Poisson Structures and Integrability

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Hamiltonian Systems

M — phase space; $\dim M = 2n$

Local coordinates: $z = (p, q) = (p^1, \ldots, p^n, q^1, \ldots, q^n)$

Canonical Hamiltonian system:

$$\frac{dz}{dt} = J\,\nabla H \qquad J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}$$

Equivalently:

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i} \qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i}$$

Lagrange Bracket (1808):

$$[\, u \,, v \,] = \sum_{i=1}^{n} \frac{\partial p^i}{\partial u}\frac{\partial q^i}{\partial v} - \frac{\partial q^i}{\partial u}\frac{\partial p^i}{\partial v}$$

(Canonical) Poisson Bracket (1809):

$$\{\, u \,, v \,\} = \sum_{i=1}^{n} \frac{\partial u}{\partial p^i}\frac{\partial v}{\partial q^i} - \frac{\partial u}{\partial q^i}\frac{\partial v}{\partial p^i}$$

Given functions $u_1, \ldots, u_{2n}$, the $(2n) \times (2n)$ matrices with respective entries

$$[\, u_i \,, u_j \,] \qquad \{\, u_i \,, u_j \,\} \qquad i, j = 1, \ldots, 2n$$

are mutually inverse.

Canonical Poisson Bracket

$$\{ F, H \} = \nabla F^T J \, \nabla H = \sum_{i=1}^{n} \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i}$$

$\Longrightarrow$ Poisson (1809)

Hamiltonian flow:

$$\frac{dz}{dt} = \{ z, H \} = J \, \nabla H$$

$\Longrightarrow$ Hamilton (1834)

First integral:

$$\{ F, H \} = 0 \quad \Longleftrightarrow \quad \frac{dF}{dt} = 0 \quad \Longleftrightarrow \quad F(z(t)) = \text{const.}$$

Poisson Brackets

$$\{\,\cdot\,,\,\cdot\,\}\colon \mathrm{C}^\infty(M,\mathbb{R}) \times \mathrm{C}^\infty(M,\mathbb{R}) \longrightarrow \mathrm{C}^\infty(M,\mathbb{R})$$

Bilinear:

$$\{\, a\,F + b\,G,\; H \,\} = a\,\{\, F,\; H \,\} + b\,\{\, G,\; H \,\}$$

$$\{\, F,\; a\,G + b\,H \,\} = a\,\{\, F,\; G \,\} + b\,\{\, F,\; H \,\}$$

Skew Symmetric: $\qquad \{\, F,\; H \,\} = -\,\{\, H,\; F \,\}$

Jacobi Identity:

$$\{\, F,\; \{\, G,\; H \,\}\,\} + \{\, H,\; \{\, F,\; G \,\}\,\} + \{\, G,\; \{\, H,\; F \,\}\,\} = 0$$

Derivation: $\qquad \{\, F,\; G\,H \,\} = \{\, F,\; G \,\}\,H + G\,\{\, F,\; H \,\}$

$$F, G, H \in \mathrm{C}^\infty(M,\mathbb{R}), \qquad a, b \in \mathbb{R}.$$

In coordinates $z = (z^1, \dots, z^m)$,

$$\{ F, H \} = \nabla F^T J(z) \nabla H$$

where $J(z)^T = -J(z)$ is a skew symmetric matrix.

The Jacobi identity imposes a system of quadratically nonlinear partial differential equations on its entries:

$$\sum_l \left(J^{il} \frac{\partial J^{jk}}{\partial z^l} + J^{jl} \frac{\partial J^{ki}}{\partial z^l} + J^{kl} \frac{\partial J^{ij}}{\partial z^l} \right) = 0$$

Given a Poisson structure, the Hamiltonian flow corresponding to $H \in \mathrm{C}^\infty(M, \mathbb{R})$ is the system of ordinary differential equations

$$\frac{dz}{dt} = \{ z, H \} = J(z) \nabla H$$

Lie's Theory of Function Groups

Used for integration of partial differential equations:

$$\{ F_i, F_j \} = G_{ij}(F_1, \ldots, F_n)$$

$\Longrightarrow$ predates Lie groups!!

Ausgezeichnete Functionen = distinguished functions
= Casimirs

$$\{ F, C \} = 0 \qquad \text{for all} \qquad F \in \mathrm{C}^\infty(M, \mathbb{R})$$

$\star\star$ All distinguished functions are first integrals (conservation laws) of any associated Hamiltonian system.

Darboux' Theorem

If $J(z)$ has constant rank, then there exist local coordinates $z = (p, q, y)$ such that the Poisson bracket is in canonical form

$$\{ F, H \} = \sum_{i=1}^{n} \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i}$$

Canonical degenerate Hamiltonian system:

$$\frac{dz}{dt} = J \nabla H \qquad J = \begin{pmatrix} O & -I & O \\ I & O & O \\ O & O & O \end{pmatrix}$$

Equivalently:

$$\frac{dp^i}{dt} = - \frac{\partial H}{\partial q^i} \qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p^i} \qquad \frac{dy^j}{dt} = 0$$

Distinguished (Casimir) functions: $F(y) = \text{const.}$

Variable rank: Weinstein, Conn.

Symplectic Structures

A two form $\Omega \in \wedge^2 T^* M$ is called *symplectic* if it is

- closed: $d\Omega = 0$, and
- nondegenerate: $\Omega \wedge \cdots \wedge \Omega \neq 0$.

Nondegeneracy implies that Ω defines an isomorphism
$TM \simeq T^* M$, mapping
the Hamiltonian vector field $\mathbf{v}_H$ to $dH = \mathbf{v}_H \,\lrcorner\, \Omega$.

In coordinates $\Omega = dz^T \wedge J(z)^{-1} dz$ and $\mathbf{v}_H = J(z) \nabla H(z) \, \partial_z$.

The associated nondegenerate Poisson bracket:

$$\{ F, H \} = \langle \mathbf{v}_F \wedge \mathbf{v}_H \, ; \Omega \rangle = \nabla F^T J(z) \, \nabla H$$

Lie–Poisson Structure

- Originally due to Lie
- Rediscovered by Kirillov, Kostant, Souriau, ...

$\mathfrak{g}$ — r-dimensional Lie algebra

$M = \mathfrak{g}^* \simeq \mathbb{R}^r$ — dual vector space

$$\{ F, H \} = \langle z ; [\nabla F(z), \nabla H(z)] \rangle$$

$z \in \mathfrak{g}^*$ $\qquad$ $F(z), H(z) \in \mathrm{C}^\infty(\mathfrak{g}^*, \mathbb{R})$

$\nabla F(z) \in \mathfrak{g}$ $\qquad$ $[\nabla F(z), \nabla H(z)]$ — Lie bracket in $\mathfrak{g}$

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In coordinates: $J_{ij}(z) = \sum_{k=1}^{r} c_{ij}^k z_k, \qquad z = z_1 \mu^1 + \cdots \ z_r \mu^r \in \mathfrak{g}^*$

c_{ij}^k — structure constants

$\mu^1, \ldots, \mu^r$ — Maurer–Cartan forms

The Euler Equations

$\Longrightarrow$ Arnold

Let $G = \mathrm{SDiff}(M)$ be the infinite-dimensional pseudo-group of volume preserving diffeomorphisms of a Riemannian manifold M.

Lie algebra: $\mathfrak{g}$ = divergence-free vector fields.

Hamiltonian functional on $\mathfrak{g}^*$: the L^2 norm.

★ The corresponding Lie–Poisson flow is equivalent to the Euler equations governing the motion of an incompressible fluid.

In $\mathbb{R}^n$:

$$u_t + u \cdot \nabla u = -\nabla p \qquad \operatorname{div} u = 0$$

u — fluid velocity $\qquad p$ — pressure

$\Longrightarrow$ Ebin, Marsden

Vorticity Equations

u — fluid velocity $\qquad\qquad$ $\omega = \nabla \wedge u$ — vorticity

$$\frac{\partial \omega}{\partial t} = \omega \cdot \nabla u - u \cdot \nabla \omega = J \, \frac{\delta H}{\delta \omega}$$

$H = \int \frac{1}{2} |\, u \,|^2 \, d\mathbf{x}$ — kinetic energy

$J = (\, \omega \cdot \nabla - \nabla \omega \,) \, \nabla \wedge \cdot$ — Poisson operator

Distinguished (Casimir) functions:

2D: $\displaystyle \iint f(\omega) \, dx \, dy$ $\qquad$ 3D: $\displaystyle \iiint (u \cdot \omega) \, dx \, dy \, dz$ — helicity

Camassa–Holm Equation and Euler–Poincaré Flows

Use the H^1 norm as the Hamiltonian for the Lie–Poisson structure on the diffeomorphism pseudo-group:

$$\mathcal{H}[u] = \int \left(u^2 + \alpha \,|\, \nabla u \,|^2 \right) dx$$

Camassa–Holm Equation:

$$u_t \pm u_{xxt} = 3\, u\, u_x \pm \left(u\, u_{xx} + \tfrac{1}{2} u_x^2 \right)_x$$

$\Longrightarrow$ nonanalytic solutions — peakons or compactons

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α models: regularized Euler; geophysics, magnetohydrodynamics, computational anatomy,mathematical morphology

Poisson Bivector Field

$\Longrightarrow$ Lichnerowicz

$$\Theta = \partial_z^T \wedge J(z)\, \partial_z \in \Lambda^2 TM$$

Poisson bracket:

$$\{\, F,\ H \,\} = \langle\, \Theta\, ; dF \wedge dH \,\rangle$$

★ Bilinearity, skew symmetry and derivation properties are automatic.

Theorem. The Jacobi identity holds if and only if

$$[\, \Theta, \Theta \,] = 0$$

where $[\,\cdot\,,\cdot\,]$ denotes the Schouten bracket.

The Schouten Bracket

The Schouten bracket $[\,\Theta, \Psi\,]$ is the natural extension of the Lie bracket $[\,\mathbf{v}, \mathbf{w}\,]$ to multi-vector fields (sections of $\bigwedge^n TM$):

- Bilinear.
- Super-symmetric:

$$[\,\Theta, \Psi\,] = (-1)^{\deg \Theta \deg \Psi} [\,\Psi, \Theta\,] \in \textstyle\bigwedge^{\deg \Theta + \deg \Psi - 1} TM$$

- Super-Jacobi identity:

$$(-1)^{\deg \Theta \deg \Xi} [\,[\,\Theta, \Psi\,], \Xi\,] + (-1)^{\deg \Xi \deg \Psi} [\,[\,\Xi, \Theta\,], \Psi\,] + \\ + (-1)^{\deg \Psi \deg \Theta} [\,[\,\Psi, \Xi\,], \Theta\,] = 0$$

- Super-derivation:

$$[\,\Theta, \Psi \wedge \Xi\,] = [\,\Theta, \Psi\,] \wedge \Xi + (-1)^{\deg \Psi (\deg \Theta - 1)} \Psi \wedge [\,\Theta, \Xi\,]$$

The Poisson Complex

$$\mathbb{R} \to \bigwedge^0 TM \xrightarrow{\delta_\Theta} \bigwedge^1 TM \xrightarrow{\delta_\Theta} \bigwedge^2 TM \xrightarrow{\delta_\Theta} \cdots$$

Suppose Θ is a Poisson bivector:

$$[\,\Theta, \Theta\,] = 0$$

Poisson derivation:

$$\delta_\Theta(\Psi) = [\,\Theta, \Psi\,]$$

Closure: By super-Jacobi:

$$\delta_\Theta^2(\Psi) = [\,\Theta, [\,\Theta, \Psi\,]\,] = -[\,\Theta, [\,\Theta, \Psi\,]\,] - (-1)^{\deg\Psi}[\,\Psi, [\,\Theta, \Theta\,]\,] = 0$$

In particular, applying δ_Θ to a function gives the associated Hamiltonian vector field:

$$\delta_\Theta(H) = [\,\Theta, H\,] = \mathbf{v}_H$$

Poisson Cohomology

Unless Θ is nondegenerate, the Poisson complex is not locally exact. If Θ has constant rank, the local cohomology involves the distinguished functions and forms.

★ Poisson cohomology is poorly understood in general.

A nondegenerate Θ defines an algebra isomorphism $\bigwedge^k TM \simeq \bigwedge^k T^*M$ via $\omega \longmapsto \mathbf{v} = \omega \lrcorner \Theta$. In this case, the Poisson complex is isomorphic to the de Rham complex:

$$\begin{array}{ccccccccc}
\mathbb{R} \to \bigwedge^0 T^*M & \xrightarrow{\quad d \quad} & \bigwedge^1 T^*M & \xrightarrow{\quad d \quad} & \bigwedge^2 T^*M & \xrightarrow{\quad d \quad} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{R} \to \bigwedge^0 TM & \xrightarrow{\quad \delta_\Theta \quad} & \bigwedge^1 TM & \xrightarrow{\quad \delta_\Theta \quad} & \bigwedge^2 TM & \xrightarrow{\quad \delta_\Theta \quad} & \cdots
\end{array}$$

BiHamiltonian Systems

Definition. A system of first order differential equations is called biHamiltonian if it can be written in Hamiltonian form in two distinct ways:

$$\frac{du}{dt} = J_1 \nabla H_1 = J_2 \nabla H_0$$

Thus both J_1 and J_2 define Poisson brackets, which we assume are not constant multiples of each other.

$\Longrightarrow$ Both Hamiltonians H_1 and H_2 are conserved.

Infinite Toda Lattice

$$H = \sum_i \left(\tfrac{1}{2} p_i^2 + e^{q_{i-1} - q_i} \right)$$

$$a_i = \tfrac{1}{2} e^{(q_{i-1} - q_i)/2} \qquad b_i = \tfrac{1}{2} p_{i-1} \qquad \Longrightarrow \quad \text{Flaschka}$$

$$\frac{da_i}{dt} = a_i (b_{i+1} - b_i) \qquad \frac{db_i}{dt} = 2(a_i^2 - a_{i-1}^2)$$

$$H_1 = \sum_i \left(a_i^2 + \tfrac{1}{2} b_i^2 \right) \qquad H_0 = \sum_i b_i$$

$$J_1 = \begin{pmatrix} 0 & a(T_+ - 1) \\ (1 - T_-)a & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} \frac{1}{2} a(T_+ - T_-)a & a(T_+ - 1)b \\ b(1 - T_-)a & 2(a^2 T_+ - T_- a^2) \end{pmatrix}$$

T_+, T_- — shift operators

Compatibility

- J_1, J_2, and $J_1 + J_2$ are all Poisson
- The Poisson bivectors satisfy

$$[\Theta_1, \Theta_1] = [\Theta_2, \Theta_2] = [\Theta_1, \Theta_2] = 0$$

- The recursion operator $R = J_2 J_1^{-1}$ satisfies the Nijenhuis torsion condition

$$R^2[\,\mathbf{v}, \mathbf{w}\,] - R[\,R\mathbf{v}, \mathbf{w}\,] - R[\,\mathbf{v}, R\mathbf{w}\,] + [\,R\mathbf{v}, R\mathbf{w}\,] = 0$$

- The symplectic forms satisfy

$$d\Omega_1 = d\Omega_2 = d(\Omega_1^{-1} + \Omega_2^{-1})^{-1} = 0$$

Theorem. (*Magri*) Suppose

$$\frac{du}{dt} = J_1 \nabla H_1 = J_2 \nabla H_0$$

is a biHamiltonian system, where J_1, J_2 form a compatible pair of Hamiltonian operators. Assume that J_1 is nondegenerate, and define the recursion operator $R = J_2 J_1^{-1}$. Then there exist an infinite sequence of conserved Hamiltonians $H_0, H_1, H_2, \ldots$ such that

- Each associated flow is a biHamiltonian system

$$\frac{du}{dt} = F_n = J_1 \nabla H_n = J_2 \nabla H_{n-1} = R\, F_{n-1}$$

- The Hamiltonians are in involution with respect to either Poisson bracket:

$$\{\, H_n, \, H_m \,\}_1 = 0 = \{\, H_n, \, H_m \,\}_2$$

  and hence conserved by all flows.

- The flows mutually commute.

Proof: *Recursion*: Starting at $n = 1$, the n^{th} flow comes from the vector field

$$\mathbf{v}_n = [\,\Theta_2, H_{n-1}\,] = [\,\Theta_1, H_n\,]$$

Set $\mathbf{v}_{n+1} = [\,\Theta_2, H_n\,]$. Then, by super-Jacobi, compatibility, and closure

$$[\,\Theta_1, \mathbf{v}_{n+1}\,] = [\,\Theta_1, [\,\Theta_2, H_n\,]\,] = -[\,\Theta_2, [\,\Theta_1, H_{n-1}\,]\,] - [\,[\,\Theta_1, \Theta_2\,], H_{n-1}\,]$$
$$= -[\,\Theta_2, \mathbf{v}_{n-1}\,] = -[\,\Theta_2, [\,\Theta_2, H_{n-2}\,]\,] = 0$$

Then, by exactness of the Θ_1–Poisson complex $\mathbf{v}_{n+1} = [\,\Theta_1, H_{n+1}\,]$ for some H_{n+1}.

Conservation: Using compatibility:

$$\mathbf{v}_n(H_m) = [\,\mathbf{v}_n, H_m\,] = [\,[\,\Theta_2, H_{n-1}\,], H_m\,] = -[\,H_{n-1}, [\,\Theta_2, H_m\,]\,]$$
$$= -[\,H_{n-1}, [\,\Theta_1, H_{m+1}\,]\,] = [\,[\,\Theta_1, H_{n-1}\,], H_{m+1}\,] = \mathbf{v}_{n-1}(H_{m+1})$$

and repeat ... to reduce to $\mathbf{v}_n(H_n) = \mathbf{v}_n(H_{n-1}) = 0$.

Completely Integrable Systems

Definition. A nondegenerate Hamiltonian system $u_t = J\,\nabla H$ on an $2n$ dimensional phase space is called completely integrable if there exist n first integrals $H = F_1, F_2, \ldots F_n$ that are in *involution*

$$\{\, F_i,\, F_j\,\} = 0$$

Is a Completely Integrable System Necessarily BiHamiltonian?

Theorem. (*Fernandes*) A completely integrable Hamiltonian system is biHamiltonian in the neighborhood of an invariant torus if and only if the graph of its Hamiltonian function is a hypersurface of translation relative to the affine structure determined by the action variables $(s^1, \ldots, s^n)$:

$$\begin{aligned} s^i &= a_1^i(y^1) + \cdots + a_n^i(y^n), \\ H(s^1, \ldots, s^n) &= \phi_1(y^1) + \cdots + \phi_n(y^n). \end{aligned}$$

Example. The perturbed Kepler problem

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - \frac{1}{r} + \frac{\varepsilon}{2r^2}$$

★ completely integrable for all ε; biHamiltonian only when $\varepsilon = 0$

Incompatible BiHamiltonian Systems

Ω_1, Ω_2 — symplectic two-forms on $\mathbb{C}^4$

$$\Omega_1 \wedge \Omega_2 \neq 0$$

Canonical forms (Debever):

$$\Omega_1 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

$$\Omega_2 = \begin{cases} dp_1 \wedge dq_1 - dp_2 \wedge dq_2 \\ \\ e^{p_1}(\, dp_1 \wedge dq_1 - dp_2 \wedge dq_2 - p_2 dp_1 \wedge dq_2 \,) \\ \\ e^{p_1+p_2}(\, dp_1 \wedge dq_1 - dp_2 \wedge dq_2 + (q_1 + q_2) dp_1 \wedge dp_2 \,) \end{cases}$$

$\Longrightarrow$ The first pair are compatible, but the latter two are not.

Bi-Hamiltonians:

$$H_1 = f(p_1, q_1) + g(p_2, q_2)$$

$$H_2 = f(p_2 e^{p_1/2}, q_2) e^{-p_1/2} + g(p_1)$$

$$H_3 = c(q_1 - q_2) + f(p_1, p_2) \qquad \frac{\partial^2 f}{\partial p_1 \, \partial p_2} + \frac{\partial f}{\partial p_1} + \frac{\partial f}{\partial p_2} = 0.$$

Poisson brackets for Field Theories

$$\mathcal{H}[u] = \int H[u]\,dx \qquad \text{— Hamiltonian functional}$$

$$\frac{\delta \mathcal{H}}{\delta u} = E(H)$$

— variational derivative = Euler–Lagrange expression

$$\{\mathcal{F}, \mathcal{H}\} = \int \left(\frac{\delta \mathcal{F}}{\delta u}\, J\, \frac{\delta \mathcal{H}}{\delta u} \right) dx \qquad \text{— Poisson bracket}$$

J — Poisson (differential) operator

$\Longrightarrow$ Formally skew–adjoint: $J^* = -J$

$\Longrightarrow$ Jacobi identity ⋆⋆

Korteweg–deVries Equation

$$\frac{\partial u}{\partial t} = u_{xxx} + u\,u_x = J_1\,\frac{\delta \mathcal{H}_1}{\delta u} = J_2\,\frac{\delta \mathcal{H}_2}{\delta u}$$

$$J_1 = D_x \qquad\qquad \mathcal{H}_1[u] = \int \left(\tfrac{1}{6}\,u^3 - \tfrac{1}{2}\,u_x^2 \right) dx$$

$$J_2 = D_x^3 + \tfrac{2}{3} u\,D_x + \tfrac{1}{3}\,u_x \qquad\qquad \mathcal{H}_2[u] = \int \tfrac{1}{2}\,u^2\,dx$$

★★ Bi–Hamiltonian system with recursion operator

$$\mathcal{R} = J_2 \cdot J_1^{-1} = D_x^2 + \tfrac{2}{3} u + \tfrac{1}{3}\,u_x\,D_x^{-1}$$

$\Longrightarrow$ Gardner, Lax, Lenard, PJO, Magri, Gel'fand–Dikii, Adler, ...

$\Longrightarrow$ Lie–Poisson structure on the Lie algebra of pseudo-differential operators (Virasoro algebra)

Poisson Functional Multi-vectors

$$\Theta = \int \theta \wedge J(\theta)\, dx$$

In general, $\int D_x \Omega\, dx = 0$ — work mod image of D_x

Schouten bracket:

$$0 = [\, \Theta, \Theta\,] = \int \operatorname{pr} \mathbf{v}_{J\theta}[\, \theta \wedge J(\theta)\,]\, dx$$

- prolonged evolutionary "vector field":

  $\operatorname{pr} \mathbf{v}_{J\theta}$ commutes with D_x and $\operatorname{pr} \mathbf{v}_{J\theta}(\theta) = 0$.

Lemma. Any constant coefficient, skew–adjoint differential operator is Poisson.

Example. Second KdV Hamiltonian operator:

$$J = D_x^3 + \tfrac{2}{3}\, u\, D_x + \tfrac{1}{3}\, u_x$$

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Functional multi-vector:

$$\Theta = \int \theta \wedge \theta_{xxx} + \tfrac{2}{3}\, u\, \theta \wedge \theta_x \, dx$$

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Functional multi-vector:

$$\Theta = \int \theta \wedge \theta_{xxx} + \tfrac{2}{3}\, u\, \theta \wedge \theta_x \, dx$$

Schouten bracket:

$$[\, \Theta, \Theta \,] = \int \operatorname{pr} \mathbf{v}_{\theta_{xxx} + \frac{2}{3} u\, \theta + \frac{1}{3} u_x\, \theta}(u) \wedge \theta \wedge \theta_x \, dx$$

$$= \int (\theta_{xxx} + \tfrac{2}{3} u\, \theta + \tfrac{1}{3} u_x\, \theta) \wedge \theta \wedge \theta_x \, dx$$

$$= \int \theta_{xxx} \wedge \theta \wedge \theta_x \, dx = 0$$

since

$$\theta_{xxx} \wedge \theta \wedge \theta_x = D_x(\theta_{xx} \wedge \theta \wedge \theta_x)$$

First Order Poisson Operators

$\Longrightarrow$ Dubrovin, Novikov

Field variables: $u(x) = (u^1(x), \ldots, u^n(x))$

$$J_{ij} = g^{ij}(u)\, D_x + b^{ij}_k(u)\, u^k_x$$

Nondegenerate Poisson operator:

- $g^{ij} = g^{ji}$ — flat (pseudo-)Riemannian metric
- $b^{ij}_k = \sum_l g^{il}\, \Gamma^j_{lk}$ — Christoffel symbols (connection)

Hyperbolic systems of hydrodynamic type:

$$\frac{\partial u}{\partial t} = J\, \frac{\delta \mathcal{H}}{\delta u}$$

Nonlinear Transport

$\Longrightarrow$ Nutku, PJO

$$\boxed{u_t = u\, u_x}$$

Conserved densities:

$$H_n(u) = \frac{1}{n}\, u^n$$

Hamiltonian structures:

$$J_0 = D_x$$

$$J_1 = 2\, u\, D_x + u_x$$

$$J_2 = u^2\, D_x + u\, u_x$$

$$J_3 = D_x\, \frac{1}{u_x}\, D_x\, \frac{1}{u_x}\, D_x$$

Hamiltonian flows:

$$u_t = V_n = u^n\, u_x = J_0 \delta H_{n+1} = J_1 \delta H_n = J_2 \delta H_{n-1} = J_3 \delta H_{n+3}$$

- J_0, J_1, J_2 are mutually compatible
- J_0, J_3 are compatible
- J_1, J_3 and J_2, J_3 are not compatible
- $J_3 J_0^{-1} = R^2$ $\qquad R = D_x \dfrac{1}{u_x}$ — recursion operator

Rational flows: $u_t = \widehat{V}_2 = \dfrac{u_{xx}}{u_x}$ $\qquad u_t = \widehat{V}_n = R^{n-2} \widehat{V}_2$

Rational conserved densities: $\widehat{H}_1 = \dfrac{1}{u_x}$

$$\widehat{V}_{2j+1} = J_0\, \delta \widehat{H}_j = J_3\, \delta \widehat{H}_{j-1}$$

2–D Hyperbolic Systems

$$\boxed{\mathbf{u}_t = J_0\,\delta\mathcal{H}/\delta\mathbf{u}}$$

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad\qquad \mathcal{H}[\mathbf{u}] = \int H(u,v)\,dx$$

$$J_0 = \sigma_1 D_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_x$$

$$\frac{\partial u}{\partial t} = D_x\,\frac{\delta H}{\delta v} \qquad\qquad \frac{\partial v}{\partial t} = D_x\,\frac{\delta H}{\delta u}$$

Examples

Gas dynamics: $H(u,v) = -\frac{1}{2}u^2 v + F(v)$

- Polytropic: $F(v) = \dfrac{v^\gamma}{\gamma(\gamma-1)}$
- Shallow water: $F(v) = \frac{1}{2}v^2, \qquad \gamma = 2.$

Elastodynamics:

$$H(u,v) = \tfrac{1}{2}u^2 + F(v)$$

- van der Waals fluid; acoustic Euler equation

Born–Infeld equation:

$$H(u,v) = \frac{u}{v} + \frac{v}{u}$$

$\Longrightarrow$ Chaplygin gas $\gamma = -1$

Zero-th Order Conservation Laws

Theorem. $F(u,v)$ is a conserved density if and only if

$$H_{uu}\,F_{vv} = H_{vv}\,F_{uu}$$

Separable Hyperbolic System:

$$\frac{H_{uu}}{H_{vv}} = \frac{\lambda(u)}{\mu(v)}$$

$\lambda(u) \equiv 1$ — generalized gas dynamics

Higher Order Hamiltonian Structure

Separable:

$$\frac{H_{uu}}{H_{vv}} = \frac{\lambda(u)}{\mu(v)}$$

$$L(u) = \int \lambda(u)\,du \qquad M(v) = \int \mu(v)\,dv$$

$$U(u,v) = \begin{pmatrix} u & M(v \\ v & L(u) \end{pmatrix} \qquad V(u,v) = \begin{pmatrix} L(u) & M(v \\ v & u \end{pmatrix}$$

Poisson operator:

$$J_3 = D_x\, V_x^{-1}\, D_x\, U_x^{-1}\, \sigma_1\, D_x$$

BiHamiltonian system:

$$\mathbf{u}_1 = J_0\,\delta H = J_3\,\delta \widehat{H}$$

Recursion operator (Sheftel'):

$$\widehat{R} = J_3 J_0^{-1} = D_x V_x^{-1} D_x U_x^{-1}$$

For gas dynamics, $U = V$ and $\widehat{R} = R^2$ where $R = D_x U_x^{-1}$ is a recursion operator

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There are also two other first order Poisson operators J_1, J_2, along with hierarchies of zeroth order polynomial conservation laws and higher order rational conservation laws, e.g.

$$\widehat{H}_1 = \frac{v_x}{u_x^2 - \mu(v)\, v_x^2}$$

$\Longrightarrow$ Verosky

Deformed Lie–Poisson Structure

$u(x) \in \mathrm{C}^\infty(\mathbb{R}, \mathfrak{g}^*)$ — curve in $\mathfrak{g}^*$ — dual to Lie algebra

Poisson bracket:

$$\{\,\mathcal{F}, \mathcal{H}\,\} = \int \left(\frac{\delta \mathcal{F}}{\delta u}\, \mathcal{P}\, \frac{\delta \mathcal{H}}{\delta u} \right) dx$$

Hamiltonian curve flow:

$$\frac{\partial u}{\partial t} = \mathcal{P}\, \frac{\delta \mathcal{H}}{\delta u} = B\, D_x \frac{\delta \mathcal{H}}{\delta u} + \mathrm{ad}^*_{\delta \mathcal{H}/\delta u}(u)$$

$B : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ — ad^*–invariant symmetric linear map

- If $\mathfrak{g}$ is semi-simple, B is a multiple of the Killing form

Noncanonical Perturbation Theory

$$\frac{dv}{dt} = J(v)\nabla H(v)$$

Perturbation expansion:

$$v = u + \varepsilon\,\varphi(u) + \varepsilon^2\,\psi(u) + \cdots$$

$$H(v) = H_0(u) + \varepsilon\,H_1(u) + \varepsilon^2\,H_2(u) + \cdots$$

$$J(v) \;\longmapsto\; J_0(u) + \varepsilon\,J_1(u) + \varepsilon^2\,J_2(u) + \cdots$$

Perturbed system:

$$\frac{du}{dt} = J_0\nabla H_0 + \varepsilon(\,J_1\nabla H_0 + J_0\nabla H_1\,) + \cdots$$

First order Hamiltonian perturbation

$$\frac{du}{dt} = (J_0 + \varepsilon J_1)\nabla(H_0 + \varepsilon H_1)$$

$$= J_0 \nabla H_0 + \varepsilon(J_1 \nabla H_0 + J_0 \nabla H_1) + \varepsilon^2 J_1 \nabla H_1$$

★ There is no guarantee that $J_0 + \varepsilon J_1$ is a Poisson operator.

Theorem. If $J_1 \nabla H_0 = \lambda J_0 \nabla H_1$ and J_0, J_1 are compatible, then the first order perturbation equation

$$\frac{du}{dt} = J_0 \nabla H_0 + \varepsilon(J_1 \nabla H_0 + J_0 \nabla H_1)$$

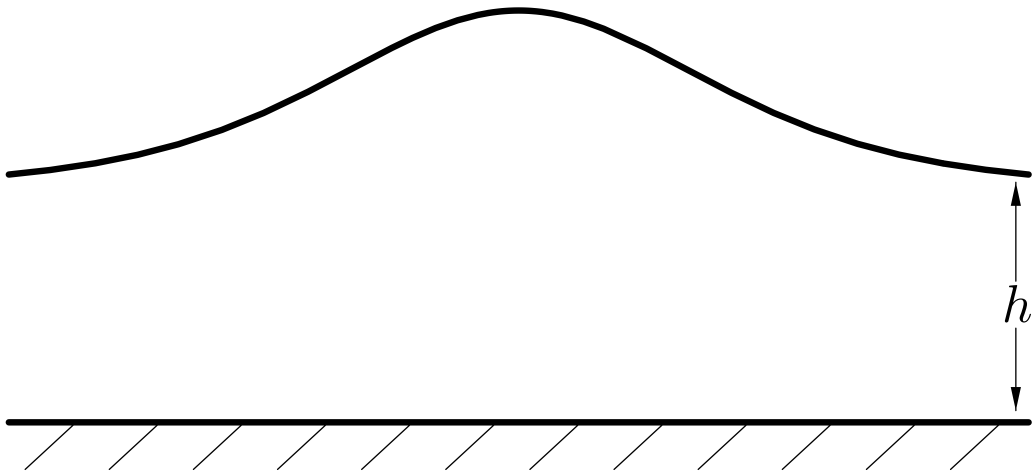
and the first order Hamiltonian perturbation

$$\frac{du}{dt} = J_0 \nabla H_0 + \varepsilon(J_1 \nabla H_0 + J_0 \nabla H_1) + \varepsilon^2 J_1 \nabla H_1$$

are biHamiltonian systems.

$\Longrightarrow$ KdV equation

2D Water Waves



$y = h + \eta(t, x)$ surface elevation

$\phi(t, x, y)$ velocity potential

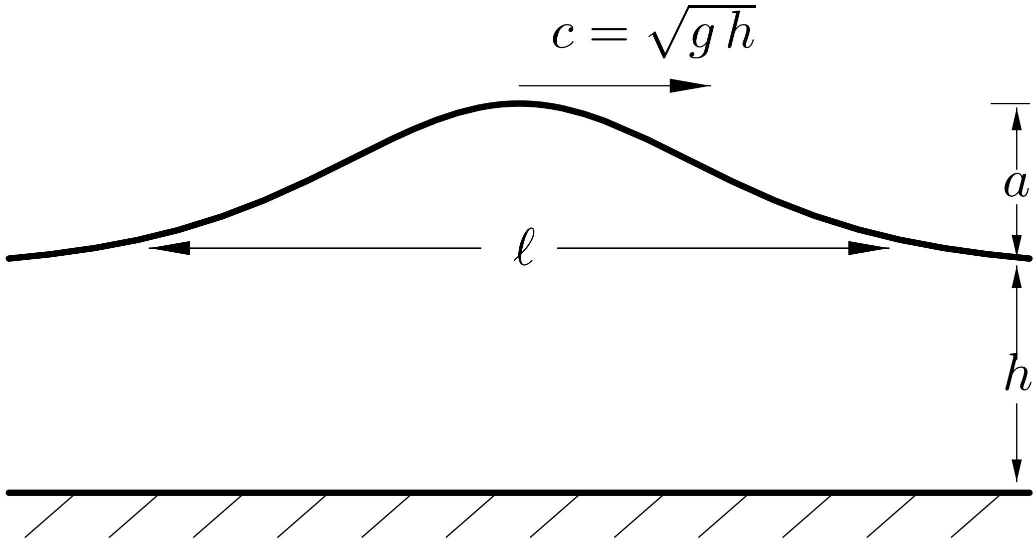
2D Water Waves

- Incompressible, irrotational fluid.
- No surface tension

$$\left.\begin{aligned} \phi_t + \tfrac{1}{2}\,\phi_x^2 + \tfrac{1}{2}\,\phi_y^2 + g\,\eta = 0 \\ \eta_t = \phi_y - \eta_x\phi_x \end{aligned}\right\} \qquad y = h + \eta(t,x)$$

$$\phi_{xx} + \phi_{yy} = 0 \qquad 0 < y < h + \eta(t,x)$$

$$\phi_y = 0 \qquad y = 0$$



Small parameters — long waves in shallow water

(KdV regime)

$$\alpha = \frac{a}{h} \qquad \beta = \frac{h^2}{\ell^2} = O(\alpha)$$

Rescale:

$$x \longmapsto \ell\, x \qquad y \longmapsto h\, y \qquad t \longmapsto \frac{\ell\, t}{c}$$

$$\eta \longmapsto a\, \eta \qquad \phi \longmapsto \frac{g\, a\, \ell\, \phi}{c} \qquad c = \sqrt{g\, h}$$

Rescaled water wave system:

$$\left.\begin{aligned} \phi_t + \frac{\alpha}{2}\,\phi_x^2 + \frac{\alpha}{2\,\beta}\,\phi_y^2 + \eta = 0 \\ \eta_t = \frac{1}{\beta}\,\phi_y - \alpha\,\eta_x\,\phi_x \end{aligned}\right\} \qquad y = 1 + \alpha\,\eta$$

$$\beta\,\phi_{xx} + \phi_{yy} = 0 \qquad 0 < y < 1 + \alpha\,\eta$$

$$\phi_y = 0 \qquad y = 0$$

Boussinesq expansion:

$$w(t,x) = \phi(t,x,0) \qquad u(t,x) = \phi_x(t,x,\theta) \qquad 0 \le \theta \le 1$$

Solve Laplace equation:

$$\phi(t,x,y) = w(t,x) - \frac{\beta^2}{2} y^2 w_{xx} + \frac{\beta^4}{4!} y^4 w_{xxxx} + \cdots$$

Plug expansion into free surface conditions: To first order

$$w_t + \eta + \frac{\alpha}{2} w_x^2 - \frac{\beta}{2} w_{xxt} = 0$$

$$\eta_t + w_x + \alpha(\eta w_x)_x - \frac{\beta}{6} w_{xxxx} = 0$$

Bidirectional Boussinesq system:

$$u_t + \eta_x + \alpha\, u\, u_x - \tfrac{1}{2}\,\beta\,(\theta^2 - 1)\, u_{xxt} = 0$$

$$\eta_t + u_x + \alpha\,(\eta\, u)_x - \tfrac{1}{6}\,\beta(3\,\theta^2 - 1)\, u_{xxx} = 0$$

★ ★ at $\theta = 1$ this system is integrable

(in fact tri-Hamiltonian!!)

$\Longrightarrow$ Kaup–Kupershmidt

Unidirectional waves:

$$u = \eta - \tfrac{1}{4}\,\alpha\,\eta^2 + \left(\tfrac{1}{3} - \tfrac{1}{2}\,\theta^2\right)\beta\,\eta_{xx}$$

Korteweg-deVries (1895) equation:

$$\eta_t + \eta_x + \tfrac{3}{2}\,\alpha\,\eta\,\eta_x + \tfrac{1}{6}\,\beta\,\eta_{xxx} = 0$$

★ ★ Due to Boussinesq in 1877!

Hamiltonian Water Waves

$\Longrightarrow$ Zakharov

$$u = \begin{pmatrix} \phi_S \\ \eta \end{pmatrix}$$

$$\phi_S(x,t) = \phi(x, h + \eta(x,t), t) \quad — \quad \text{surface potential}$$

Hamiltonian functional:

$$\mathcal{H}[u] = \iint_D |\nabla\phi|^2 \, dx \, dy + \int_S \tfrac{1}{2} g \, \eta^2 \, dx$$

kinetic potential

energy

$$\frac{\partial u}{\partial t} = J \, \frac{\delta \mathcal{H}}{\delta u} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta\mathcal{H}/\delta\phi_S \\ \delta\mathcal{H}/\delta\eta \end{pmatrix}$$

Water Waves

Symmetries and Conservation Laws

# spatial dimensions	surface tension	dim. symm. group	# cons. laws
2		9	8
2	$\surd$	8	7
3		13	12
3	$\surd$	12	13

$\Longrightarrow$ T.B. Benjamin - PJO

Symmetries of 2D Water Waves

(1) Horizontal translation: $\frac{\partial}{\partial x}$ $(x+\alpha, y, t, \varphi)$

(2) Time translations: $\frac{\partial}{\partial t}$ $(x, y, t+\alpha, \varphi)$

(3) Change in potential: $\frac{\partial}{\partial \varphi}$ $(x, y, t, \varphi+\alpha)$

(4) Vertical translation: $\frac{\partial}{\partial y} - g\,t\,\frac{\partial}{\partial t}$ $(x, y+\alpha, t, \varphi - \alpha\,g\,t)$

(5) Horizontal Galilean boost:

$$t\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial \varphi} \qquad (\,x+\alpha\,t, y, t, \varphi + \alpha\,x + \tfrac{1}{2}\,\alpha^2\,t\,)$$

(6) Vertical Galilean boost:

$$t\,\frac{\partial}{\partial y} + (\,y - \tfrac{1}{2}\,g\,t^2\,)\,\frac{\partial}{\partial \varphi} \qquad (\,x, y+\alpha\,t, t, \varphi + \alpha\,(\,y - \tfrac{1}{2}\,g\,t^2\,) + \tfrac{1}{2}\,\alpha^2\,t\,)$$

(7) Vertical acceleration:

$$g\,t^2 \frac{\partial}{\partial y} - t\,\frac{\partial}{\partial t} + (\,\varphi + 2\,g\,t\,y - \tfrac{1}{2}\,g^2\,t^3\,)\,\frac{\partial}{\partial \varphi}$$

$$(\,x, y + \tfrac{1}{2}\,g\,t^2(1-\lambda^{-2}), \lambda^{-1}t, \lambda\,\varphi + g\,t\,y(\lambda - \lambda^{-1}) + \tfrac{1}{6}\,g^2\,t^3(\lambda - 3\,\lambda^{-1} + 2\,\lambda^{-3})\,)$$

(8) Gravity–free rotation:

$$(\,y + \tfrac{1}{2}\,g\,t^2\,)\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y} + g\,t\,x\,\frac{\partial}{\partial \varphi}$$

$$(\,x\cos\alpha + (\,y + \tfrac{1}{2}\,g\,t^2\,)\sin\alpha, -\,x\sin\alpha + (\,y + \tfrac{1}{2}\,g\,t^2\,)\cos\alpha - \tfrac{1}{2}\,g\,t^2,$$

$$t, \varphi + g\,t(x\sin\alpha + (\,y + \tfrac{1}{2}\,g\,t^2\,)\cos\alpha)\,)$$

(9) Rescaling: $x\,\dfrac{\partial}{\partial x} + y\,\dfrac{\partial}{\partial y} + \tfrac{1}{2}\,t\,\dfrac{\partial}{\partial t} + \tfrac{3}{2}\,\varphi\,\dfrac{\partial}{\partial \varphi}$ $\quad(\,\lambda\,x, \lambda\,y, \sqrt{\lambda}\,t, \lambda^{3/2}\,\varphi\,)$

Conservation Laws of 2D Water Waves

(1) Horizontal momentum: $\displaystyle\int_S \varphi\, dy = P_x$

(2) Energy: $\displaystyle\int_S \left(\tfrac{1}{2}\, \varphi\, \nu\, ds + \tfrac{1}{2}\, g^2\, dx \right) = E$

(3) Mass: $\displaystyle\int_S y\, dx = M$

(4) Vertical momentum: $\displaystyle\int_S \varphi\, dx = -\, g\, M\, t + P_y$

(5) Horizontal Center of Mass: $\displaystyle\int_S x\, y\, dx = -\, P_x\, t + C_x$

(6) Potential Energy: $\displaystyle\int_S \tfrac{1}{2}\, y^2\, dx = -\, \tfrac{1}{2}\, g\, M\, t^2 + P_y\, t + U$

(7) Radial Momentum:

$$\int_S \varphi(x\, dy - y\, dx) = -\, \tfrac{7}{6}\, g^2\, M\, t^3 + \tfrac{7}{2}\, g\, P_y\, t^2 + \left(\tfrac{7}{2}\, g\, U - 4\, E \right) t + P_r$$

(8) Angular Momentum: $\displaystyle\int_S \varphi(x\, dx - y\, dy) = \tfrac{1}{2}\, y\, P_x\, t^2 - g\, C_x\, t + P_\theta$

Infinite Depth Ocean

(1) Horizontal momentum: $$\int_S \varphi\, dy = P_x$$

(2) Energy: $$\int_S \left(\tfrac{1}{2}\, \varphi\, \nu\, ds + \tfrac{1}{2}\, g^2\, dx \right) = E$$

(3) Mass: $$\int_S y\, dx = M$$

(4) Vertical momentum: $$\int_S \varphi\, dx = -\, g\, M\, t + P_y$$

(5) Horizontal Center of Mass: $$\int_S x\, y\, dx = -\, P_x\, t + C_x + B^5_\infty$$

(6) Potential Energy: $$\int_S \tfrac{1}{2}\, y^2\, dx = -\, \tfrac{1}{2}\, g\, M\, t^2 + P_y\, t + U + B^6_\infty$$

(7) Radial Momentum:

$$\int_S \varphi(x\, dy - y\, dx) = -\, \tfrac{7}{6}\, g^2\, M\, t^3 + \tfrac{7}{2}\, g\, P_y\, t^2 + \left(\tfrac{7}{2}\, g\, U - 4\, E \right) t + P_r + B^7_\infty$$

(8) Angular Momentum: $$\int_S \varphi(x\, dx - y\, dy) = \tfrac{1}{2}\, y\, P_x\, t^2 - g\, C_x\, t + P_\theta + B^8_\infty$$

$$B^6_\infty = \lim_{y \to \infty} \int_0^t \int_{-\infty}^{\infty} y \left[\, v(x, y, t) - v(x, y, 0) \,\right] dx\, d\tau$$

First Order Hamiltonian KdV Model

For surface elevation $\eta(t, x)$:

Energy:

$$\mathcal{H}[\eta] = \int \left[\tfrac{1}{2}\eta^2 + \tfrac{1}{8}\alpha\,\eta^3 \right] dx$$

Poisson operator:

$$J = D_x + \tfrac{1}{6}\beta\,D_x^3 + \tfrac{1}{4}\alpha\,(\eta\,D_x + D_x\,\eta)$$

Hamiltonian flow:

$$\eta_t + J\,\frac{\delta\mathcal{H}}{\delta\eta} = 0$$

Unidirectional model:

$$\eta_t + \eta_x + \tfrac{3}{2}\alpha\,\eta\,\eta_x + \tfrac{1}{6}\beta\,\eta_{xxx} + \tfrac{1}{16}\alpha\,\beta\,(\eta^2)_{xxx} + \tfrac{15}{32}\alpha^2\,\eta^2\,\eta_x = 0$$

First Order Hamiltonian KdV Model

For horizontal velocity at height $0 \leq \theta \leq 1$:

$$u(t,x) = \varphi_x(t,x,y)$$

Energy:

$$\mathcal{H}[u] = \int \left[\tfrac{1}{2} u^2 + \tfrac{3}{8} \alpha u^3 + \tfrac{1}{6} \beta (2 - 3\theta^2) u_x^2 \right] dx$$

Poisson operator:

$$J = D_x + \beta \left(\tfrac{5}{6} - \theta^2 \right) D_x^3 - \tfrac{1}{4} \alpha (u D_x + D_x u)$$

Hamiltonian flow:

$$u_t + J \frac{\delta \mathcal{H}}{\delta u} = 0$$

Unidirectional model:

$$u_t + u_x + \tfrac{3}{2} \alpha u u_x + \tfrac{1}{6} \beta u_{xxx} - \tfrac{1}{18} \beta^2 \left(\tfrac{5}{9} - \tfrac{3}{2} \theta^2 + \theta^4 \right) u_{xxxxx}$$
$$+ \alpha \beta \left(\tfrac{53}{24} - \tfrac{11}{4} \theta^2 \right) u u_{xxx} + \alpha \beta \left(\tfrac{139}{24} - 7 \theta^2 \right) u_x u_{xx} - \tfrac{45}{32} \alpha^2 u^2 u_x = 0$$

A "Magic" Depth

At depth

$$\theta^\star = \sqrt{\frac{11}{12} - \frac{3}{4}\tau}$$

τ — surface tension

- The bidirectional Boussinesq model has a Hamiltonian structure
- The unidirectional Hamiltonian model is a version of KdV 5, and has a family of exact sech^2 soliton solutions
- The first order expansion of the water wave Poisson structure gives the Korteweg-deVries biHamiltonian structure exactly.

Vortex Filaments

The motion of a vortex filament $C \subset \mathbb{R}^3$ is described by $\dfrac{\partial C}{\partial t} = \kappa\,\mathbf{b}$, where κ is the curvature, and $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are the Frenet frame on the curve.

Theorem. (*Hasimoto*) The curvature and torsion of a vortex filament evolve according to the completely integrable nonlinear Schrödinger equation

$$-\,\mathrm{i}\,\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial s^2} + \tfrac{1}{2}\,|\,\psi\,|^2\,\psi$$

where

$$\psi(t,s) = \kappa(t,s)\exp\left(\,\mathrm{i}\int \tau(t,s)\,ds\,\right).$$

★ Integrable biHamiltonian systems appear in a surprising range of geometric curve flows. (Lamb, Langer, Singer, Perline, Marí–Beffa, Sanders, Wang, Qu, Chou, Anco, . . .)

Euclidean plane curves

$$\frac{\partial C}{\partial t} = J\,\mathbf{n}, \qquad \frac{\partial \kappa}{\partial t} = \mathcal{R}(J) \qquad G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{B} = -\,\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2\kappa_s$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$\frac{\partial C}{\partial t} = J\,\mathbf{n}, \qquad \frac{\partial \kappa}{\partial t} = \mathcal{R}(J) \qquad G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2, \qquad \mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa,$$

$$\begin{aligned}\mathcal{R} &= \mathcal{A} - \kappa_s\mathcal{D}^{-1}\mathcal{B} \\ &= \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s\mathcal{D}^{-1}\cdot\kappa\end{aligned}$$

$$\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + \tfrac{5}{3}\,\kappa\,\kappa_{sss} + \tfrac{5}{3}\,\kappa_s\kappa_{ss} + \tfrac{5}{9}\,\kappa^2\kappa_s$$

$\Longrightarrow$ Sawada–Kotera equation

Euclidean space curves

$$\frac{\partial C}{\partial t} = J_1\,\mathbf{n} + J_2\,\mathbf{b}, \qquad \frac{\partial}{\partial t}\begin{pmatrix}\kappa\\ \tau\end{pmatrix} = \mathcal{R}(J)$$

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \dfrac{2\tau}{\kappa}D_s^2 + \dfrac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2}D_s + \dfrac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \dfrac{1}{\kappa}D_s^3 - \dfrac{\kappa_s}{\kappa^2}D_s^2 + \dfrac{\kappa^2 - \tau^2}{\kappa}D_s + \dfrac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

Recursion operator: $\begin{pmatrix}\kappa_t\\ \tau_t\end{pmatrix} = \mathcal{R}\begin{pmatrix}\kappa_s\\ \tau_s\end{pmatrix} \qquad \mathcal{R} = \mathcal{A} - \begin{pmatrix}\kappa_s\\ \tau_s\end{pmatrix}\mathcal{D}^{-1}(\,\kappa \quad 0\,)$

$\Longrightarrow$ vortex filament flow

Tri–Hamiltonian Duality

$\Longrightarrow$ Fokas–Fuchssteiner; Camassa–Holm; PJO–Rosenau

There are, in fact, three mutually compatible Poisson structures associated with the Korteweg–deVries equation:

$$J_1 = D_x, \qquad J_2 = u\,D_x + D_x\,u, \qquad J_3 = D_x^3$$

- The Hamiltonian pair $J_1, J_2 + J_3$ produces the KdV biHamiltonian flow.
- The Hamiltonian pair $J_1 + J_3, J_2$ produces the Camassa–Holm biHamiltonian flow.

★ The same construction applies to other integrable systems, producing compacton/peakon integrable duals.