Symmetry–Preserving Numerical Methods

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Symmetry–Preserving Numerical Methods via Moving Frames

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$\Longrightarrow$ Pilwon Kim, Martin Welk

Geometric Integration

Many differential equations arising in applications possess rich symmetry groups, which can be determined by the classical Lie infinitesimal algorithm.

Geometric integration is concerned with designing numerical algorithms that preserve structure of differential equations.

So, one goal is to design numerical approximations that preserve some or all symmetries of the underlying differential equation.

Invariantization

A new approach to the classical Cartan theory of moving frames provides a systematic way for designing such algorithms using the process of invariantization based on a choice of cross-section.

With this approach, any standard numerical algorithm can be invariantized to produce a corresponding symmetry-preserving algorithm.

A clever choice of cross-section defining the invariantization procedure can, in many cases, produce an invariant algorithm that performs significantly better than the original.

Moreover, thanks to the way the invariantization process is implemented, it is extremely easy to modify the numerical code for the original algorithm to produce a symmetry-preserving version.

The Geometry of Differential Equations

Although in use since the time of Lie and Darboux, jet space was first formally defined by Ehresmann in 1950.

Jet space is the proper setting for the geometry of partial differential equations.

Question: What is the proper setting for the geometry of numerical analysis?

Jet Space

M — smooth m-dimensional manifold

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — (extended) jet bundle

- Equivalence classes of p-dimensional submanifolds $u = f(x)$ under n^{th} order contact at a point.

- Coordinates $(x, u^{(n)})$ are given by the derivatives of u with respect to x up to order n.

Differential Equations

An n^{th} order system of differential equations

$$\Delta(x, u^{(n)}) = 0$$

defines a submanifold $\mathcal{S}_\Delta \subset \mathrm{J}^n$.

Symmetry Groups

G — Lie group acting on M

Definition. G is a symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$ if it maps solutions to solutions.

Theorem. G is a symmetry group of $\Delta = 0$ if and only if the submanifold $\mathcal{S}_\Delta \subset \mathrm{J}^n$ is invariant under the prolonged action $G^{(n)}$ of the group on J^n.

$\Longrightarrow$ Lie's infinitesimal algorithm

Differential Invariants

A differential invariant is an invariant of the prolonged action: $\qquad I : \mathrm{J}^n \longrightarrow \mathbb{R}$

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$$\Longrightarrow \quad \text{curvature, torsion, } \ldots$$

Theorem. A (regular) system of differential equations admits G as a symmetry group if and only if it can be expressed in terms of the differential invariants.

Euclidean Differential Invariants

Euclidean curvature:

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}$$

Euclidean arc length:

$$ds = \sqrt{1 + u_x^2}\, dx$$

Higher order differential invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Euclidean–invariant differential equation:

$$F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0$$

Affine Differential Invariants

Affine curvature

$$\kappa = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{9(u_{xx})^{8/3}}$$

Affine arc length

$$ds = \sqrt[3]{u_{xx}}\, dx$$

Higher order affine invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Affine–invariant differential equation:

$$F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0$$

Finite Difference Approximations

Key remark: Every (finite difference) numerical approximation to the derivatives of a function or, geometrically depend on evaluating the function at several points $z_i = (x_i, u_i)$ where $u_i = f(x_i)$.

In other words, we seek to approximate the n^{th} order jet of a submanifold $N \subset M$ by a function $F(z_0, \ldots, z_n)$ defined on the $(n+1)$-fold Cartesian product space $M^{\times(n+1)} = M \times \cdots \times M$, or, more correctly, on the "off-diagonal" part

$$M^{\diamond(n+1)} = \{\, z_i \neq z_j \text{ for all } i \neq j \,\}$$

$\Longrightarrow$ *distinct* $(n+1)$-tuples of points.

Multi–Space

- The proper setting for the geometry of finite difference approximations to differential equations is multi-space $M^{(n)}$.

- To include both derivatives and their finite difference approximations, multi-space should contain both the jet space and the off-diagonal Cartesian product space as submanifolds:

$$\left.\begin{array}{c} M^{\diamondsuit(n+1)} \\[2em] \downarrow \\[2em] \mathrm{J}^n(M,p) \end{array}\right\} \quad \subset \quad M^{(n)}$$

Multi–Space for Curves

M — smooth m-dimensional manifold

$M^{(n)}$ — n^{th} order multi-space

- Equivalence classes of $n+1$–pointed curves

$$(z_0, \ldots, z_n; C) \qquad z_i \in C$$

under n^{th} order multi-contact at a point.

- Coordinates are given by the divided differences up to order n.

$$C = \{ u = f(x) \} \subset \mathbb{R}^m \qquad \text{— curve (graph)}$$

$$z_i = (x_i, u_i) \in C \qquad \text{(may coalesce)}$$

Local coordinates for $M^{(n)}$ consist of the independent variables along with all the divided differences

$$x_0, \ldots, x_n \qquad u^{(0)} = u_0 = [\, z_0 \,]_C \qquad u^{(1)} = [\, z_0 z_1 \,]_C$$

$$u^{(2)} = 2\, [\, z_0 z_1 z_2 \,]_C \quad \ldots \quad u^{(n)} = n!\, [\, z_0 z_1 \ldots z_n \,]_C$$

- The $n!$ factor is included so that $u^{(n)}$ agrees with the usual derivative coordinate when restricted to J^n.

Finite Difference Approximations

An $(n+1)$-*point numerical approximation of order* k to a differential function $\Delta\colon \mathrm{J}^n \to \mathbb{R}$ is a k^{th} order extension $F\colon M^{(n)} \to \mathbb{R}$ of Δ to multi-space, based on the inclusion $\mathrm{J}^n \subset M^{(n)}$.

$$\begin{aligned} &F(x_0, \ldots, x_n, u^{(0)}, \ldots, u^{(n)}) \\ &\qquad \longrightarrow \quad F(x, \ldots, x, u^{(0)}, \ldots, u^{(n)}) = \Delta(x, u^{(n)}) \end{aligned}$$

Joint Invariants

A joint invariant is an invariant of the Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) = I(z_1, \ldots, z_k)}$$

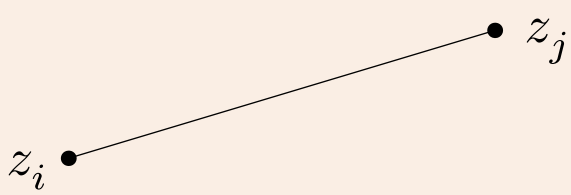
A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) = I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

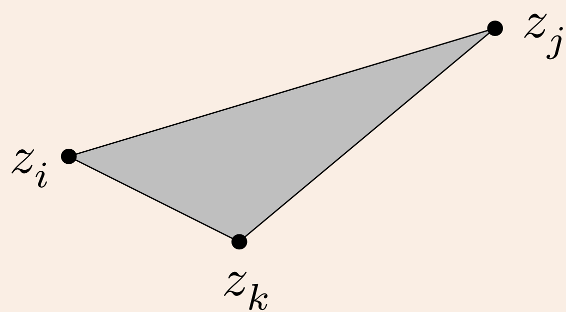
$$d(z_i, z_j) = \| \, z_i - z_j \, \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

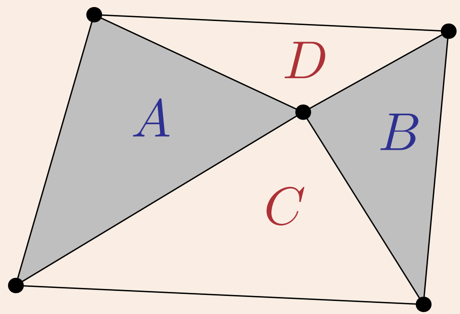
$$[\, i\ j\ k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

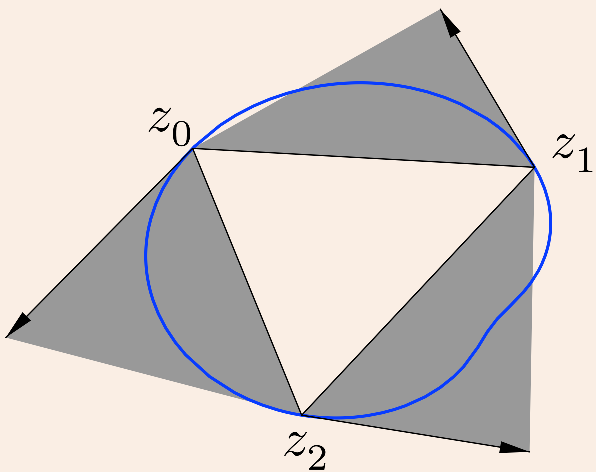
Theorem. Every joint projective invariant is a function of the planar cross-ratios

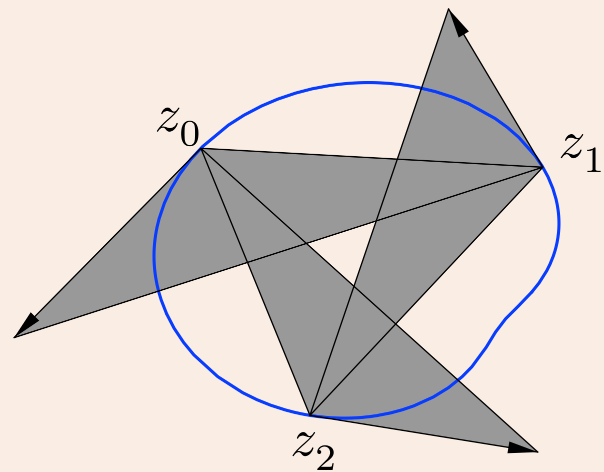
$$[\, z_i, z_j, z_k, z_l, z_m \,] \;=\; \frac{A\,B}{C\,D}$$



- Three–point projective joint differential invariant
    — tangent triangle ratio:

$$\frac{[\,0\;2\;\dot 0\,]\;[\,0\;1\;\dot 1\,]\;[\,1\;2\;\dot 2\,]}{[\,0\;1\;\dot 0\,]\;[\,1\;2\;\dot 1\,]\;[\,0\;2\;\dot 2\,]}$$



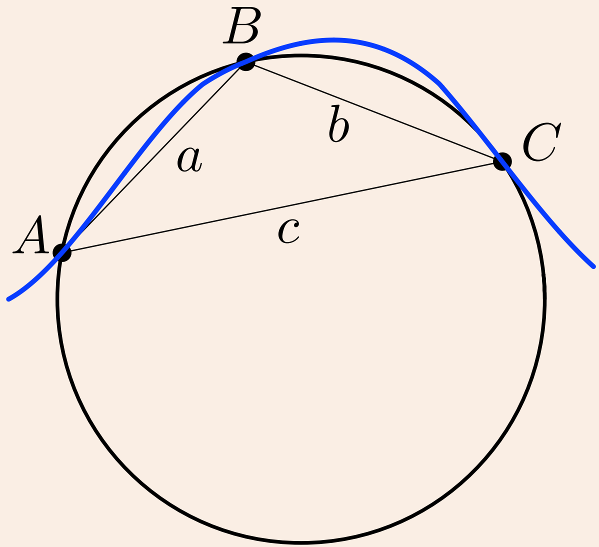


Invariant Numerical Approximations

Basic Idea:

Every invariant finite difference approximation to a differential invariant must expressible in terms of the joint invariants of the transformation group.

Numerical approximation to curvature



Heron's formula

$$\tilde{\kappa}(A, B, C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$s = \frac{a+b+c}{2} \qquad \text{—} \qquad \text{semi-perimeter}$$

Expansion:

$$\widetilde{\kappa} = \kappa + \frac{1}{3}(b-a)\,\frac{d\kappa}{ds} + \frac{1}{12}(b^2 - ab + a^2)\,\frac{d^2\kappa}{ds^2} + \\ + \frac{1}{60}(b^3 - ab^2 + a^2 b - a^3)\,\frac{d^3\kappa}{ds^3} + \\ + \frac{1}{120}(b-a)\,(3b^2 + 5ab + 3a^2)\,\kappa^2\,\frac{d\kappa}{ds} + \cdots.$$

Higher order invariants

$$\kappa_s = \frac{d\kappa}{ds}$$

Invariant finite difference approximation:

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_i, P_{i-1})}$$

Unbiased centered difference:

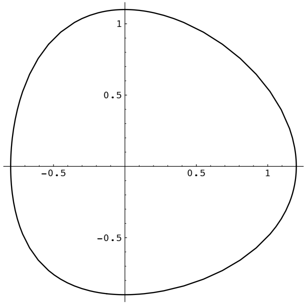
$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}) = \frac{\widetilde{\kappa}(P_i, P_{i+1}, P_{i+2}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_{i+1}, P_{i-1})}$$

Better approximation (M. Boutin):

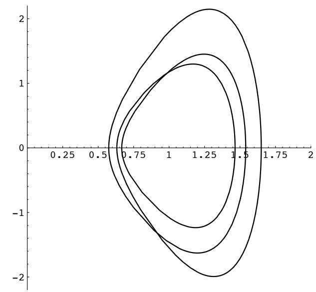
$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = 3\ \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}_{i-2} + 2\,\mathbf{d}_{i-1} + 2\,\mathbf{d}_i + \mathbf{d}_{i+1}}$$

$$\mathbf{d}_j = \mathbf{d}(P_j, P_{j+1})$$

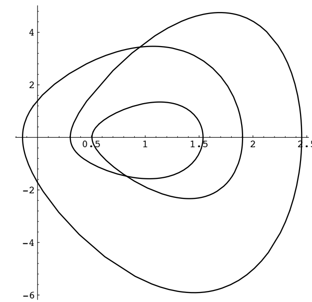
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



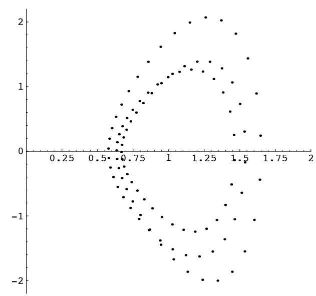
The Original Curve



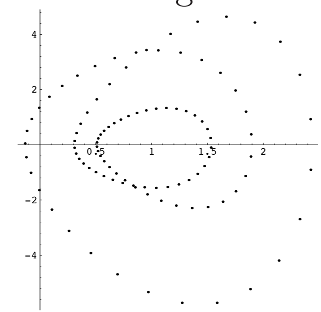
Euclidean Signature



Affine Signature

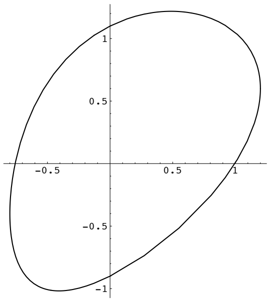


Discrete Euclidean Signature

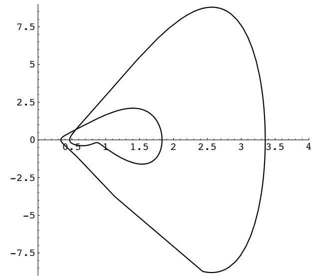


Discrete Affine Signature

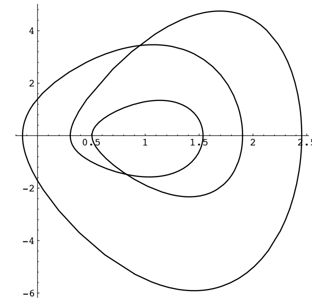
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \;\; y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



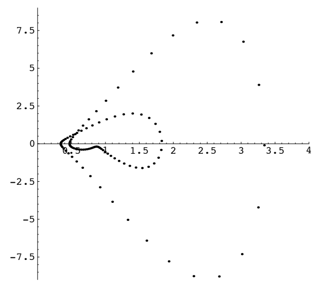
The Original Curve



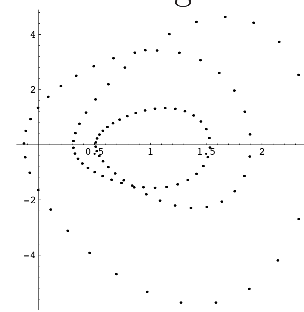
Euclidean Signature



Affine Signature



Discrete Euclidean Signature



Discrete Affine Signature

$\star \quad \star \quad \star$

Moving frames provide a systematic algorithm for constructing invariants:

- differential invariants
- joint invariants
- invariant numerical approximations

$\star \quad \star \quad \star$

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint invariants and semi-differential invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory

- Computer vision
    - object recognition
    - symmetry detection
- Invariant variational problems
- Invariant numerical methods
- Poisson geometry & solitons
- Killing tensors in relativity
- Invariants of Lie algebras in quantum mechanics
- Lie pseudogroups

Moving Frames

Definition.

A moving frame is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

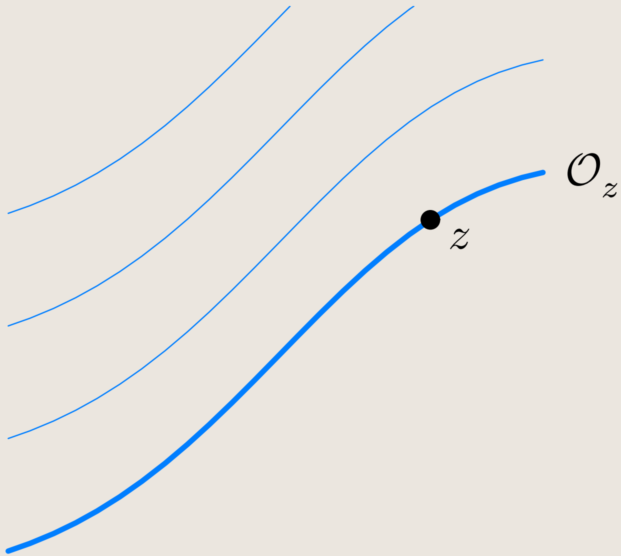
Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup: $\quad G_z = \{ g \mid g \cdot z = z \} \quad$ for $\; z \in M$

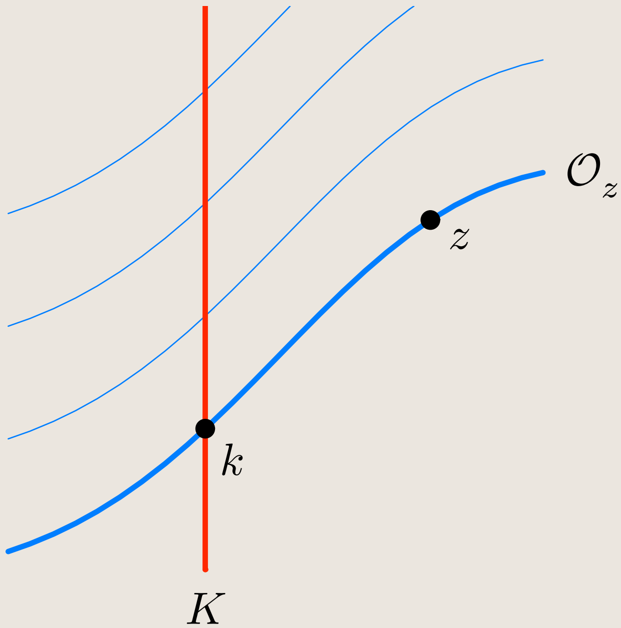
- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity: $\quad \Longrightarrow \quad G_z = \{e\}$ for all $z \in M$.
- locally free — the orbits all have the same dimension as G: $\quad \Longrightarrow \quad G_z$ is a discrete subgroup of G.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  $\not\approx\;$ irrational flow on the torus

Geometric Construction



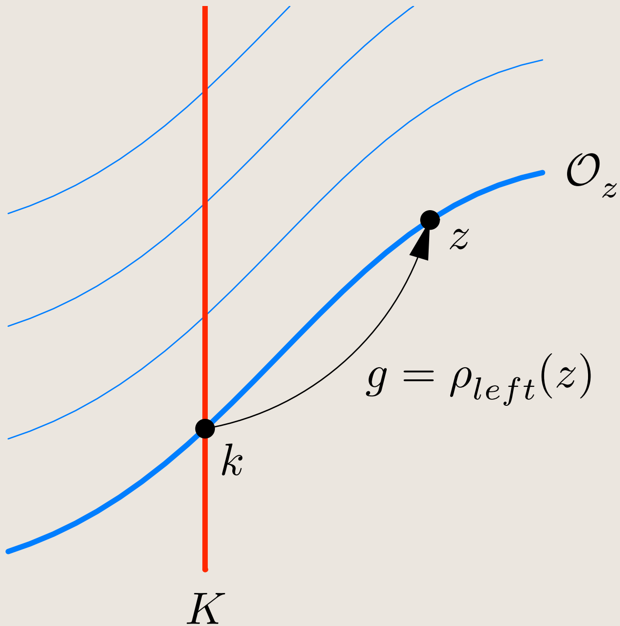
$\boxed{\text{Normalization} = \text{choice of cross-section to the group orbits}}$

Geometric Construction



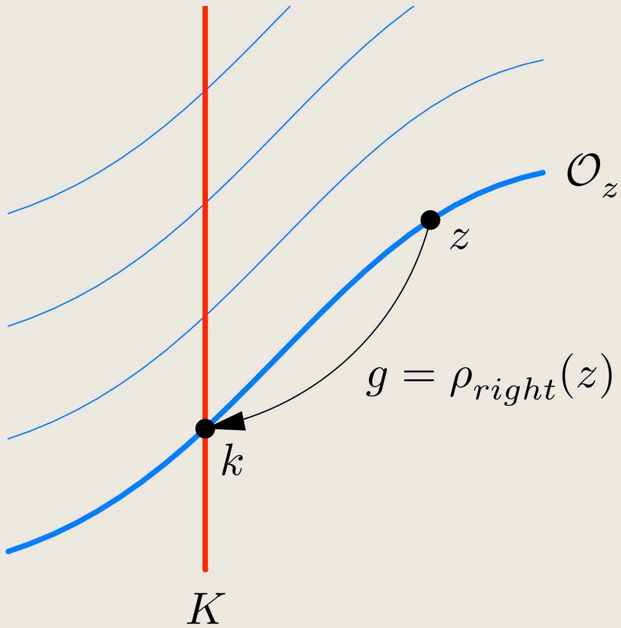
Normalization = choice of cross-section to the group orbits

Geometric Construction



$$\boxed{\text{Normalization} = \text{choice of cross-section to the group orbits}}$$

Geometric Construction



Normalization = choice of cross-section to the group orbits

Algebraic Construction

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\; , z_r = c_r \,\}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g,z) = c_1 \qquad \ldots \qquad w_r(g,z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Completeness of Invariants

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \; \ldots \; , I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame $g = \rho(z)$ is the the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

$$\iota(z_1) = c_1, \ \ldots \ \iota(z_r) = c_r, \quad \iota(z_{r+1}) = I_1(z), \ \ldots \ \iota(z_r) = I_{m-r}(z).$$

cross-section variables

"phantom invariants"

fundamental invariants

$$\boxed{\iota\,[\,F(z_1, \ldots, z_m)\,] \ = \ F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))}$$

Invariantization amounts to restricting F to the cross-section

$$I \mid K = F \mid K$$

and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota \; : \; \text{functions} \quad \longmapsto \quad \text{invariants}$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

- An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants

$\Longrightarrow$ invariant numerical approximations

Euclidean Plane Curves

Special Euclidean group: $\quad G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

acts on $M = \mathbb{R}^2$ via rigid motions: $\ w = R\,z + b$

To obtain the classical (left) moving frame we invert the group transformations:

$$\left.\begin{aligned} y &= \quad \cos\theta\,(x-a) + \sin\theta\,(u-b) \\ v &= -\sin\theta\,(x-a) + \cos\theta\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-b)$$

Assume for simplicity the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

$\Longrightarrow$ extensions to parametrized curves are straightforward

Prolong the action to J^n via implicit differentiation:

$$
\begin{aligned}
y &= \quad \cos\theta\,(x-a) + \sin\theta\,(u-b) \\
v &= -\sin\theta\,(x-a) + \cos\theta\,(u-b) \\
v_y &= \frac{-\sin\theta\; + u_x\cos\theta}{\cos\theta\; + u_x\sin\theta} \\
v_{yy} &= \frac{u_{xx}}{(\cos\theta\; + u_x\sin\theta\,)^3} \\
v_{yyy} &= \frac{(\cos\theta\; + u_x\sin\theta\,)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta\; + u_x\sin\theta\,)^5} \\
&\;\vdots
\end{aligned}
$$

Prolong the action to J^n via implicit differentiation:

$$
\begin{aligned}
y &= \cos\theta\,(x-a) + \sin\theta\,(u-b) \\
v &= -\sin\theta\,(x-a) + \cos\theta\,(u-b) \\
v_y &= \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta} \\
v_{yy} &= \frac{u_{xx}}{(\cos\theta + u_x\sin\theta)^3} \\
v_{yyy} &= \frac{(\cos\theta + u_x\sin\theta)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta + u_x\sin\theta)^5} \\
&\vdots
\end{aligned}
$$

Normalization: $\qquad r = \dim G = 3$

$$y = \cos\theta\,(x-a) + \sin\theta\,(u-b) = 0$$

$$v = -\sin\theta\,(x-a) + \cos\theta\,(u-b) = 0$$

$$v_y = \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta} = 0$$

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x\sin\theta)^3}$$

$$v_{yyy} = \frac{(\cos\theta + u_x\sin\theta)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta + u_x\sin\theta)^5}$$

$$\vdots$$

Solve for the group parameters:

$$y = \quad \cos\theta\,(x-a) + \sin\theta\,(u-b) = 0$$

$$v = -\sin\theta\,(x-a) + \cos\theta\,(u-b) = 0$$

$$v_y = \frac{-\sin\theta\ + u_x\cos\theta}{\cos\theta\ + u_x\sin\theta} = 0$$

$\Longrightarrow$ Left moving frame $\qquad \rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x \qquad b = u \qquad \theta\ = \tan^{-1} u_x$$

$$a = x \qquad b = u \qquad \theta = \tan^{-1} u_x$$

Differential invariants

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x \sin\theta)^3} \longmapsto \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$v_{yyy} = \cdots \qquad \longmapsto \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}$$

$$v_{yyyy} = \cdots \qquad \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

Invariant one-form — arc length

$$dy = (\cos\theta + u_x \sin\theta)\, dx \qquad \longmapsto \qquad ds = \sqrt{1+u_x^2}\; dx$$

Dual invariant differential operator

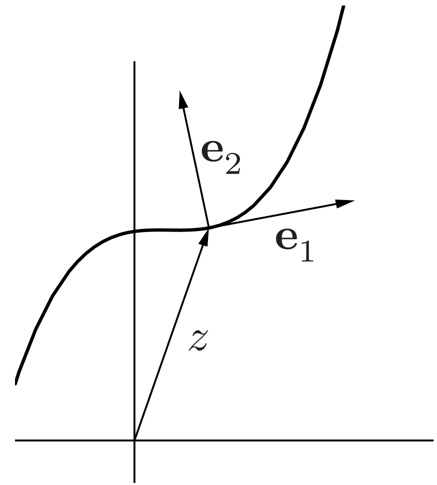
— arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\theta + u_x \sin\theta}\,\frac{d}{dx} \qquad \longmapsto \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

The Classical Picture:



Moving frame $\qquad \rho : \ (x, u, u_x) \longmapsto (R, \mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\, \mathbf{e}_1, \ \mathbf{e}_2 \,) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

Example. $G = \mathbb{R}^2 \ltimes \mathbb{R}$

$$(x, u) \quad \longmapsto \quad (\lambda^{-1} x + a, \lambda u + b)$$

Multi-prolonged action: compute the divided differences of the basic lifted invariants

$$y_k = \lambda^{-1} x_k + a,$$

$$v_k = \lambda u_k + b,$$

$$v^{(1)} = [\, w_0 w_1 \,] = \frac{v_1 - v_0}{y_1 - y_0}$$

$$= \lambda^2 \frac{u_1 - u_0}{x_1 - x_0} = \lambda^2 [\, z_0 z_1 \,] = \lambda^2 u^{(1)},$$

$$v^{(n)} = \lambda^{n+1} u^{(n)}.$$

Moving frame cross-section

$$y_0 = 0, \qquad v_0 = 0, \qquad v^{(1)} = 1.$$

Solve for the group parameters

$$a = -\sqrt{u^{(1)}}\; x_0, \qquad b = -\frac{u_0}{\sqrt{u^{(1)}}}, \qquad \lambda = \frac{1}{\sqrt{u^{(1)}}}.$$

Multi-invariants: ι — invariantization

$$\iota(x_k) = H_k = (x_k - x_0)\sqrt{u^{(1)}} = (x_k - x_0)\sqrt{\frac{u_1 - u_0}{x_1 - x_0}}$$

$$\iota(u_k) = K_k = \frac{u_k - u_0}{\sqrt{u^{(1)}}} = (u_k - u_0)\sqrt{\frac{x_1 - x_0}{u_1 - u_0}}$$

$$\iota(u^{(n)}) = K^{(n)} = \frac{u^{(n)}}{(u^{(1)})^{(n+1)/2}} = \frac{n!\,[\,z_0 z_1 \ldots z_n\,]}{[\,z_0 z_1 z_2\,]^{(n+1)/2}}$$

$$K^{(0)} = K_0 = 0 \qquad K^{(1)} = 1$$

Coalescent limit

$$K^{(n)} \quad \longrightarrow \quad I^{(n)} = \frac{u^{(n)}}{(u^{(1)})^{(n+1)/2}}$$

$\Longrightarrow$ $K^{(n)}$ is a first order invariant numerical approximation to the differential invariant $I^{(n)}$.

$\Longrightarrow$ Higher order invariant numerical approximations are obtained by invariantization of higher order divided difference approximations.

$$F(\ldots, x_k, \ldots, u^{(n)}, \ldots) \longrightarrow F(\ldots, H_k, \ldots, K^{(n)}, \ldots)$$

To construct an invariant numerical scheme for any similarity-invariant ordinary differential equation

$$F(x, u, u^{(1)}, u^{(2)}, \ldots u^{(n)}) = 0,$$

we merely invariantize the defining differential function, leading to the *invariantized* numerical approximation

$$F(0, 0, 1, K^{(2)}, \ldots, K^{(n)}) = 0.$$

Example. Euclidean group SE(2)

$$y = x\cos\theta - u\sin\theta + a \qquad v = x\sin\theta + u\cos\theta + b$$

Multi-prolonged action on $M^{(1)}$:

$$y_0 = x_0\cos\theta - u_0\sin\theta + a \qquad v_0 = x_0\sin\theta + u_0\cos\theta + b$$

$$y_1 = x_1\cos\theta - u_1\sin\theta + a \qquad v^{(1)} = \frac{\sin\theta + u^{(1)}\cos\theta}{\cos\theta - u^{(1)}\sin\theta}$$

Cross-section

$$y_0 = v_0 = v^{(1)} = 0$$

Right moving frame

$$a = -\, x_0 \cos\theta + u_0 \sin\theta = -\frac{x_0 + u^{(1)}\, u_0}{\sqrt{1 + (u^{(1)})^2}}$$

$$\tan\theta = -\, u^{(1)} \,.$$

$$b = -x_0 \sin\theta - u_0 \cos\theta = \frac{x_0\, u^{(1)} - u_0}{\sqrt{1 + (u^{(1)})^2}}$$

Euclidean multi-invariants

$$(y_k, v_k) \quad \longrightarrow \quad I_k = (H_k, K_k)$$

$$H_k = \frac{(x_k - x_0) + u^{(1)}\,(u_k - u_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{1 + [\,z_0 z_1\,]\,[\,z_0 z_k\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}$$

$$K_k = \frac{(u_k - u_0) - u^{(1)}\,(x_k - x_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{[\,z_0 z_k\,] - [\,z_0 z_1\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}$$

Difference quotients

$$[\,I_0 I_k\,] = \frac{K_k - K_0}{H_k - H_0} = \frac{K_k}{H_k} = \frac{(x_k - x_1)[\,z_0 z_1 z_k\,]}{1 + [\,z_0 z_k\,]\,[\,z_0 z_1\,]}$$

$$I^{(2)} = 2\,[\,I_0 I_1 I_2\,] = 2\,\frac{[\,I_0 I_2\,] - [\,I_0 I_1\,]}{H_2 - H_1}$$

$$= \frac{2\,[\,z_0 z_1 z_2\,]\sqrt{1 + [\,z_0 z_1\,]^2}}{(\,1 + [\,z_0 z_1\,]\,[\,z_1 z_2\,]\,)(\,1 + [\,z_0 z_1\,]\,[\,z_0 z_2\,]\,)}$$

$$= \frac{u^{(2)}\sqrt{1 + (u^{(1)})^2}}{\left[\,1 + (u^{(1)})^2 + \frac{1}{2}u^{(1)}u^{(2)}(x_2 - x_0)\right]\left[\,1 + (u^{(1)})^2 + \frac{1}{2}u^{(1)}u^{(2)}(x_2 - x_1)\,\right]}$$

Euclidean–invariant numerical approximation to the Euclidean curvature:

$$\lim_{z_1, z_2 \longrightarrow z_0} \; I^{(2)} = \kappa = \frac{u^{(2)}}{(1 + (u^{(1)})^2)^{3/2}}$$

Similarly, the third order multi-invariant

$$I^{(3)} = 6\,[\,I_0 I_1 I_2 I_3\,] = 6\,\frac{[\,I_0 I_1 I_3\,] - [\,I_0 I_1 I_2\,]}{H_3 - H_2}$$

will form a Euclidean–invariant approximation for the normalized differential invariant

$$\kappa_s = \iota(u_{xxx})$$

Invariantization of Numerical Schemes

Suppose we are given a numerical scheme for integrating a differential equation, e.g., a Runge–Kutta Method for ordinary differential equations, or the Crank–Nicolson method for parabolic partial differential equations.

If G is a symmetry group of the differential equation, then one can use an appropriate moving frame to invariantize the numerical scheme, leading to an invariant numerical scheme that preserves the symmetry group.

In challenging regimes, the resulting invariantized numerical scheme can, with an inspired choice of moving frame, perform significantly better than its progenitor.

The group G acts on M — the space of independent and dependent variables, and hence on the joint space

$$M^{\diamondsuit(n+1)} = \{\, z_i \neq z_j \text{ for all } i \neq j \,\} \subset M^{\times(n+1)} = M \times \cdots \times M$$

All finite difference numerical schemes are prescribed by suitable functions

$$F \colon M^{\diamondsuit(n+1)} \longrightarrow \mathbb{R}$$

An invariantized scheme is simply obtained by invariantizing the functions $I = \iota(F)$, which is, thus, automatically a function of the joint invariants of G.

- The moving frame and hence the induced invariantization process depends on the choice of cross-section $K \subset M^{\diamond(n+1)}$ to the group orbits.
- Intelligent choice of cross-section requires some understanding of the error terms in the numerical schemes.
- For ordinary differential equations, if the scheme is of order n in the underlying step size h, so that the local truncation error term is $\mathrm{O}(h^{n+1})$ one may, in very favorable circumstances, be able to choose the cross-section to eliminate all the lowest order error terms and thereby produce an invariantized scheme of order $n+1$.

- A more common scenario is that, because of the large number of terms forming the local error, one cannot eliminate all the low order terms, but one may be able to eliminate a large fraction of them by choice of cross-section. For instance, many terms in the error formula depend on u_x, and so a good choice of cross-section would be to set (among other normalizations)

$$u_x = 0$$

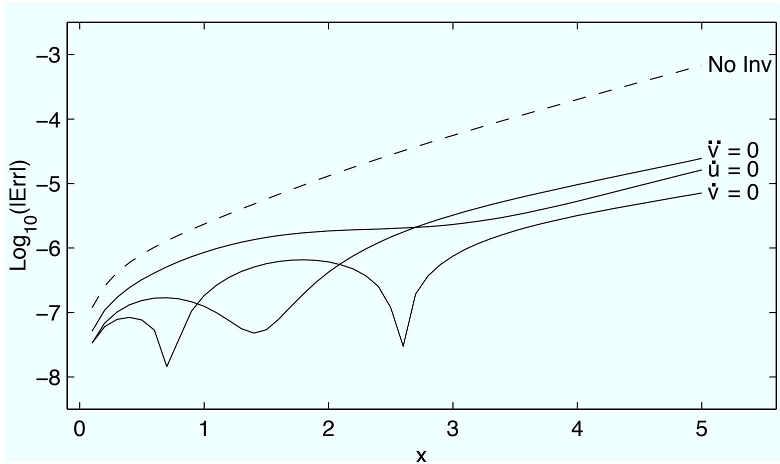
  and thereby eliminate all such terms in the error. The result is a scheme of the same order, but one in which the error tends be be less. And in many case, the result is a much better scheme.

- The choice of cross-section is crucial. In some cases, we have used adaptively invariantized schemes, where the cross-section depends on the local behavior of the solution, to facilitate the construction & improve the accuracy.

- An invariantized scheme is easy to program!

  One merely replaces all stored quantities by their invariantizations and then runs the original non-invariant program on the invariantized data.

Invariantized Runge–Kutta Schemes

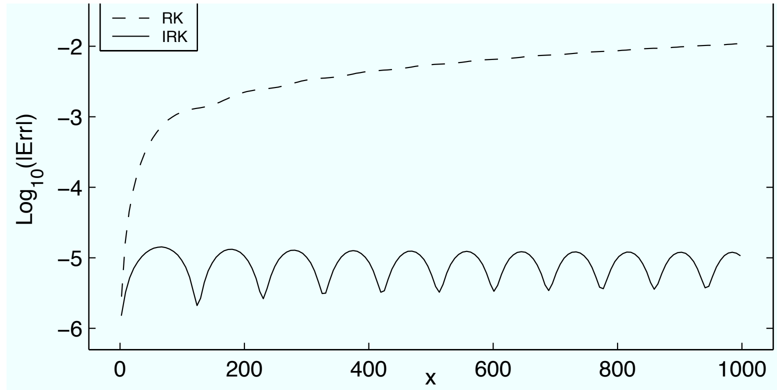


$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

Symmetry group:

$$(x, u) \quad \longmapsto \quad (x, u + \lambda\, e^x)$$

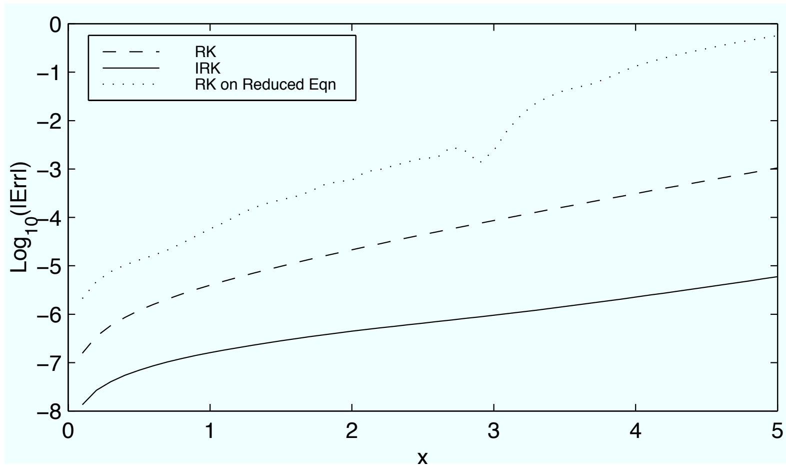
Invariantization for Driven Oscillator



$$u_{xx} + u = \sin x^{\alpha}, \qquad \alpha = .99$$

$$(x, u) \quad \longmapsto \quad (x, u + \lambda \cos x + \mu \sin x)$$

Comparison of symmetry reduction and invariantization

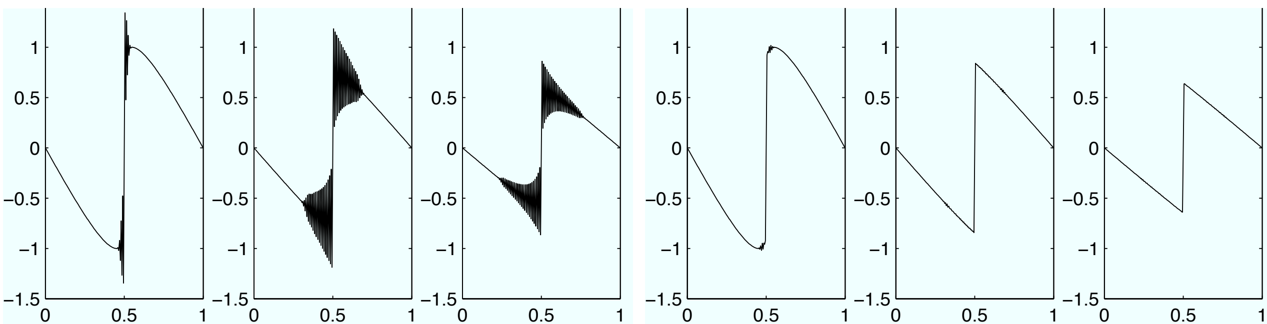


$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

Adaptive Invariantization of Crank–Nicolson for Burgers' Equation

$$u_t = \varepsilon\, u_{xx} + u\, u_x$$

$$(t, x, u) \quad \longmapsto \quad \begin{cases} (t,\, x + \lambda\, t,\, u - \lambda) \quad \text{Galilean boost} \\[1em] \left(\dfrac{t}{1 - \mu\, t}, \dfrac{x}{1 - \mu\, t}, (1 - \mu\, t) u - \mu\, x \right) \end{cases}$$



Non-invariant

Invariantized.

Mathematical Morphology

$u(x)$ — B/W image

Dilation : $$u \oplus S(x) = \max_{y \in S} \; u(x+y)$$

Erosion : $$u \ominus S(x) = \min_{y \in S} \; u(x+y)$$

e.g. $S = \text{disk}$

Morphological PDEs

Hamilton–Jacobi partial differential equation:

$$u_t = \pm\,|\nabla u|$$

Symmetry Group:

$$u \quad \longmapsto \quad \varphi(u)$$

Here, we focus on the one-parameter subgroup

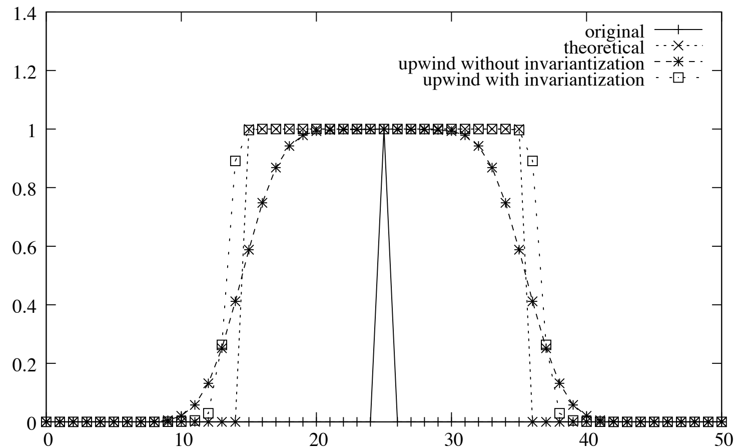
$$u \;\longmapsto\; \frac{\lambda u}{1 + (\lambda - 1)u}$$

Invariantization of 1D Morphology

Upwind scheme:

$$u_t = |u_x|$$

$$u_i^{k+1} = u_i^k + \frac{\Delta t}{\Delta x} \max\{u_{i+1}^k - u_i^k, u_{i-1}^k - u_i^k, 0\} \, .$$

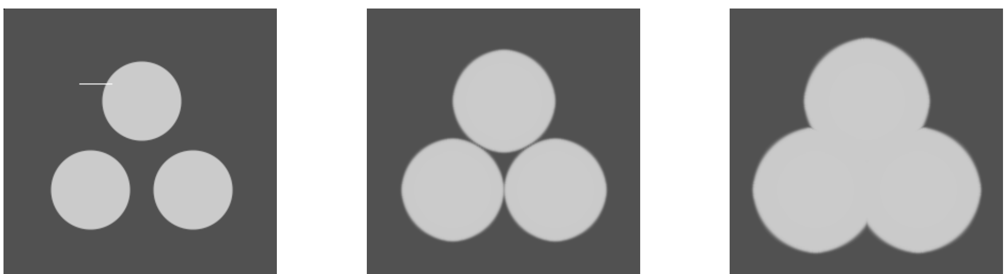


1D dilation of a single peak, 20 iterations, $\Delta t = \Delta x = 0.5$, without and with invariantization.

Invariantization of 2D Morphology

Non-invariant upwind scheme:

Invariantized upwind scheme:



Dilation with invariantized upwind scheme