} + (2 Q_y Q_{xy} + Q_x Q_{yy}) Q_{xxy} -$$

$$- (Q_y Q_{xx} + 2 Q_x Q_{xy}) Q_{xxy} + Q_x Q_{xx} Q_{yyy}$$

$$U = Q_x T_y - Q_y T_x \qquad \text{weight } 4$$

$$\vdots$$

Transvectants

$$(Q,\ R)^{(0)} = Q\,R$$

$$(Q,\ R)^{(1)} = Q_x R_y - Q_y R_x$$

$$(Q,\ R)^{(2)} = Q_{xx} R_{yy} - 2\,Q_{xy} R_{xy} + Q_{yy} R_{xx}$$

$$\vdots$$

$$(Q,\ R)^{(r)} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r Q}{\partial x^{r-i} \partial y^i} \frac{\partial^r R}{\partial x^i \partial y^{r-i}}$$

Note:

$$H = \tfrac{1}{2}\,(Q,\ Q)^{(2)} = Q_{xx} Q_{yy} - Q_{xy}^2$$

Transvectants

Projective version: $\qquad \deg Q = n, \qquad \deg R = m$

$$(Q,\, R)^{(r)} = r! \, \sum_{k=0}^{r} (-1)^k \binom{n-r+k}{k} \binom{m-k}{r-k} \, Q^{(r-k)}(p) R^{(k)}(p)$$

$$(Q,\, R)^{(0)} = QR$$

$$(Q,\, R)^{(1)} = m\, Q' R - n\, Q R'$$

$$(Q,\, R)^{(2)} = m(m-1) Q'' R - 2(m-1)(n-1) Q' R' + n(n-1) Q R''$$

$$(Q,\, R)^{(3)} = m(m-1)(m-2) Q''' R - 3(m-1)(m-2)(n-2) Q'' R' + $$
$$+ 3(m-2)(n-1)(n-2) Q' R'' - n(n-1)(n-2) Q R'''$$

The First Fundamental Theorem

Theorem. *All* polynomial covariants and invariants of any system of binary forms can be expressed as linear combinations of iterated transvectants.

Gordan's Theorem

Theorem. The invariants and covariants of a binary form admit a finite generating basis.

$\Longrightarrow$ Constructive

$\Longrightarrow$ Hilbert!

Counting Invariants and Covariants

Sylvester's Table

degree	2	3	4	5	6	7	8	9	10	12
# invariants	1	1	2	4	5	26 (30)	9	89	104	109
# covariants	2	4	5	23	26	124 (130)	69	415	475	949

- degree 7 — Dixmier & Lazard (1986)
- degree 8 — Shioda (1967), Bedratyuk (2006)

A Rational Basis for Covariants

Let

$$S_j = (Q,\, Q)^{(2j)} \qquad j = 1, \ldots m$$

$$T_k = (S_k,\, Q)^{(1)} \qquad k = 1, \ldots m'$$

where

$$4 \leq \deg Q = n = m + m' = \begin{cases} 2\,m & \text{even} \\ 2\,m + 1 & \text{odd} \end{cases}$$

Theorem. (*Stroh, Hilbert*)

Every polynomial covariant C can be written as

$$C = \frac{1}{Q^N}\, P(Q, S_1, \ldots, S_m, T_1, \ldots, T_{m'})$$

where P is a polynomial and N an integer.

Moving Frames

Definition.

A moving frame is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

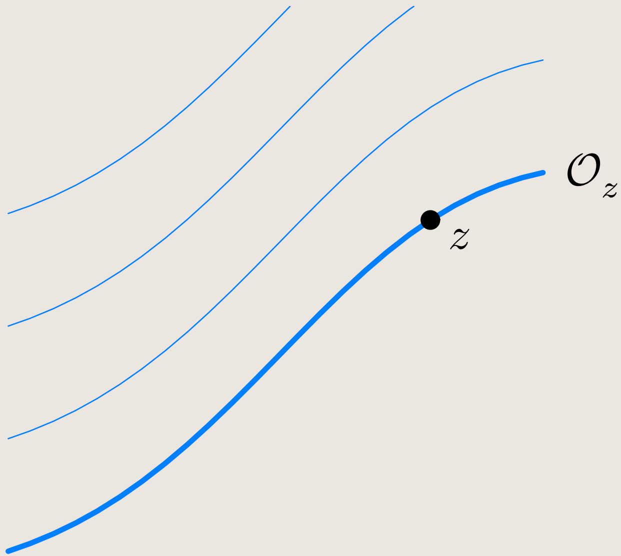
Isotropy & Freeness

Isotropy subgroup: $G_z = \{ g \mid g \cdot z = z \}$ for $z \in M$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity: $\Longrightarrow$ $G_z = \{e\}$ for all $z \in M$.
- locally free — the orbits all have the same dimension as G: $\Longrightarrow$ G_z is a discrete subgroup of G.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once

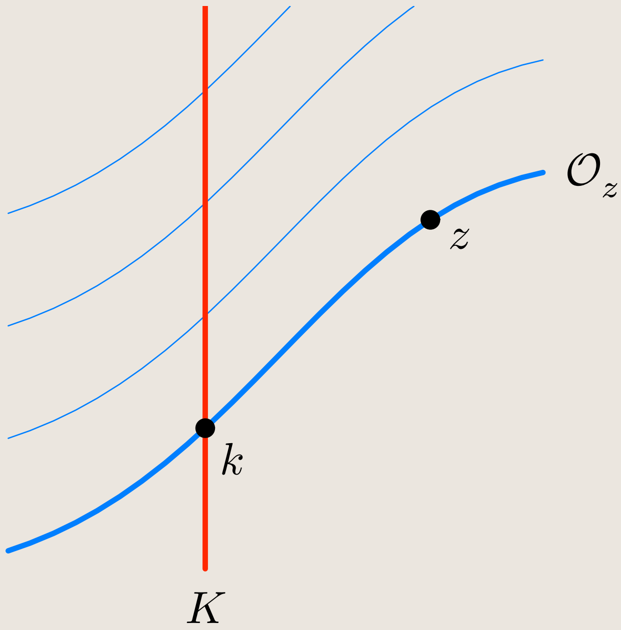
  $\not\approx$ irrational flow on the torus

Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



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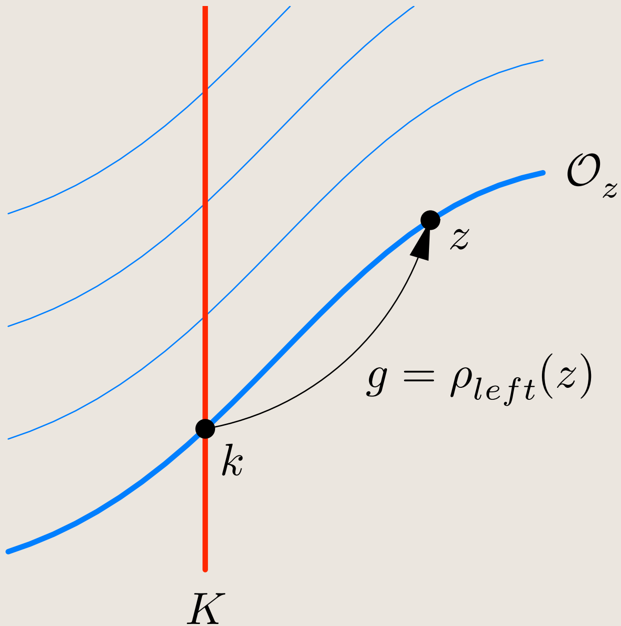

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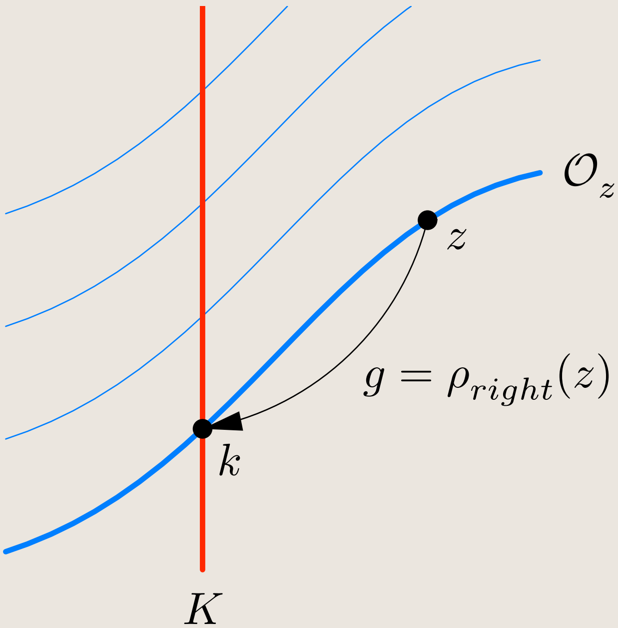

Normalization = choice of cross-section to the group orbits

Algebraic Construction

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\; , z_r = c_r \,\}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Completeness of Invariants

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ \ldots \ , I_{m-r}(z))$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

- An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants

$\Longrightarrow$ invariant numerical approximations

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$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

Euclidean Plane Curves

Special Euclidean group: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

acts on $M = \mathbb{R}^2$ via rigid motions: $w = R\,z + c$

To obtain the classical (left) moving frame we invert the group transformations:

$$\left.\begin{aligned} y &= \cos\theta\,(x - a) + \sin\theta\,(u - b) \\ v &= -\sin\theta\,(x - a) + \cos\theta\,(u - b) \end{aligned}\right\} \qquad w = R^{-1}(z - c)$$

Assume for simplicity the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

$\Longrightarrow$ extensions to parametrized curves are straightforward

Prolong the action to J^n via implicit differentiation:

$$y = \cos\theta\,(x-a) + \sin\theta\,(u-b)$$

$$v = -\sin\theta\,(x-a) + \cos\theta\,(u-b)$$

$$v_y = \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x\sin\theta)^3}$$

$$v_{yyy} = \frac{(\cos\theta + u_x\sin\theta)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta + u_x\sin\theta)^5}$$

$$\vdots$$

Choice of cross-section: $r = \dim G = 3$

$$y = \cos\theta\,(x - a) + \sin\theta\,(u - b) = 0$$

$$v = -\sin\theta\,(x - a) + \cos\theta\,(u - b) = 0$$

$$v_y = \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta} = 0$$

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x\sin\theta)^3}$$

$$v_{yyy} = \frac{(\cos\theta + u_x\sin\theta)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta + u_x\sin\theta)^5}$$

$$\vdots$$

Solve for the group parameters:

$$y = \cos\theta\,(x-a) + \sin\theta\,(u-b) = 0$$

$$v = -\sin\theta\,(x-a) + \cos\theta\,(u-b) = 0$$

$$v_y = \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta} = 0$$

$\Longrightarrow$ Left moving frame $\qquad \rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x \qquad b = u \qquad \theta = \tan^{-1} u_x$$

$$a = x \qquad b = u \qquad \theta = \tan^{-1} u_x$$

Differential invariants

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x \sin\theta)^3} \longmapsto \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$v_{yyy} = \cdots \longmapsto \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}$$

$$v_{yyyy} = \cdots \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

Invariant one-form — arc length

$$dy = (\cos\theta + u_x \sin\theta)\, dx \longmapsto ds = \sqrt{1+u_x^2}\; dx$$

Dual invariant differential operator

— arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\theta + u_x \sin\theta}\,\frac{d}{dx} \qquad \longmapsto \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

However, unless an invariant is constant

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

it carries little information in isolation, since an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

However, a functional dependency or syzygy among multiple invariants *is* intrinsic

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\bar{s}} = \overline{\kappa}^3 - 1$$

Equivalence & Syzygies

Theorem. (Cartan) Two submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, we need only know a single syzygy between κ and κ_s in order to establish equivalence!

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants:

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Theorem. Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry and Signature

Theorem. Let $\mathcal{S}$ denote the signature of the p-dimensional submanifold N. Then the dimension of its symmetry group

$$G_N = \{ g \mid g \cdot N \subset N \}$$

equals

$$\dim G_N = \dim N - \dim \mathcal{S}$$

Corollary. For a regular submanifold $N \subset M$,

$$0 \le \dim G_N \le \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point: $\dim \mathcal{S} = 0$
- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ Euclidean geometry: circles, lines, helices, spheres, cylinders & planes.

$\Longrightarrow$ Equi-affine plane geometry: conic sections.

$\Longrightarrow$ Projective plane geometry: W curves (*Lie & Klein*)

Discrete Symmetries

Definition. The index of a submanifold N equals the number of points in N which map to a generic point of its signature:

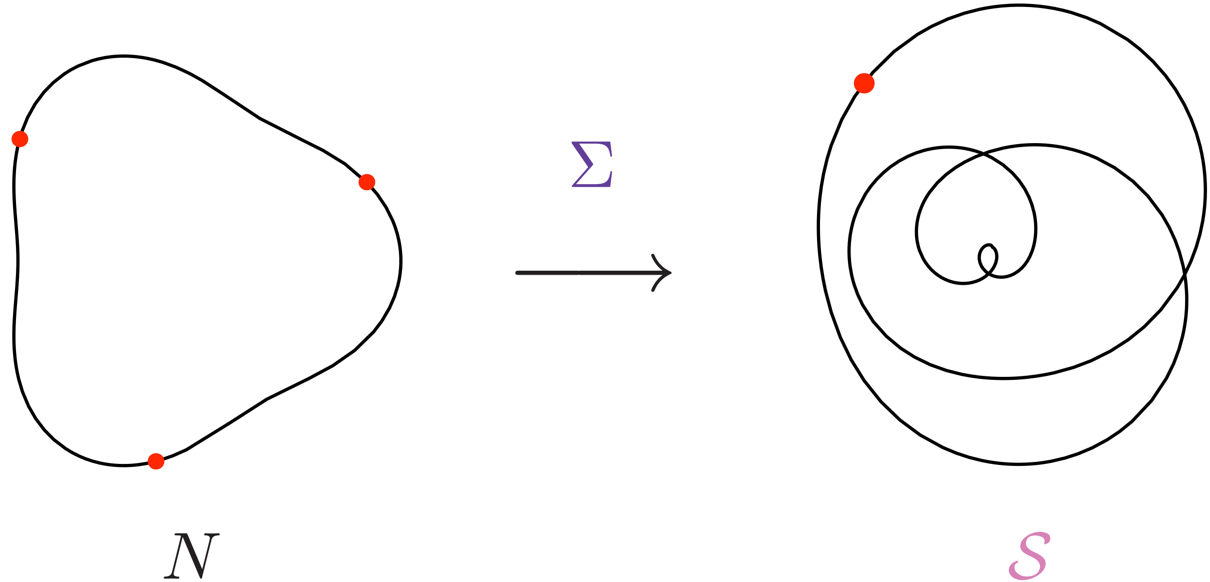
$$\iota_N = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

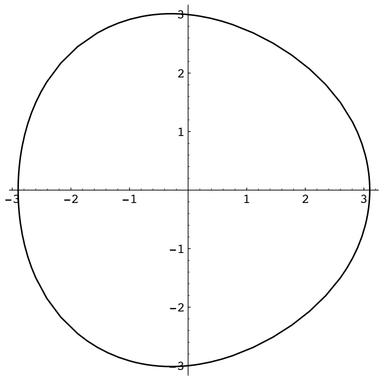
Theorem. The cardinality of the symmetry group of a submanifold N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

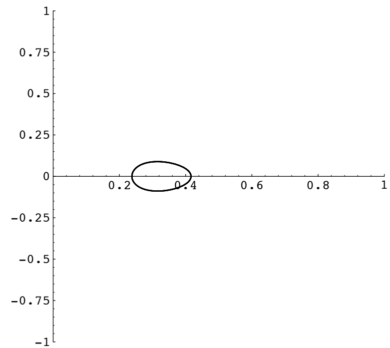
The Index



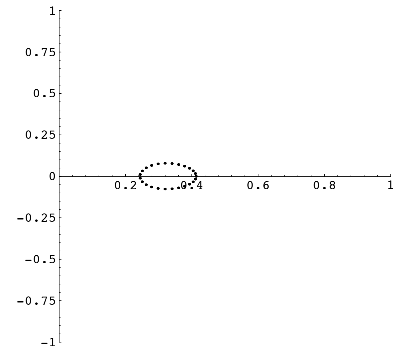
The polar curve $r = 3 + \frac{1}{10}\cos 3\theta$



The Original Curve

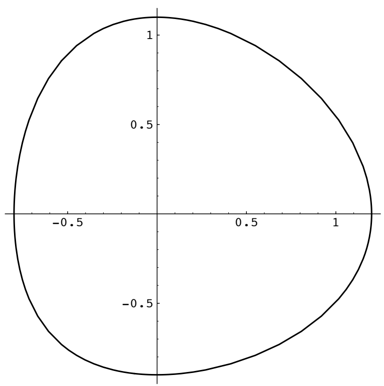


Euclidean Signature

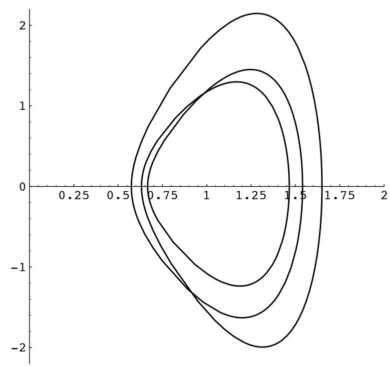


Numerical Signature

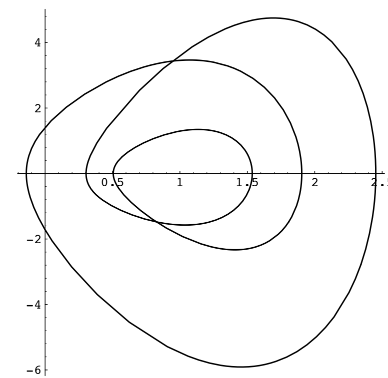
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

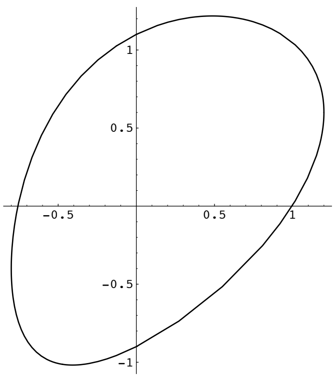


Euclidean Signature

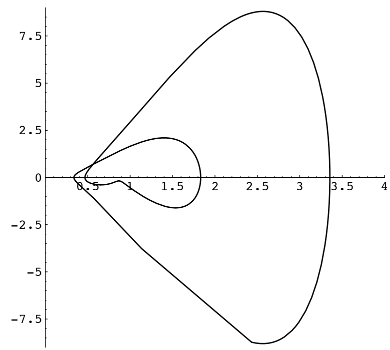


Affine Signature

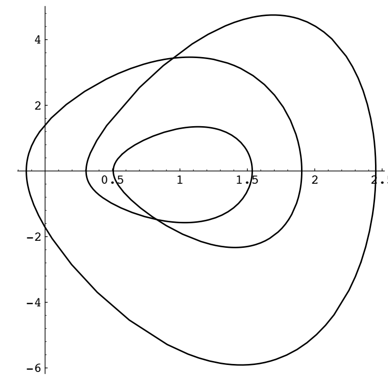
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve



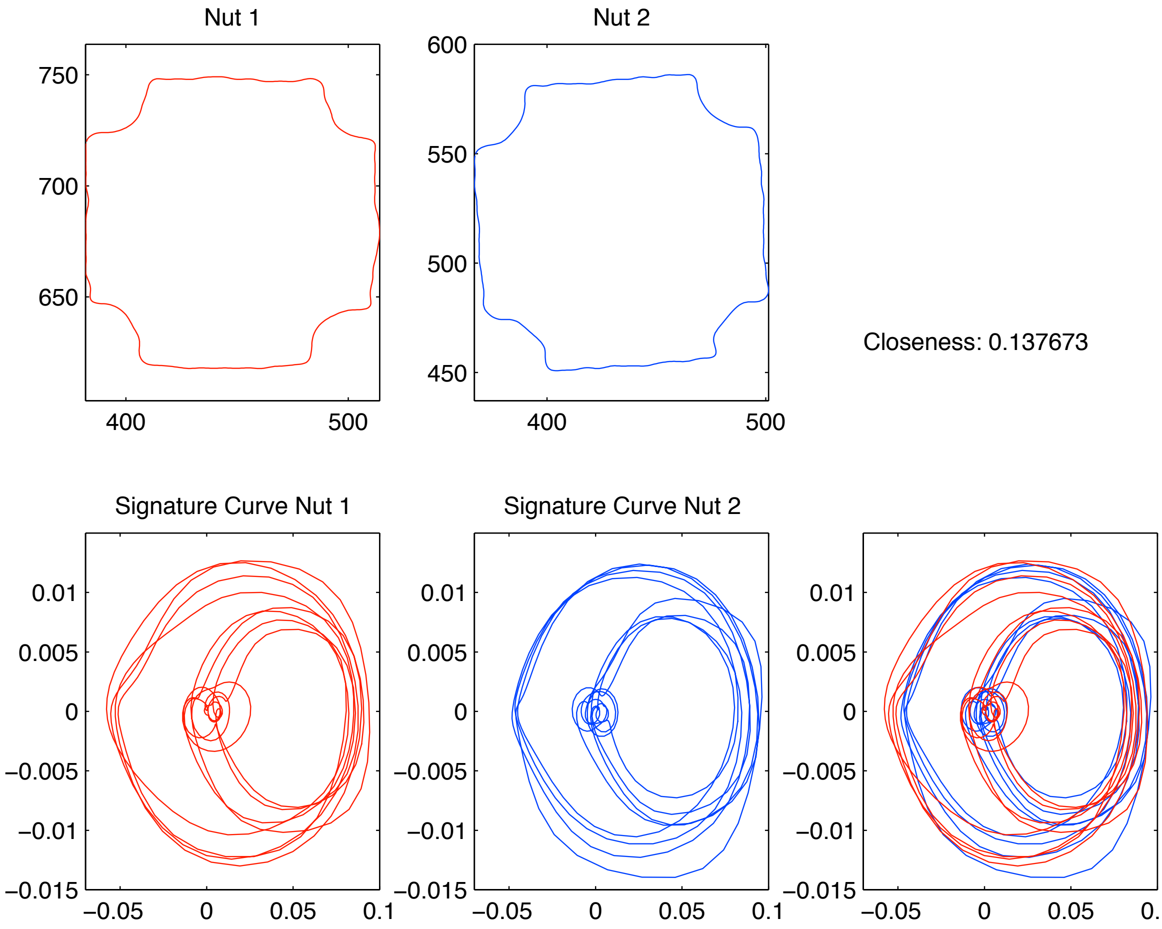
Euclidean Signature



Affine Signature

“Industrial Mathematics”

$\Longrightarrow$ Steve Haker



Nut 1
750
700
650
400
500
Nut 2
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
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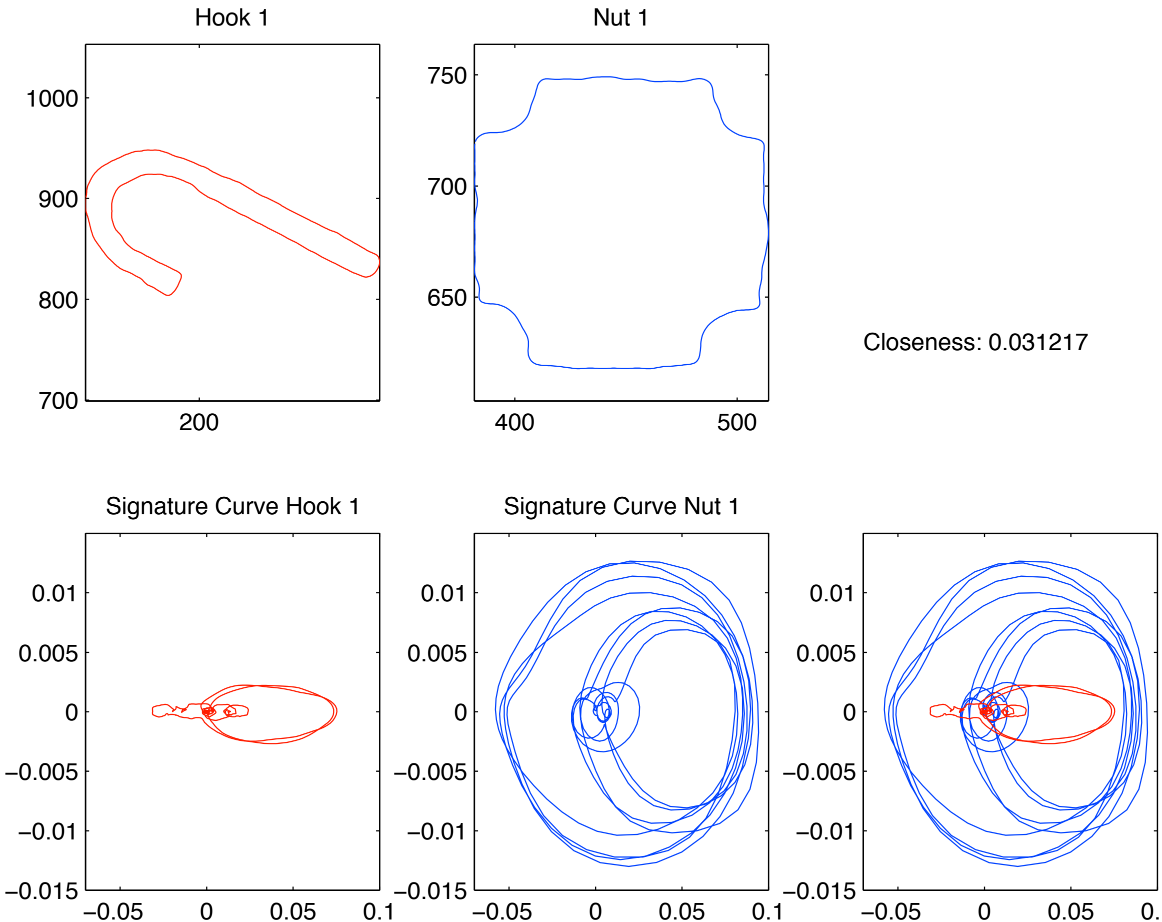
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1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Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
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900
800
700
200
750
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650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

Moving Frames and Binary Forms

Projective equivalence of binary forms of degree n:

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q} \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right)$$

Transformation group:

$$g : (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta} \, , \; \frac{u}{(\gamma x + \delta)^n} \right)$$

Equivalence of functions $\iff$ equivalence of their graphs

$$\Gamma_Q = \{ \, (x, u) = (x, Q(x)) \, \} \subset \mathbb{C}^2$$

Moving Frame Calculation

$$M = \mathbb{R}^2 \setminus \{u = 0\}$$

$$G = \mathrm{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \,\middle|\, \Delta = \alpha\,\delta - \beta\,\gamma \neq 0 \right\}$$

$$(x, u) \longmapsto \left(\frac{\alpha\, x + \beta}{\gamma\, x + \delta}, \frac{u}{(\gamma\, x + \delta)^n} \right) \qquad n \neq 0, 1$$

Prolongation:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \qquad\qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u \qquad\qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2\,(n-1)\,\gamma\,\sigma\, u_x + n\,(n-1)\,\gamma^2\, u}{\Delta^2\, \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

Choice of cross-section: $r = \dim G = 4$

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} = 0 \qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u = 1 \qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}} = 0$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2\,(n-1)\,\gamma\,\sigma\, u_x + n\,(n-1)\,\gamma^2\, u}{\Delta^2\, \sigma^{n-2}} = \frac{1}{n\,(n-1)}$$

$$v_{yyy} = \cdots$$

Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\,u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\,u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\,x u^{(1-n)/n}$$

Hessian:

$$H = n(n-1)u\,u_{xx} - (n-1)^2 u_x^2 \neq 0$$

Note: $H \equiv 0$ if and only if $Q(x) = (a\,x+b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} \approx \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3\,(n-2)}{n^3(n-1)} \approx \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad\qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}\,(Q,\,Q)^{(2)} \;= n(n-1)QQ'' - (n-1)^2 Q'^2 \;\sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = \;(Q,\,H)^{(1)} \;= (2n-4)Q'\,H - nQ\,H' \;\sim Q_x H_y - Q_y H_x$$

$$U = \;(Q,\,T)^{(1)} \;= (3n-6)Q'\,T - nQ\,T' \;\sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \qquad \deg H = 2n-4 \qquad \deg T = 3n-6 \qquad \deg U = 4n-8$$

Signatures of Binary Forms

Signature curve of a nonsingular binary form $Q(x)$:

$$\mathcal{S}_Q = \left\{ (J(x)^2, K(x)) = \left(\frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $\quad H(x) \neq 0 \;$ and $\; (J'(x), K'(x)) \neq 0.$

Signature map:

$$\Sigma : \Gamma_Q \longrightarrow \mathcal{S}_Q \qquad\qquad \Sigma(x) = (J(x)^2, K(x))$$

Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

Theorem. A binary form of degree $n \geq 3$ is complex-equivalent to a sum of two n^{th} powers

$$Q(x,y) \sim x^n + y^n$$

if and only if its signature curve is a straight line:

$$K = -\frac{n-3}{n-2}\,J^2 + \frac{2\,n(n-2)}{(n-1)^2}$$

or, equivalently,

$$H\,U - \frac{n-3}{n-2}\,T^2 + \frac{2\,n(n-2)}{(n-1)^2}\,H^3 = 0.$$

$\Longrightarrow$ In particular, a quartic is the sum of two fourth powers if and only if $j = 0$.

Complex Binary Cubics

- $H \equiv 0$ $\quad Q \sim x^3$ or 1

  $\Longrightarrow$ degenerate

- $T^2 = -H^3$ $\quad Q \sim x^2$ or x $\quad \mathcal{S}_Q = \{ (-1, 0) \}$

  $\Longrightarrow$ point

- $U = -\frac{3}{2} H^2$: $\quad Q \sim x^2 - 1$ $\quad \mathcal{S}_Q = \{ (t, -\frac{3}{2}) \}$

  $\Longrightarrow$ line

Real Binary Cubics

Syzygy: $$T^2 + H^3 = 2^4\,3^6\,\Delta\,Q^2$$

Δ — discriminant of Q

$\Delta < 0$: $H < 0$ $Q \sim x^2 - 1$

$$\mathcal{S}_Q = \left\{ \left(t, -\tfrac{3}{2}\right) \;\middle|\; -1 \le t \le 0 \right\}$$

$\Delta > 0$: H indefinite $Q \sim x^2 + 1$

$$\mathcal{S}_Q = \left\{ \left(t, -\tfrac{3}{2}\right) \;\middle|\; t \ge 0 \right\} \cup \left\{ \left(t, \tfrac{3}{2}\right) \;\middle|\; t < -1 \right\}$$

Complex Binary Quartics

Syzygies:

$$T^2 = -\tfrac{16}{9}H^3 + 2^{10}3^2\, i\, Q^2 H - 2^{14}3^4\, j\, Q^3,$$

$$U = -\tfrac{8}{3}H^2 + 2^9 3^2\, i\, Q^2.$$

where

$$i = a_0 a_4 - 4a_1 a_3 + 3a_2^2 \qquad j = \det \begin{vmatrix} a_4 & a_3 & a_2 \\ a_3 & a_2 & a_1 \\ a_2 & a_1 & a_0 \end{vmatrix}$$

$$s = 48\, Q/H, \qquad J = T^2/H^3, \qquad K = U/H^2, \qquad r = j^2/i^3.$$

Signature curve:

$$J^2 = -\tfrac{16}{9} + 4\, i\, s^2 - 12\, j\, s^3, \qquad K = -\tfrac{8}{3} + 2\, i\, s^2.$$

or

$$\tfrac{9}{2}\, r \left(K + \tfrac{8}{3}\right)^3 = \left(K - \tfrac{1}{2}J^2 + \tfrac{16}{9}\right)^2.$$

Classification of Complex Quartics

Type I: $Q \sim x^4 + \mu\, x^2 + 1, \quad \mu \neq \pm 2$

$$\Longrightarrow \quad J \text{ not constant}$$

$$r = \frac{j^2}{i^3} = \frac{\mu^2(36-\mu^2)^2}{27(12+\mu^2)^3}\,.$$

Note:

$$\pm\,\mu, \quad \pm\frac{12-2\mu}{2+\mu}, \quad \pm\frac{12+2\mu}{2-\mu},$$

all give the same value for r, so the associated quartics are equivalent.

Type II:	$Q \sim p^2 + 1$	J not constant	$r = \frac{1}{27}$	
Type III:	$Q \sim p^2$	$J = 0$	$K = 0$	$(\mu = \pm 2)$
Type IV:	$Q \sim p$	$J^2 = -\frac{16}{9}$	$K = -\frac{8}{3}$	
Type V:	$Q \sim 1$	degenerate		

Rational Basis for Covariants

Q — binary form of degree $n \geq 4$

$$S_j = (Q,\, Q)^{(2j)} \qquad j = 1, \ldots m$$

$$T_k = (S_k,\, Q)^{(1)} \qquad k = 1, \ldots m'$$

where

$$n = m + m' = \begin{cases} 2m & \text{even} \\ 2m+1 & \text{odd} \end{cases}$$

Theorem. (*Stroh, Hilbert*)

Every polynomial covariant C can be written as

$$C = \frac{1}{Q^N} P(Q, S_1, \ldots, S_m, T_1, \ldots, T_{m'})$$

where P is a polynomial and N an integer.

$$s_j = \frac{Q^{2j-2} S_j}{H^j} \qquad\qquad t_k = \frac{Q^{2k-2} T_k}{H^{k+1/2}}$$

Then

$$J^2 = \frac{T^2}{H^3} = t_1^2, \qquad\qquad K = -\frac{1}{2} - J^2 + \frac{n-3}{6(n-2)}\, s_2$$

Independent invariants:

$$i_\nu = \Psi_\nu(s_1, \ldots, s_m, t_1, \ldots, t_{m'})$$

The signature curve is obtained by eliminating the parameters $s_1, \ldots, s_m, t_1, \ldots, t_{m'}$.

A *null form* in one with all zero invariants.

Theorem. Two non-null binary forms are equivalent if and only if they have the same absolute rational invariants.

Symmetries of Binary Forms

Theorem. The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree n is:

- A two-parameter group if and only if $H \equiv 0$ if and only if Q is equivalent to a constant. $\Longrightarrow$ totally singular
- A one-parameter group if and only if $H \not\equiv 0$ and $T^2 = c\,H^3$ if and only if Q is complex-equivalent to a monomial x^k, with $k \neq 0, n$. $\Longrightarrow$ maximally symmetric
- In all other cases, a finite group whose cardinality equals the index of the signature curve, and is bounded by

$$\iota_Q \quad \leq \quad \begin{cases} 6\,n - 12 & U = cH^2 \\ 4\,n - 8 & \text{otherwise} \end{cases}$$

Equations for Symmetries

$\Longrightarrow$ *Irina Kogan*

Theorem. Let $Q(x)$ be a binary form of degree n which is not complex equivalent to a monomial. Then the projective symmetries

$$y = \varphi(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$$

of $Q(x)$ are the common solutions to the two rational equations

$$J(y) = J(x), \qquad K(y) = K(x).$$

Or, equivalently, the common roots

$$y = \varphi(x) = \frac{\alpha\, x + \beta}{\gamma\, x + \delta}$$

to the polynomial equations

	degree
$H(y)^3\, T(x)^2 = T(y)^2\, H(x)^3$	$6\,(n-2)$
$H(y)^2\, U(x) = U(y)\, H(x)^2$	$4\,(n-2)$

where

$$H = \tfrac{1}{2}\,(Q,\, Q)^{(2)} \qquad T = (Q,\, H)^{(1)} \qquad U = (Q,\, T)^{(1)}$$

Cubic Example #1

$$Q = p^3 + 1$$

Projective symmetry group:

$$p \qquad \frac{1}{p} \qquad \omega\, p \qquad \omega^2\, p \qquad \frac{\omega}{p} \qquad \frac{\omega^2}{p}$$

$$\omega = -\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}$$

Matrix generators:

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$$

$\#$ symmetries $= 18$

Cubic Example #2

$$Q(p) = p^3 + p$$

Projective symmetry group:

$$p, \quad -p, \quad \frac{\mathrm{i}\,p+1}{3\,p+\mathrm{i}}, \quad \frac{\mathrm{i}\,p-1}{-3\,p+\mathrm{i}}, \quad \frac{-\mathrm{i}\,p+1}{-3\,p+\mathrm{i}}, \quad \frac{-\mathrm{i}\,p+1}{3\,p+\mathrm{i}}.$$

Matrix generators:

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -\frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ -3\,\mathrm{i} & 1 \end{pmatrix}.$$

$\#$ symmetries $= 18$

Finite Subgroups of PSL(2)

- Abelian $\qquad \# \mathcal{A}_n = n$

$$\alpha : \; p \longmapsto \omega\, p, \qquad \omega^n = 1 \;\; \text{— primitive}$$

- Dihedral $\qquad \# \mathcal{D}_n = 2\,n$

$$\alpha, \qquad p \longmapsto 1/p$$

- Tetrahedral $\qquad \# \mathcal{T} = 12$

$$\sigma : \; p \longmapsto -p, \qquad \tau : \; p \longmapsto \frac{\mathrm{i}\,(p+1)}{p-1},$$

- Octahedral $\qquad \# \mathcal{O} = 24$

$$\tau : \; p \longmapsto \frac{\mathrm{i}\,(p+1)}{p-1}, \qquad \iota : \; p \longmapsto \mathrm{i}\,p$$

- Icosahedral $\qquad \# \mathcal{I} = 60$

$$\sigma, \qquad \tau, \qquad \rho : \; p \longmapsto \frac{2\,p - (1-\sqrt{5})\,\mathrm{i} \; - (1+\sqrt{5})}{\left[(1-\sqrt{5})\,\mathrm{i} \; - (1+\sqrt{5})\right] p - 2}$$

Quartics

$$Q(p) = p^4 + \mu\, p^2 + 1 \quad \text{or} \quad p^2 + 1 \quad \text{where} \quad \mu \neq \pm 2.$$

General μ: the projective symmetry group is a dihedral group $\mathcal{D}_2$, generated by $-p$ and $1/p$.

$\mu = 0$: dihedral group $\mathcal{D}_4$, generated by $\,\mathrm{i}\,p$ and $1/p$.

$\mu \;=\; \pm 2\,\mathrm{i}\,\sqrt{3}$: the projective symmetry group is the 12 element octahedral group $\mathcal{O}$, generated by $-p$ and $\,\mathrm{i}\,(p-1)/(p+1)$.

Projective Symmetry Groups of Quintics	
$p^5 + 1$	$\mathcal{D}_5$
$p^5 + p$	$\mathcal{A}_4$
$p^5 + p^2$	$\mathcal{A}_3$
$p^5 + p^3$	$\mathcal{A}_2$
$p^5 + p^2 + 1$	$\{e\}$
$p^5 - 4p - 2$	$\{e\}$

Quintic Computation

$$Q(p) = p^5 + p$$

Initially MAPLE produces symmetries which involve square roots and so do not look like linear fractional transformations. However, after some simplifications under the radical, we obtain the group of linear fractional transformations generated by

$$\mathrm{i}\, p \qquad \frac{\sqrt{2}\,(1+\mathrm{i}\,)p - 2}{\sqrt{2}\,(1-\mathrm{i}\,) + 2\,p}$$

with corresponding matrices

$$\begin{pmatrix} \mathrm{i}^{\,5/6} & 0 \\ 0 & \mathrm{i}^{\,-1/6} \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -\sqrt{2} \\ \sqrt{2} & 1-\mathrm{i} \end{pmatrix}.$$

Ternary forms

— see work of Irina Kogan and Marc Moreno Maza.