Moving Frames and their Applications

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Moving Frames

Classical contributions:

M. Bartels ($\sim$1800), J. Serret, J. Frénet, G. Darboux, É. Cotton,

Élie Cartan

Modern developments: (1970's)

S.S. Chern, M. Green, P. Griffiths, G. Jensen, ...

The equivariant approach: (1997 –)

M. Fels & PJO, Moving coframes. I. A practical algorithm, Acta Appl. Math. **51** (1998) 161-213; II. Regularization and theoretical foundations, Acta Appl. Math. **55** (1999) 127-208.

E.L. Mansfield, *A Practical Guide to the Invariant Calculus*, Cambridge University Press, Cambridge, 2010

Equivariant Moving Frames

Definition.

A moving frame is a G-equivariant map (section)

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup: $G_z = \{\, g \mid g \cdot z = z \,\}$ for $z \in M$

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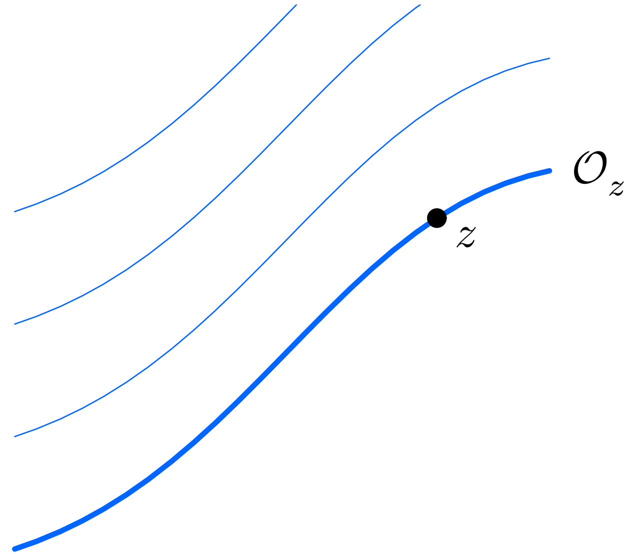
- locally free — the orbits all have the same dimension as G
  $$\Longrightarrow \quad G_z \subset G \text{ is discrete for all } z \in M$$
- regular — the orbits form a regular foliation
  $$\not\approx \quad \text{irrational flow on the torus}$$

Geometric Construction



Normalization = choice of cross-section to the group orbits

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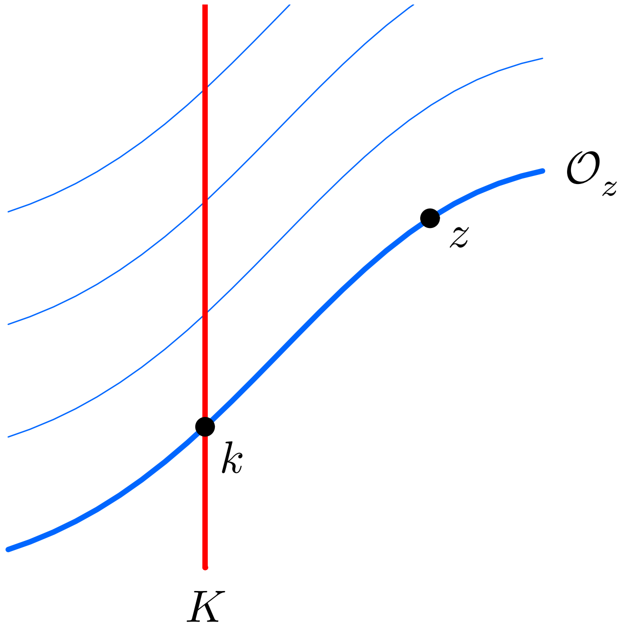

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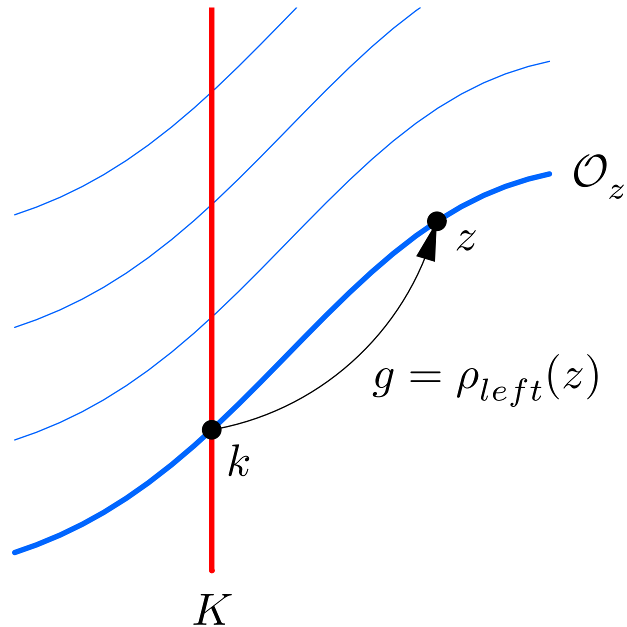

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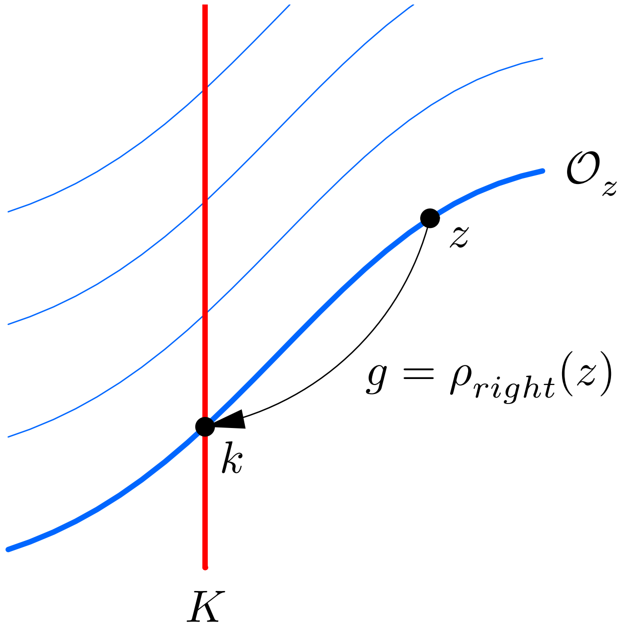

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Algebraic Construction

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\;, z_r = c_r \,\}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g,z) = c_1 \qquad \ldots \qquad w_r(g,z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ Coordinates of the canonical form $k \in K$.

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame $g = \rho(z)$ is the the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

$$\iota(z_1) = c_1 \quad \ldots \quad \iota(z_r) = c_r, \quad \iota(z_{r+1}) = I_1(z) \quad \ldots \quad \iota(z_m) = I_{m-r}(z).$$

cross-section variables

"phantom invariants"

fundamental invariants

$$\boxed{\iota\left[\,F(z_1, \ldots, z_m)\,\right] \;=\; F(c_1, \ldots, c_r,\; I_1(z), \ldots, I_{m-r}(z))}$$

Invariantization amounts to restricting F to the cross-section

$$I \mid K = F \mid K$$

and then requiring $I = \iota(F)$ be constant on orbits.

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In particular, if $I(z)$ is an invariant, then $\iota(I) = I$:

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$\Longrightarrow$ Rewrite Rule

The Replacement Theorem

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants.

Invariantization

Invariantization defines a canonical projection

$$\iota : \text{functions} \longmapsto \text{invariants}$$

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$$\iota : \begin{array}{lcl} \text{functions} & & \text{invariants} \\ \text{differential forms} & & \text{invariant differential forms} \\ \text{differential operators} & \longmapsto & \text{invariant differential operators} \end{array}$$

Invariantization

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	functions		invariants
	differential forms		invariant differential forms
ι :	differential operators		invariant differential operators
	variational principles	$\longmapsto$	invariant variational principles

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♡ Existence of a moving frame requires freeness of the group action.

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- An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

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Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\, z + c \qquad\qquad A \in \mathrm{SL}(2), \qquad c \in \mathbb{R}^2$$

Invert for left moving frame:

$$\left.\begin{aligned} y &= \delta\,(x-a) - \beta\,(u-b) \\ v &= -\gamma\,(x-a) + \alpha\,(u-b) \end{aligned}\right\} \qquad w = A^{-1}(z-c)$$

$$\alpha\,\delta - \beta\,\gamma = 1$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - \beta\, u_x)\; dx \qquad\qquad D_y = \frac{1}{\delta - \beta\, u_x}\, D_x$$

Prolongation:

$$y = \delta\,(x - a) - \beta\,(u - b)$$

$$v = -\,\gamma\,(x - a) + \alpha\,(u - b)$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x}$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\,u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Normalization: $\qquad r = \dim G = 5$

$$y = \delta\,(x - a) - \beta\,(u - b) = 0$$

$$v = -\gamma\,(x - a) + \alpha\,(u - b) = 0$$

$$v_y = -\,\frac{\gamma - \alpha\, u_x}{\delta - \beta\, u_x} = 0$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\, u_x)^3} = 1$$

$$v_{yyy} = -\,\frac{(\delta - \beta\, u_x)\, u_{xxx} + 3\,\beta\, u_{xx}^2}{(\delta - \beta\, u_x)^5} = 0$$

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Equi-affine Moving Frame

$$\rho : \ (x, u, u_x, u_{xx}, u_{xxx}) \ \longmapsto \ (A, \mathbf{b}) \in \mathrm{SA}(2)$$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Nondegeneracy condition: $\qquad u_{xx} \neq 0.$

Equi-affine arc length

$$dy = (\delta - \beta\, u_x)\; dx \quad\longmapsto\quad ds = \sqrt[3]{u_{xx}}\; dx$$

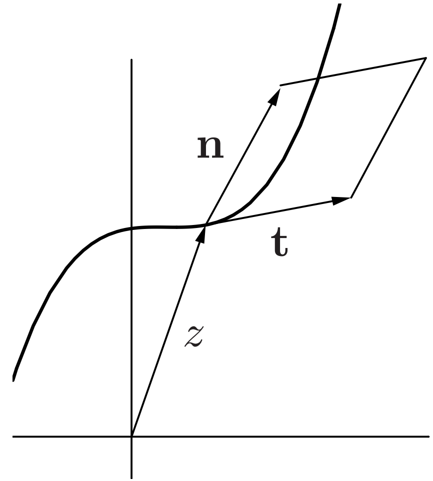
Equi-affine curvature

$$v_{yyyy} \quad\longmapsto\quad \kappa = \frac{5\,u_{xx}u_{xxxx} - 3\,u_{xxx}^2}{9\,u_{xx}^{8/3}}$$

$$v_{yyyyy} \quad\longmapsto\quad \frac{d\kappa}{ds}$$

$$v_{yyyyyy} \quad\longmapsto\quad \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

The Classical Picture:



$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix} = (\,\mathbf{t},\ \mathbf{n}\,)$$

$$\mathbf{b} = \begin{pmatrix} x \\ u \end{pmatrix} = z$$

Frenet frame

$$\mathbf{t} = \frac{dz}{ds}, \qquad \mathbf{n} = \frac{d^2 z}{ds^2}.$$

Frenet equations = Pulled-back Maurer–Cartan forms:

$$\frac{dz}{ds} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = \kappa\,\mathbf{t}.$$

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q} \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

Action of $G = \mathrm{GL}(2)$ on $\mathbb{R}^2$ (or $\mathbb{C}^2$):

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \; \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

Prolongation:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u \qquad \Delta = \alpha\, \delta - \beta\, \gamma$$

$$v_y = \frac{\sigma\, u_x - n\, \gamma\, u}{\Delta\, \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\gamma\, \sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

Normalization:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} = 0 \qquad\qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u = 1 \qquad\qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}} = 0$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\,\gamma\,\sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}} = \frac{1}{n(n-1)}$$

$$v_{yyy} = \cdots$$

Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\,u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\,u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\,x u^{(1-n)/n}$$

Hessian:

$$H = n(n-1)u\,u_{xx} - (n-1)^2 u_x^2 \neq 0$$

Note: $H \equiv 0$ if and only if $Q(x) = (a\,x + b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} = \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3(n-2)}{n^3(n-1)} = \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} \; = n(n-1)QQ'' - (n-1)^2 Q'^2 \;\sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = \;(Q,\, H)^{(1)} \; = (2n-4)Q'H - nQH' \;\sim Q_x H_y - Q_y H_x$$

$$U = \;(Q,\, T)^{(1)} \; = (3n-6)Q'T - nQT' \;\sim Q_x T_y - Q_y T_x$$

★★ The equivalence and symmetry properties of nonsingular binary forms (of arbitrary degree) are entirely determined by their signature curves parametrized by (κ, κ_s) or, equivalently (J, K).

Differential Invariants

A differential invariant is an invariant function $I : \mathrm{J}^n \to \mathbb{R}$ for the prolonged (pseudo-)group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$\Longrightarrow$ curvature, torsion, ...

Invariant differential operators:

$\mathcal{D}_1, \ldots, \mathcal{D}_p \qquad \Longrightarrow$ arc length derivative

- If I is a differential invariant, so is $\mathcal{D}_j I$.

$\mathcal{I}_G$ — the algebra of differential invariants

The Basis Theorem

Theorem. Given a Lie group (or Lie pseudo-group*) acting on p-dimensional submanifolds, the corresponding differential invariant algebra $\mathcal{I}_G$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and p invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_k = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_k.$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Kruglikov–Lychagin, Muñoz–Muriel–Rodríguez, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae

  for the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{jk}^{i}\, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\;\ldots\; \mathcal{D}_J I_\kappa \;\ldots\;) \equiv 0$$

Recurrence Formulae

$$\mathcal{D}_i\, \iota(F) = \iota(D_i F) \; + \; \sum_{\kappa=1}^{r} R_i^\kappa \, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

ι	—	invariantization map
$F(x, u^{(n)})$	—	differential function
$I = \iota(F)$	—	differential invariant
D_i	—	total derivative with respect to x^i
$\mathcal{D}_i = \iota(D_i)$	—	invariant differential operator
$\mathbf{v}_\kappa^{(n)}$	—	infinitesimal generators of prolonged action of G on jets
R_i^κ	—	Maurer–Cartan invariants (coefficients of pulled-back Maurer–Cartan forms)

Recurrence Formulae

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♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_i^κ.

♡ Once the Maurer–Cartan invariants R_i^κ are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}_G$!

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}_G$ can be determined, using linear differential algebra, without knowing the explicit formulae for the differential invariants, or the invariant differential operators, or the moving frame, or, even, the group transformations!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

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The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}_G$ is a rational, non-commutative differential algebra.

Generating Differential Invariants

Theorem. (*Fels–O*) If the moving frame has order n, then the set of normalized differential invariants of order $\leq n + 1$ forms a generating set.

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Theorem. (*O–Hubert*) Given a *minimal order cross-section*, meaning that, for each $k = 0, 1, \ldots, n$,

$$Z_1(x, u^{(k)}) = c_1, \quad \ldots \quad Z_{r_k}(x, u^{(k)}) = c_{r_k},$$

defines a cross-section for the action of $G^{(k)}$ on J^k, then the differential invariants $\iota(D_i Z_j)$ for $i = 1, \ldots, p,\ j = 1, \ldots, r$ and, in the intransitive case, the order zero invariants, form a generating set.

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Theorem. (*Hubert*) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants serve to generate the differential invariant algebra $\mathcal{I}_G$.

Curves

Theorem. Let G be an ordinary* Lie group acting on the m-dimensional manifold M. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the G-invariant arc length element ds.

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$\Longrightarrow \quad m = 3$ — curvature κ & torsion τ

Euclidean Surfaces

Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ acts on surfaces $S \subset \mathbb{R}^3$.

For simplicity, we assume the surface is (locally) the graph of a function

$$z = u(x, y)$$

Infinitesimal generators:

$$\mathbf{v}_1 = -y\,\partial_x + x\,\partial_y, \qquad \mathbf{v}_2 = -u\,\partial_x + x\,\partial_u, \qquad \mathbf{v}_3 = -u\,\partial_y + y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u.$$

- The translations $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ will be ignored, as they play no role in the higher order recurrence formulae.

Cross-section (Darboux frame):

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0$$

Principal curvatures

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Higher order differential invariants — invariantized jet coordinates:

$$I_{jk} = \iota(u_{jk}) \qquad \text{where} \qquad u_{jk} = \frac{\partial^{j+k} u}{\partial x^j \partial y^k}$$

Power series (Monge) normal form:

$$u(x,y) = \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{2}\kappa_2 y^2 + \tfrac{1}{6} I_{30} x^3 + \tfrac{1}{2} I_{21} x^2 y + \tfrac{1}{2} I_{12} x y^2 + \tfrac{1}{6} I_{03} x^3 + \cdots$$

★ ★ Nondegeneracy condition: non-umbilic point $\kappa_1 \neq \kappa_2$.

Algebra of Euclidean Differential Invariants

Principal curvatures:

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Invariant differentiation operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y)$$

$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

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Invariant differentiation operators:

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$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

The recurrence formulae enable one to express the higher order differential invariants in terms of the principal curvatures, or, equivalently, the mean and Gauss curvatures, and their invariant derivatives:

$$\begin{aligned} I_{jk} = \iota(u_{jk}) &= \widetilde{\Phi}_{jk}(\kappa_1, \kappa_2, \mathcal{D}_1\kappa_1, \mathcal{D}_2\kappa_1, \mathcal{D}_1\kappa_2, \mathcal{D}_2\kappa_2, \mathcal{D}_1^2\kappa_1, \ \ldots\ ) \\ &= \Phi_{jk}(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ ) \end{aligned}$$

Recurrence Formulae

$$\iota(D_i u_{jk}) = \mathcal{D}_i\,\iota(u_{jk}) - \sum_{\kappa=1}^{3} R_i^\kappa\,\iota[\,\varphi_\kappa^{jk}(x,y,u^{(j+k)})\,], \qquad j+k \geq 1$$

$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

Recurrence Formulae

$$\iota(D_i u_{jk}) = \mathcal{D}_i\, \iota(u_{jk}) - \sum_{\kappa=1}^{3} R_i^\kappa\, \iota[\, \varphi_\kappa^{jk}(x,y,u^{(j+k)})\,], \qquad j+k \geq 1$$

$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

$\varphi_\kappa^{jk}(0,0,I^{(j+k)}) = \iota[\, \varphi_\kappa^{jk}(x,y,u^{(j+k)})\,]$

— invariantized prolonged infinitesimal generator coefficients.

$$I_{j+1,k} = \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_1^\kappa$$

$$I_{j,k+1} = \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_2^\kappa$$

Prolonged infinitesimal generators:

$$\begin{aligned}
\mathrm{pr}\,\mathbf{v}_1 &= -y\,\partial_x + x\,\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y} \\
&\quad - 2\,u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\,\partial_{u_{xy}} - 2\,u_{xy}\partial_{u_{yy}} + \;\cdots\;, \\
\mathrm{pr}\,\mathbf{v}_2 &= -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + u_x\,u_y\partial_{u_y} \\
&\quad + 3\,u_x\,u_{xx}\partial_{u_{xx}} + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\partial_{u_{xy}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\,\partial_{u_{yy}} + \;\cdots\;, \\
\mathrm{pr}\,\mathbf{v}_3 &= -u\,\partial_y + y\,\partial_u + u_x\,u_y\partial_{u_x} + (1 + u_y^2)\,\partial_{u_y} \\
&\quad + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\partial_{u_{xx}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\,\partial_{u_{xy}} + 3\,u_y\,u_{yy}\partial_{u_{yy}} + \;\cdots\;.
\end{aligned}$$

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&\qquad - 2\,u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\,\partial_{u_{xy}} - 2\,u_{xy}\partial_{u_{yy}} + \cdots, \\
\operatorname{pr} \mathbf{v}_2 &= -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + u_x u_y \partial_{u_y} \\
&\qquad + 3\,u_x u_{xx}\partial_{u_{xx}} + (u_y u_{xx} + 2\,u_x u_{xy})\,\partial_{u_{xy}} + (2\,u_y u_{xy} + u_x u_{yy})\,\partial_{u_{yy}} + \cdots, \\
\operatorname{pr} \mathbf{v}_3 &= -u\,\partial_y + y\,\partial_u + u_x u_y \partial_{u_x} + (1 + u_y^2)\,\partial_{u_y} \\
&\qquad + (u_y u_{xx} + 2\,u_x u_{xy})\,\partial_{u_{xx}} + (2\,u_y u_{xy} + u_x u_{yy})\,\partial_{u_{xy}} + 3\,u_y u_{yy}\partial_{u_{yy}} + \cdots.
\end{aligned}$$

$$I_{jk} = \iota(u_{jk})$$

Phantom differential invariants:

$$I_{00} = I_{10} = I_{01} = I_{11} = 0$$

Principal curvatures:

$$I_{20} = \kappa_1 \qquad I_{02} = \kappa_2$$

Phantom recurrence formulae:

$$\begin{aligned}
\kappa_1 = I_{20} &= \mathcal{D}_1 I_{10} - R_1^2 = -R_1^2, \\
0 = I_{11} &= \mathcal{D}_1 I_{01} - R_1^3 = -R_1^3, \\
I_{21} &= \mathcal{D}_1 I_{11} - (\kappa_1 - \kappa_2)\, R_1^1 = -(\kappa_1 - \kappa_2)\, R_1^1, \\
0 = I_{11} &= \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2, \\
\kappa_2 = I_{02} &= \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3, \\
I_{12} &= \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.
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$$0 = I_{11} = \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2,$$

$$\kappa_2 = I_{02} = \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3,$$

$$I_{12} = \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.$$

Maurer–Cartan invariants:

$$R_1^1 = -Y_1, \qquad R_1^2 = -\kappa_1, \qquad R_1^3 = 0,$$

$$R_1^2 = -Y_2, \qquad R_2^2 = 0, \qquad R_3^2 = -\kappa_2.$$

Commutator invariants:

$$Y_1 = \frac{I_{21}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad\qquad Y_2 = \frac{I_{12}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

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$$\kappa_1 = I_{20} = \mathcal{D}_1 I_{10} - R_1^2 = -R_1^2,$$

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$$\kappa_2 = I_{02} = \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3,$$

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Maurer–Cartan invariants:

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$$R_1^2 = -Y_2, \qquad R_2^2 = 0, \qquad R_3^2 = -\kappa_2.$$

Commutator invariants:

$$Y_1 = \frac{I_{21}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad Y_2 = \frac{I_{12}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

$$\boxed{[\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1\, \mathcal{D}_2 - \mathcal{D}_2\, \mathcal{D}_1 = Y_2\, \mathcal{D}_1 - Y_1\, \mathcal{D}_2,}$$

Third order recurrence relations:

$$I_{30} = \mathcal{D}_1\kappa_1 = \kappa_{1,1}, \quad I_{21} = \mathcal{D}_2\kappa_1 = \kappa_{1,2}, \quad I_{12} = \mathcal{D}_1\kappa_2 = \kappa_{2,1}, \quad I_{03} = \mathcal{D}_2\kappa_2 = \kappa_{2,2},$$

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Fourth order recurrence relations:

$$\begin{aligned}
I_{40} &= \kappa_{1,11}-\frac{3\,\kappa_{1,2}^2}{\kappa_1-\kappa_2}+3\,\kappa_1^3,\\
I_{31} &= \kappa_{1,12}-\frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1-\kappa_2} &&= \kappa_{1,21}+\frac{\kappa_{1,1}\kappa_{1,2}-2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1-\kappa_2},\\
I_{22} &= \kappa_{1,22}+\frac{\kappa_{1,1}\kappa_{2,1}-2\,\kappa_{2,1}^2}{\kappa_1-\kappa_2}+\kappa_1\kappa_2^2 &&= \kappa_{2,11}-\frac{\kappa_{1,2}\kappa_{2,2}-2\,\kappa_{1,2}^2}{\kappa_1-\kappa_2}+\kappa_1^2\kappa_2,\\
I_{13} &= \kappa_{2,21}+\frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1-\kappa_2} &&= \kappa_{2,12}-\frac{\kappa_{2,1}\kappa_{2,2}-2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1-\kappa_2},\\
I_{04} &= \kappa_{2,22}+\frac{3\,\kappa_{2,1}^2}{\kappa_1-\kappa_2}+3\,\kappa_2^3.
\end{aligned}$$

★ The two expressions for I_{31} and I_{13} follow from the commutator formula.

Fourth order recurrence relations

$$I_{40} = \kappa_{1,11} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3,$$

$$I_{31} = \kappa_{1,12} - \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} = \kappa_{1,21} + \frac{\kappa_{1,1}\kappa_{1,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2},$$

$$I_{22} = \kappa_{1,22} + \frac{\kappa_{1,1}\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 = \kappa_{2,11} - \frac{\kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2,$$

$$I_{13} = \kappa_{2,21} + \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} = \kappa_{2,12} - \frac{\kappa_{2,1}\kappa_{2,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2},$$

$$I_{04} = \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3.$$

★ ★ The two expressions for I_{22} imply the Codazzi syzygy

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\,\kappa_{2,1} + \kappa_{1,2}\,\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1\,\kappa_2\,(\kappa_1 - \kappa_2) = 0,$$

which can be written compactly as

$$K = \kappa_1\kappa_2 = -\,(\mathcal{D}_1 + Y_1)\,Y_1 - (\mathcal{D}_2 + Y_2)\,Y_2.$$

$\Longrightarrow$ Gauss' Theorema Egregium

Generating Differential Invariants

♡ From the general structure of the recurrence relations, one proves that the Euclidean differential invariant algebra $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by the principal curvatures κ_1, κ_2 or, equivalently, the mean and Gauss curvatures, H, K, through the process of invariant differentiation:

$$I = \Phi(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots \ )$$

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♢ Remarkably, for suitably generic surfaces, the Gauss curvature can be written as a universal rational function of the mean curvature and its invariant derivatives of order ≤ 4:

$$K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1^2 H, \ \ldots \ , \mathcal{D}_2^4 H)$$

and hence $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by mean curvature alone!

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♠ To prove this, given

$$K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)\, Y_1 - (\mathcal{D}_2 + Y_2)\, Y_2$$

it suffices to write the commutator invariants Y_1, Y_2 in terms of H.

The Commutator Trick

$$\boxed{K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2}$$

To determine the commutator invariants:

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H &= Y_2\,\mathcal{D}_1 H - Y_1\,\mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_2\mathcal{D}_J H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_J H &= Y_2\,\mathcal{D}_1\mathcal{D}_J H - Y_1\,\mathcal{D}_2\mathcal{D}_J H \end{aligned} \qquad (*)$$

Non-degeneracy condition:

$$\det \begin{pmatrix} \mathcal{D}_1 H & \mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_J H & \mathcal{D}_2\mathcal{D}_J H \end{pmatrix} \neq 0,$$

Solve $(*)$ for Y_1, Y_2 in terms of derivatives of H, producing a universal formula

$$\boxed{K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots\ )}$$

for the Gauss curvature as a rational function of the mean curvature and its invariant derivatives!

Definition. A surface $S \subset \mathbb{R}^3$ is mean curvature degenerate if, near any non-umbilic point $p_0 \in S$, there exist scalar functions $F_1(t), F_2(t)$ such that

$$\mathcal{D}_1 H = F_1(H), \qquad \mathcal{D}_2 H = F_2(H).$$

- surfaces with symmetry: rotation, helical;
- minimal surfaces;
- constant mean curvature surfaces;
- ???

Theorem. If a surface is mean curvature non-degenerate then the algebra of Euclidean differential invariants is generated entirely by the mean curvature and its successive invariant derivatives.

Minimal Generating Invariants

A set of differential invariants is a generating system if all other differential invariants can be written in terms of them and their invariant derivatives.

Euclidean curves $C \subset \mathbb{R}^3$:	curvature κ and torsion τ
Equi–affine curves $C \subset \mathbb{R}^3$:	affine curvature κ and torsion τ
Euclidean surfaces $S \subset \mathbb{R}^3$:	mean curvature H
Equi–affine surfaces $S \subset \mathbb{R}^3$:	Pick invariant P.
Conformal surfaces $S \subset \mathbb{R}^3$:	third order invariant J_3.
Projective surfaces $S \subset \mathbb{R}^3$:	fourth order invariant K_4.

$\Longrightarrow$ For any $n \geq 1$, there exists a Lie group G_N acting on surfaces $S \subset \mathbb{R}^3$ such that its differential invariant algebra requires n generating invariants!

♠ Finding a minimal generating set appears to be a very difficult problem.

(No known bound on order of syzygies.)

Invariant Variational Problems

According to Lie, any G–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ldots\ )\ \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$\mathbf{E}(L) \simeq F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0$$

Main Problem:

Construct F directly from P.

(*P. Griffiths, I. Anderson*)

Planar Euclidean group $\quad G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \qquad \text{— curvature (differential invariant)}$$

$$ds = \sqrt{1+u_x^2}\,dx \qquad \text{— arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx} \qquad \text{— arc length derivative}$$

Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\,dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\,ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) \simeq F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[\,u\,] = \int ds = \int \sqrt{1+u_x^2}\ dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

$\Longrightarrow$ straight lines

The Elastica (Euler):

$$\mathcal{I}[\,u\,] = \int \tfrac{1}{2}\,\kappa^2\,ds = \int \frac{u_{xx}^2\,dx}{(1+u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General Euclidean–invariant variational problem

$$\mathcal{I}[\,u\,] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \;\ldots\;)\, ds$$

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\,dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\,ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \, \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\, (-\mathcal{D})^j \, \frac{\partial P}{\partial \kappa_i} - P$$

From the Invariant Variational Bicomplex

ϑ — invariant contact form (invariant variation)

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta)$$

Invariant variation of curvature

$$\mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{A}^* = \mathcal{D}^2 + \kappa^2$$

$$d_{\mathcal{V}}\,(ds) = \mathcal{B}(\vartheta) \wedge ds$$

Invariant variation of arc length:

$$\mathcal{B} = -\,\kappa \qquad\qquad \mathcal{B}^* = -\,\kappa$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = (\mathcal{D}^2 + \kappa^2)\,\mathcal{E}(P) + \kappa\,\mathcal{H}(P).$$

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \; \ldots \;)\, ds$$

Euclidean–invariant Euler-Lagrange formula

$$\boxed{\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(P) + \kappa\, \mathcal{H}(P) = 0}$$

The Elastica: $\qquad \mathcal{I}[u] = \int \frac{1}{2}\kappa^2\, ds \qquad P = \frac{1}{2}\kappa^2$

$$\mathcal{E}(P) = \kappa \qquad\qquad \mathcal{H}(P) = -\,P = -\,\frac{1}{2}\kappa^2$$

$$\boxed{\begin{aligned} \mathbf{E}(L) &= (\mathcal{D}^2 + \kappa^2)\, \kappa + \kappa\, (\,-\,\tfrac{1}{2}\kappa^2\,) \\ &= \kappa_{ss} + \tfrac{1}{2}\kappa^3 = 0 \end{aligned}}$$

The shape of a Möbius strip

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

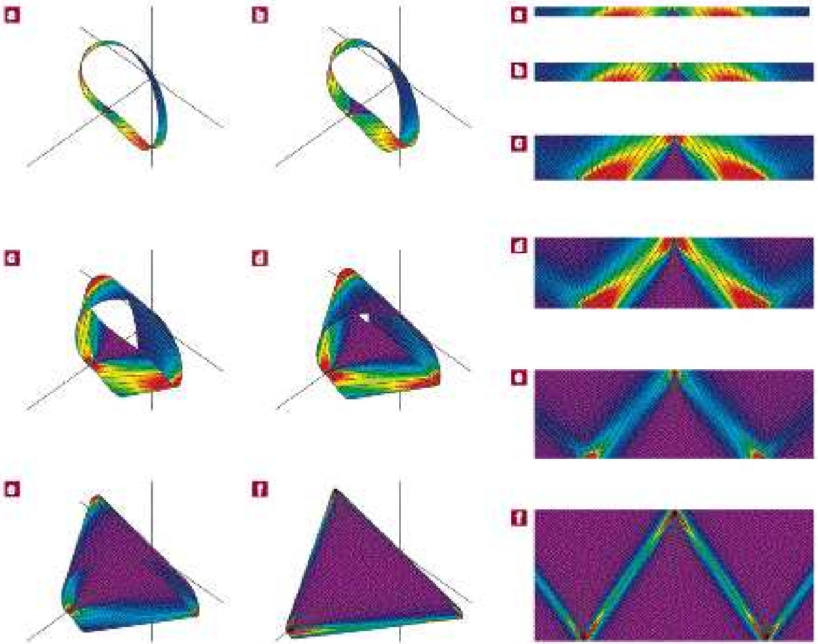


Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

The Infinite Jet Bundle

Jet bundles

$$M = \mathrm{J}^0 \longleftarrow \mathrm{J}^1 \longleftarrow \mathrm{J}^2 \longleftarrow \cdots$$

Inverse limit

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

$\Longrightarrow$ Taylor series

Differential Forms

Coframe — basis for the cotangent space $T^*\mathrm{J}^\infty$:

- Horizontal one-forms

$$dx^1, \ldots, dx^p$$

- Contact (vertical) one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha \, dx^i$$

Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum A_J^\alpha \, \theta_J^\alpha$$

The Variational Bicomplex

$\Longrightarrow$ *Dedecker, Vinogradov, Tsujishita, I. Anderson, ...*

Bigrading of the differential forms on J^∞:

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

$r = \#$ horizontal forms

$s = \#$ contact forms

Vertical and Horizontal Differentials

$$d = d_H + d_V$$

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

Vertical and Horizontal Differentials

$$F(x, u^{(n)}) \qquad \text{— differential function}$$

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total differential}$$

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha \qquad \text{— first variation}$$

$$d_H(dx^i) = d_V(dx^i) = 0,$$

$$d_H(\theta_J^\alpha) = \sum_{i=1}^{p} dx^i \wedge \theta_{J,i}^\alpha \qquad d_V(\theta_J^\alpha) = 0$$

The Simplest Example

$$(x, u) \in M = \mathbb{R}^2$$

x — independent variable

u — dependent variable

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x \, dx$$

$$\theta_x = du_x - u_{xx} \, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx} \, dx$$

$$\vdots$$

$$\theta = du - u_x\, dx, \qquad \theta_x = du_x - u_{xx}\, dx, \qquad \theta_{xx} = du_{xx} - u_{xxx}\, dx$$

Differential:

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial u}\, du + \frac{\partial F}{\partial u_x}\, du_x + \frac{\partial F}{\partial u_{xx}}\, du_{xx} + \cdots \\ &= (D_x F)\, dx + \frac{\partial F}{\partial u}\theta + \frac{\partial F}{\partial u_x}\theta_x + \frac{\partial F}{\partial u_{xx}}\theta_{xx} + \cdots \\ &= d_H F + d_V F \end{aligned}$$

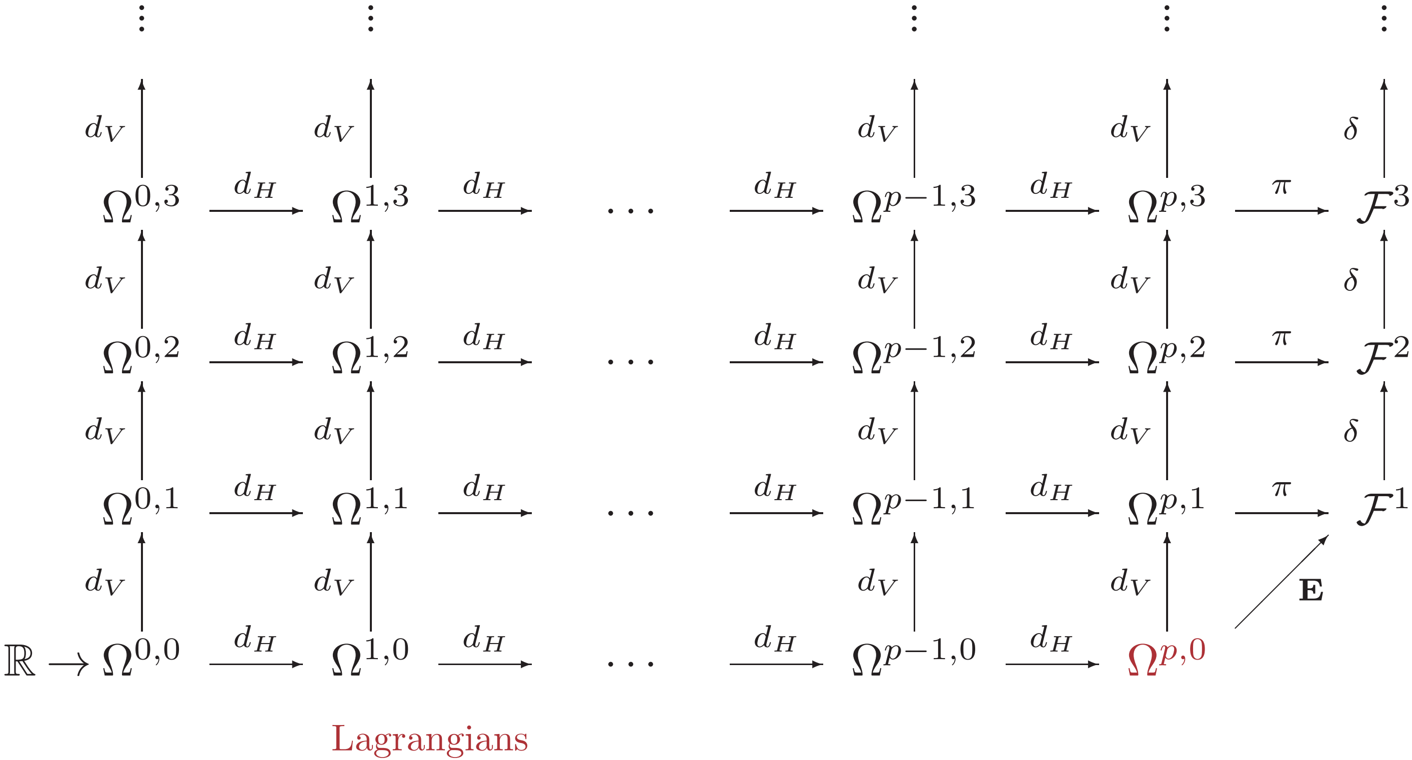
Total derivative:

$$D_x F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\, u_x + \frac{\partial F}{\partial u_x}\, u_{xx} + \frac{\partial F}{\partial u_{xx}}\, u_{xxx} + \cdots$$

The Variational Bicomplex

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & {\scriptstyle \delta}\uparrow \\
 & & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & & & & {\scriptstyle d_V}\uparrow & & {\scriptstyle d_V}\uparrow & \nearrow{\scriptstyle \mathbf{E}} & \\
\mathbb{R} & \to & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

The Variational Bicomplex

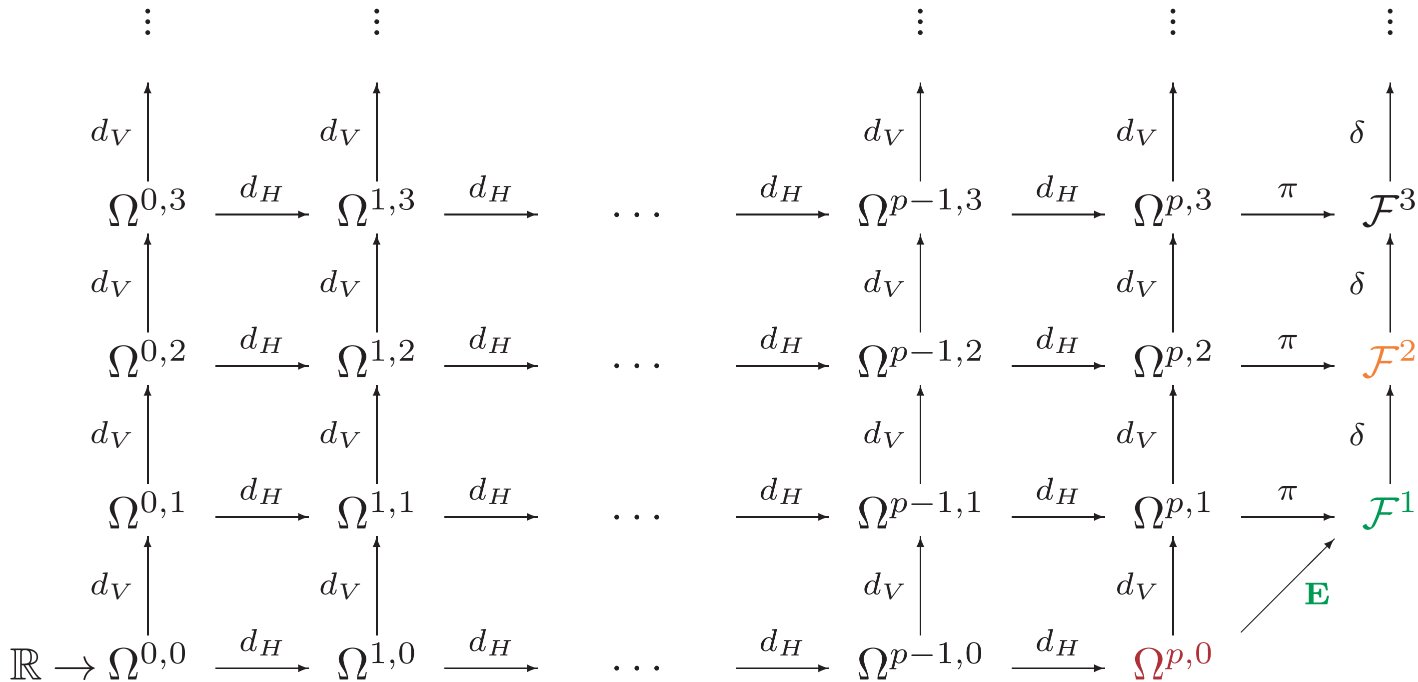


The Variational Bicomplex

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,3} & \xrightarrow{d_H} & \Omega^{1,3} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,3} & \xrightarrow{d_H} & \Omega^{p,3} & \xrightarrow{\pi} & \mathcal{F}^3 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,2} & \xrightarrow{d_H} & \Omega^{p,2} & \xrightarrow{\pi} & \mathcal{F}^2 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle \delta} \\
 & & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,1} & \xrightarrow{d_H} & \Omega^{p,1} & \xrightarrow{\pi} & \mathcal{F}^1 \\
 & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & \nearrow{\scriptstyle \mathbf{E}} & \\
\mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \Omega^{p-1,0} & \xrightarrow{d_H} & \Omega^{p,0} & &
\end{array}
$$

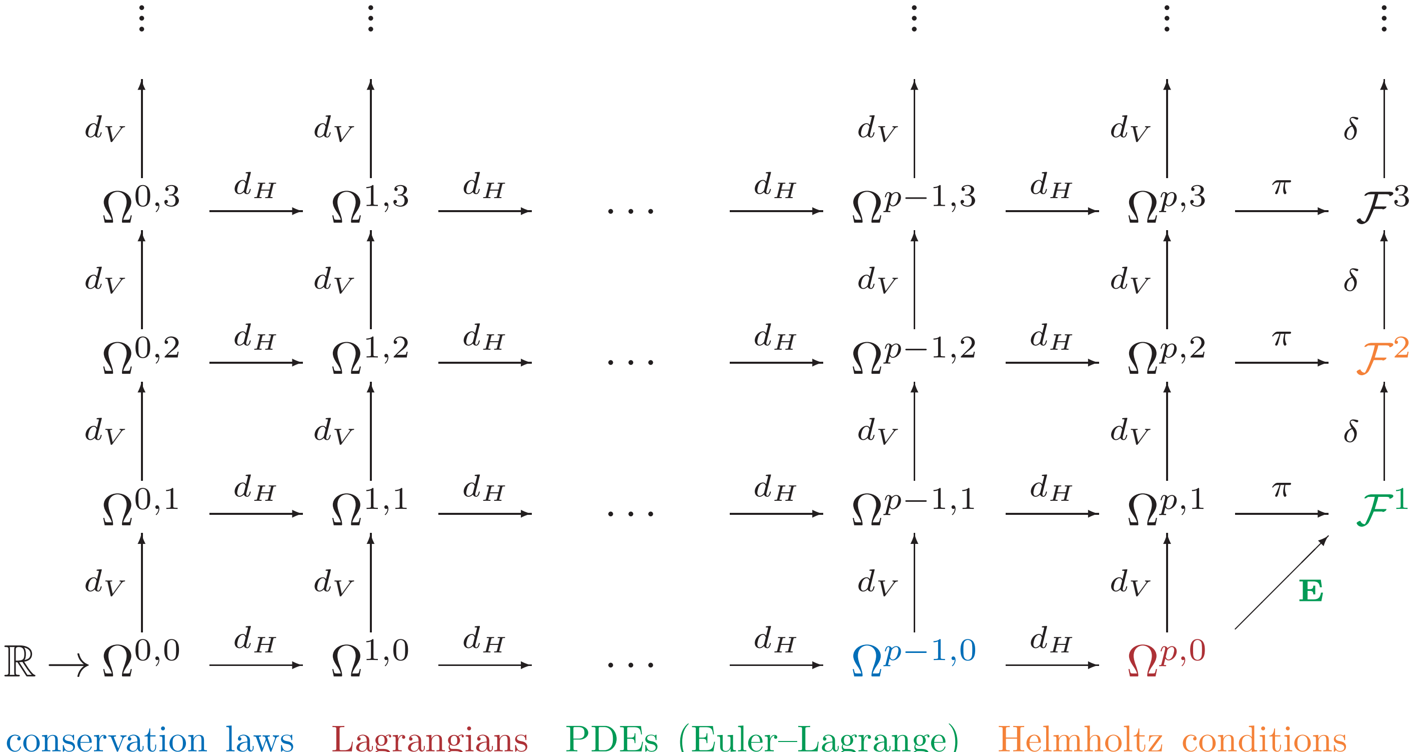
Lagrangians PDEs (Euler–Lagrange)

The Variational Bicomplex



Lagrangians PDEs (Euler–Lagrange) Helmholtz conditions

The Variational Bicomplex



The Variational Derivative

$$\boxed{\mathbf{E} = \pi \circ d_V}$$

d_V — first variation

π — integration by parts = mod out by image of d_H

$$\begin{array}{ccccc} \Omega^{p,0} & \xrightarrow{\;d_V\;} & \Omega^{p,1} & \xrightarrow{\;\pi\;} & \mathcal{F}^1 = \Omega^{p,1}/\,d_H\,\Omega^{p-1,1} \\ \lambda = L\,d\mathbf{x} & \longrightarrow & \displaystyle\sum_{\alpha,J} \frac{\partial L}{\partial u^\alpha_J}\,\theta^\alpha_J \wedge\, d\mathbf{x} & \longrightarrow & \displaystyle\sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\;\theta^\alpha \wedge\, d\mathbf{x} \end{array}$$

Variational problem $\longrightarrow$ First variation $\longrightarrow$ Euler–Lagrange source form

The Simplest Example: $(x, u) \in M = \mathbb{R}^2$

Lagrangian form: $\lambda = L(x, u^{(n)})\, dx \ \in \ \Omega^{1,0}$

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Lagrangian form: $\lambda = L(x, u^{(n)})\, dx \ \in \ \Omega^{1,0}$

First variation — vertical derivative:

$$d\lambda = d_V \lambda = d_V L \wedge dx$$

$$= \left(\frac{\partial L}{\partial u} \theta + \frac{\partial L}{\partial u_x} \theta_x + \frac{\partial L}{\partial u_{xx}} \theta_{xx} + \cdots \right) \wedge dx \ \in \ \Omega^{1,1}$$

The Simplest Example: $(x, u) \in M = \mathbb{R}^2$

Lagrangian form: $\lambda = L(x, u^{(n)})\, dx \ \in\ \Omega^{1,0}$

First variation — vertical derivative:

$$d\lambda = d_V\, \lambda = d_V\, L \wedge dx$$

$$= \left(\frac{\partial L}{\partial u}\, \theta + \frac{\partial L}{\partial u_x}\, \theta_x + \frac{\partial L}{\partial u_{xx}}\, \theta_{xx} + \cdots \right) \wedge\, dx \ \in\ \Omega^{1,1}$$

Integration by parts — compute modulo im d_H:

$$d\lambda \ \sim\ \delta\lambda = \left(\frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge\, dx \ \in\ \mathcal{F}^1$$

$$= \mathbf{E}(L)\, \theta \wedge\, dx$$

$\Longrightarrow$ Euler-Lagrange source form.

To analyze invariant variational problems, invariant conservation laws, etc., we apply the moving frame invariantization process to the variational bicomplex:

The Invariant Variational Complex

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The Invariant Variational Complex

ι — invariantization associated with moving frame ρ.

$\iota \qquad \iota$

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The Invariant Variational Complex

ι — invariantization associated with moving frame ρ.

- Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(n)}) = \iota(u^\alpha_K)$$

ι

ι

The Invariant Variational Complex

ι — invariantization associated with moving frame ρ.

- Fundamental differential invariants

$$H^i(x,u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x,u^{(n)}) = \iota(u^\alpha_K)$$

- Invariant horizontal forms

$$\varpi^i = \iota(dx^i)$$

- Invariant contact forms

$$\vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

The Invariant "Quasi–Tricomplex"

Differential forms

$$\Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$$

Differential

$$d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$$

$$\begin{aligned} d_{\mathcal{H}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r+1,s} \\ d_{\mathcal{V}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r,s+1} \\ d_{\mathcal{W}} : \quad & \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r-1,s+2} \end{aligned}$$

Key fact: invariantization and differentiation *do not commute*:

$$d\,\iota(\Omega) \neq \iota(d\Omega)$$

The Universal Recurrence Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\,[\mathbf{v}_{\kappa}(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$ — infinitesimal generators

$\nu^1, \ldots, \nu^r$ — invariantized dual Maurer–Cartan forms

$\Longrightarrow$ uniquely determined by the recurrence formulae for the phantom differential invariants

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\,[\mathbf{v}_{\kappa}(\Omega)]$$

★★★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal formula by letting Ω range over the basic functions and differential forms!

$$\boxed{\boxed{\; d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \nu^{\kappa} \wedge \iota\,[\mathbf{v}_{\kappa}(\Omega)] \;}}$$

★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal formula by letting Ω range over the basic functions and differential forms!

★ ★ ★ Moreover, determining the structure of the differential invariant algebra and invariant variational bicomplex requires only linear differential algebra, and not any explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!

Euclidean plane curves

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa \qquad \iota(u_{xxx}) = I_3 = \kappa_s \qquad \iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

In general:

$$\iota(\, F(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, \ldots\,)) = F(0, 0, 0, \kappa, \kappa_s, \kappa_{ss} + 3\kappa^3, \ldots\,)$$

Invariant arc length form

$$dy = (\cos\phi - u_x \sin\phi)\, dx - (\sin\phi)\, \theta$$

$$\varpi = \iota(dx) = \qquad \omega \qquad + \qquad \eta$$

$$= \sqrt{1+u_x^2}\; dx + \frac{u_x}{\sqrt{1+u_x^2}}\, \theta$$

$$\Longrightarrow \quad \theta = du - u_x\, dx$$

Invariant contact forms

$$\vartheta = \iota(\theta) = \frac{\theta}{\sqrt{1+u_x^2}} \qquad \vartheta_1 = \iota(\theta_x) = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}$$

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_u, \qquad \mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3\,u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

Basic recurrence formula

$$d\iota(F) = \iota(dF) + \iota(\mathbf{v}_1(F))\,\nu^1 + \iota(\mathbf{v}_2(F))\,\nu^2 + \iota(\mathbf{v}_3(F))\,\nu^3$$

Use phantom invariants

$$0 = dH = \iota(dx) + \iota(\mathbf{v}_1(x))\,\nu^1 + \iota(\mathbf{v}_2(x))\,\nu^2 + \iota(\mathbf{v}_3(x))\,\nu^3 = \varpi + \nu^1,$$

$$0 = dI_0 = \iota(du) + \iota(\mathbf{v}_1(u))\,\nu^1 + \iota(\mathbf{v}_2(u))\,\nu^2 + \iota(\mathbf{v}_3(u))\,\nu^3 = \vartheta + \nu^2,$$

$$0 = dI_1 = \iota(du_x) + \iota(\mathbf{v}_1(u_x))\,\nu^1 + \iota(\mathbf{v}_2(u_x))\,\nu^2 + \iota(\mathbf{v}_3(u_x))\,\nu^3 = \kappa\,\varpi + \vartheta_1 + \nu^3,$$

to solve for the Maurer–Cartan forms:

$$\boxed{\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\,\varpi - \vartheta_1.}$$

$$\boxed{\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\varpi - \vartheta_1.}$$

Recurrence formulae:

$$\begin{aligned} d\kappa = d\iota(u_{xx}) &= \iota(du_{xx}) + \iota(\mathbf{v}_1(u_{xx}))\,\nu^1 + \iota(\mathbf{v}_2(u_{xx}))\,\nu^2 + \iota(\mathbf{v}_3(u_{xx}))\,\nu^3 \\ &= \iota(u_{xxx}\,dx + \theta_{xx}) - \iota(3\,u_x u_{xx})\,(\kappa\varpi + \vartheta_1) = I_3\varpi + \vartheta_2. \end{aligned}$$

Therefore,

$$\mathcal{D}\kappa = \kappa_s = I_3, \qquad d_{\mathcal{V}}\,\kappa = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

where the final formula follows from the contact form recurrence formulae

$$d\vartheta = d\iota(\theta_x) = \varpi \wedge \vartheta_1, \qquad d\vartheta_1 = d\iota(\theta) = \varpi \wedge (\vartheta_2 - \kappa^2\,\vartheta) - \kappa\,\vartheta_1 \wedge \vartheta$$

which imply

$$\vartheta_1 = \mathcal{D}\vartheta, \qquad \vartheta_2 = \mathcal{D}\vartheta_1 + \kappa^2\,\vartheta = (\mathcal{D}^2 + \kappa^2)\,\vartheta$$

Similarly,

$$\begin{aligned} d\varpi &= \iota(d^2 x) + \nu^1 \wedge \iota(\mathbf{v}_1(dx)) + \nu^2 \wedge \iota(\mathbf{v}_2(dx)) + \nu^3 \wedge \iota(\mathbf{v}_3(dx)) \\ &= (\kappa\,\varpi + \vartheta_1) \wedge \iota(u_x\,dx + \theta) = \kappa\,\varpi \wedge \vartheta + \vartheta_1 \wedge \vartheta. \end{aligned}$$

In particular,

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Key recurrence formulae:

$$\boxed{d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta} \qquad \boxed{d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi}$$

Plane Curves

Invariant Lagrangian:

$$\widetilde{\lambda} = L(x, u^{(n)})\, dx = P(\kappa, \kappa_s, \ldots)\, \varpi$$

Euler–Lagrange form:

$$d_{\mathcal{V}}\, \widetilde{\lambda} \;\sim\; \mathbf{E}(L)\, \vartheta \wedge \varpi$$

Invariant Integration by Parts Formula

$$F\, d_{\mathcal{V}}\, (\mathcal{D}H) \wedge \varpi \;\sim\; -\,(\mathcal{D}F)\, d_{\mathcal{V}}\, H \wedge \varpi - (F \cdot \mathcal{D}H)\, d_{\mathcal{V}}\, \varpi$$

$$\begin{aligned} d_{\mathcal{V}}\, \widetilde{\lambda} &= d_{\mathcal{V}}\, P \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &= \sum_n \frac{\partial P}{\partial \kappa_n}\; d_{\mathcal{V}}\, \kappa_n \wedge \varpi + P\, d_{\mathcal{V}}\, \varpi \\ &\sim\; \mathcal{E}(P)\; d_{\mathcal{V}}\, \kappa \wedge \varpi + \mathcal{H}(P)\; d_{\mathcal{V}}\, \varpi \end{aligned}$$

Vertical differentiation formulae

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta) \qquad \mathcal{A} \ \text{— “Eulerian operator”}$$

$$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi \qquad \mathcal{B} \ \text{— “Hamiltonian operator”}$$

$$\begin{aligned} d_{\mathcal{V}}\,\widetilde{\lambda} &\sim \ \mathcal{E}(P)\ \mathcal{A}(\vartheta) \wedge \varpi + \mathcal{H}(P)\ \mathcal{B}(\vartheta) \wedge \varpi \\ &\sim \ \Big[\,\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P)\Big]\ \vartheta \wedge \varpi \end{aligned}$$

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = 0}$$

Evolution of Invariants and Signatures

G — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

G–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- I, J — differential invariants
- $\mathbf{t}$ — "unit tangent"
- $\mathbf{n}$ — "unit normal"

$\mathbf{t}, \ \mathbf{n}$ — basis of the invariant vector fields dual to the invariant one-forms:

$$\langle\, \mathbf{t}\,; \varpi \,\rangle = 1, \qquad \langle\, \mathbf{n}\,; \varpi \,\rangle = 0,$$

$$\langle\, \mathbf{t}\,; \vartheta \,\rangle = 0, \qquad \langle\, \mathbf{n}\,; \vartheta \,\rangle = 1.$$

$$C_t = \mathbf{V} = I\, \mathbf{t} + J\, \mathbf{n}$$

- The tangential component $I\,\mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set I to be anything we like without affecting the curve evolution.
- There are two principal choices of tangential component:

Normal Curve Flows

$$C_t = J\,\mathbf{n}$$

Examples — Euclidean–invariant curve flows

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa\,\mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3}\,\mathbf{n}$ — equi-affine invariant curve shortening flow:
  $$C_t = \mathbf{n}_{\text{equi-affine}}\,;$$
- $C_t = \kappa_s\,\mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss}\,\mathbf{n}$ — thermal grooving of metals.

Intrinsic Curve Flows

Theorem. The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}\,I = 0.$$

$\mathcal{D}$ — invariant arc length derivative

$\mathcal{B}$ — invariant arc length variation

$$d_{\mathcal{V}}\,(ds) = \mathcal{B}(\vartheta) \wedge\, ds$$

Normal Evolution of Differential Invariants

Theorem. Under a normal flow $C_t = J\,\mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

Invariant variations:

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_\kappa(\vartheta), \qquad d_{\mathcal{V}}\,\kappa_s = \mathcal{A}_{\kappa_s}(\vartheta).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant variation of curvature;

$\mathcal{A}_{\kappa_s} = \mathcal{D}\,\mathcal{A}_\kappa + \kappa\,\kappa_s$ — invariant variation of κ_s.

Euclidean–invariant Curve Evolution

Normal flow: $C_t = J\,\mathbf{n}$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2)\,J,$$

$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,J.$$

Warning: For non-intrinsic flows, ∂_t and ∂_s *do not commute*!

Grassfire flow: $J = 1$

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = 3\,\kappa\,\kappa_s, \qquad \ldots$$

$\Longrightarrow$ caustics

Euclidean Signature Evolution

Evolution of the Euclidean signature curve

$$\kappa_s = \Phi(t, \kappa).$$

Grassfire flow:

$$\frac{\partial \Phi}{\partial t} = 3\,\kappa\,\Phi - \kappa^2\,\frac{\partial \Phi}{\partial \kappa}\ .$$

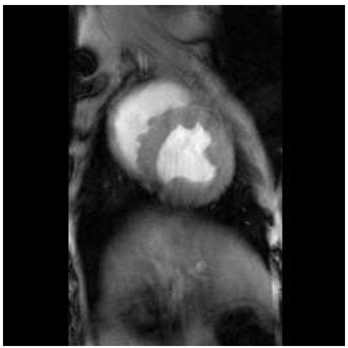
Curve shortening flow:

$$\frac{\partial \Phi}{\partial t} = \Phi^2\,\Phi_{\kappa\kappa} - \kappa^3 \Phi_\kappa + 4\,\kappa^2 \Phi.$$

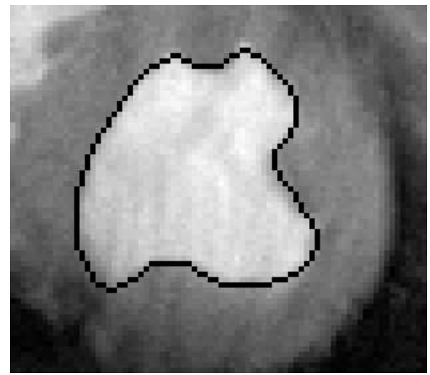
Modified Korteweg-deVries flow:

$$\frac{\partial \Phi}{\partial t} = \Phi^3\,\Phi_{\kappa\kappa\kappa} + 3\,\Phi^2\,\Phi_\kappa\,\Phi_{\kappa\kappa} + 3\,\kappa\,\Phi^2.$$

Canine Left Ventricle Signature

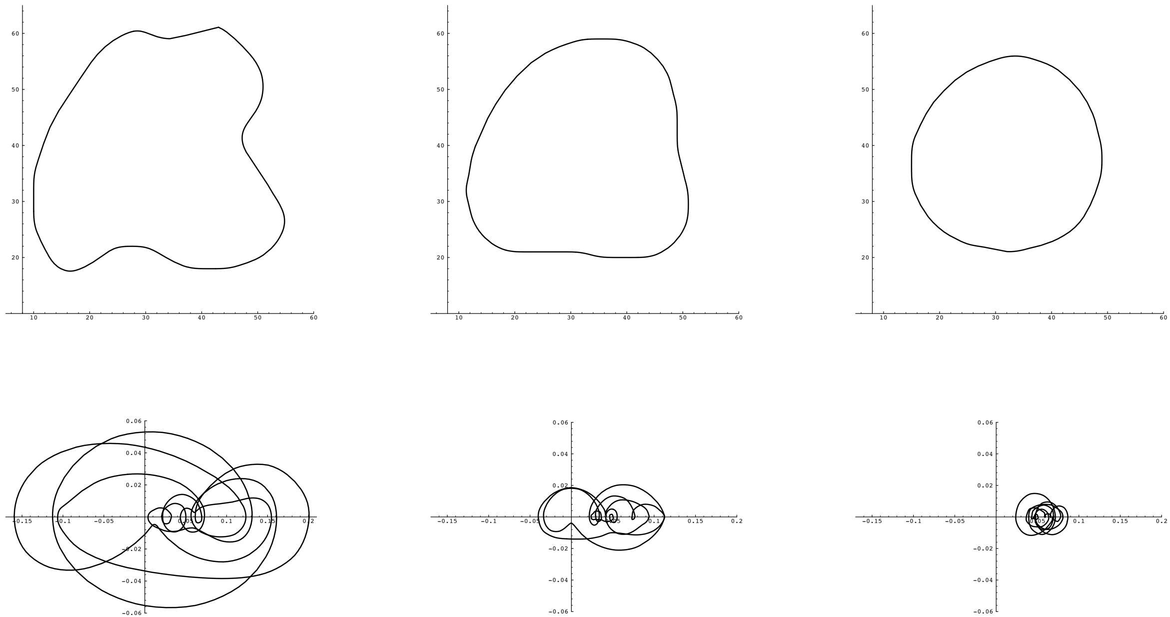


Original Canine Heart
MRI Image



Boundary of Left Ventricle

Smoothed Ventricle Signature



Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \qquad (*)$$

Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1}\mathcal{B} \qquad (*)$$

In surprisingly many situations, (*) is a well-known integrable evolution equation, and $\mathcal{R}$ is its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang

$\Longrightarrow$ Qu, Chou, Anco, and many more ...

Euclidean plane curves

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\,\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta, \qquad d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

$$\Longrightarrow \qquad \mathcal{A} = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{B} = -\,\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\,\kappa^2 \kappa_s$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$d_{\mathcal{V}}\, \kappa = \mathcal{A}(\vartheta), \qquad d_{\mathcal{V}}\, \varpi = \mathcal{B}(\vartheta) \wedge \varpi$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s \mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2, \qquad \mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa,$$

$$\begin{aligned} \mathcal{R} &= \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} \\ &= \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s \mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s \mathcal{D}^{-1} \cdot \kappa \end{aligned}$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + \tfrac{5}{3}\,\kappa\,\kappa_{sss} + \tfrac{5}{3}\,\kappa_s \kappa_{ss} + \tfrac{5}{9}\,\kappa^2 \kappa_s$$

$\Longrightarrow$ Sawada–Kotera equation

Recursion operator:

$$\widehat{\mathcal{R}} = \mathcal{R} \cdot (\mathcal{D}^2 + \tfrac{1}{3}\,\kappa + \tfrac{1}{3}\,\kappa_s \mathcal{D}^{-1}).$$

Euclidean space curves

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\begin{pmatrix} d_{\mathcal{V}}\,\kappa \\ d_{\mathcal{V}}\,\tau \end{pmatrix} = \mathcal{A}\begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B}\begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} \wedge \varpi$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \dfrac{2\tau}{\kappa} D_s^2 + \dfrac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \dfrac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \dfrac{1}{\kappa} D_s^3 - \dfrac{\kappa_s}{\kappa^2} D_s^2 + \dfrac{\kappa^2 - \tau^2}{\kappa} D_s + \dfrac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} \kappa & 0 \end{pmatrix}$$

Recursion operator:

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix}$$

$\Longrightarrow$ vortex filament flow

$\Longrightarrow$ nonlinear Schrödinger equation (Hasimoto)

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