Invariant Signatures for Recognition and Symmetry

Peter J. Olver
University of Minnesota
`http://www.math.umn.edu/~olver`

I.MA., March, 2006

$\Longrightarrow$ E. Calabi, C. Shakiban, A. Tannenbaum, S. Haker

Object Recognition

Goal: recognize when two visual objects are equivalent

$$g : \mathcal{O} \longmapsto \widetilde{\mathcal{O}}$$

Symmetry

Goal: find all self-equivalences of a visual object

$$g : \mathcal{O} \longmapsto \mathcal{O}$$

Equivalence, Symmetry & Groups

Basic fact:

Equivalence and symmetry transformations

$$g : \mathcal{O} \longmapsto \widetilde{\mathcal{O}}$$

belong to a group:

$$g \in G$$

.

Computer Vision Groups

Euclidean *Preserves lengths and angles*

Translations

Rotations

Reflections

Similarity *Preserves length ratios*

Euclidean + Scaling

Equi-affine *Preserves area (volume)*

$$\mathbf{x} \longmapsto A\,\mathbf{x} + b, \qquad \det A = 1$$

Affine *Preserves area (volume) ratios*

Equi–affine + Scaling

Projective *Preserves cross-ratios*

$$(x, y) \;\longmapsto\; \left(\frac{ax + by + c}{gx + hy + j},\; \frac{dx + ey + f}{gx + hy + j}\right)$$

$$\det A = \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = 1$$

Camera Rotations

Projective orthogonal transformations:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \in \mathrm{SO}(3)$$

Video Groups

$$(x, y, t) \;\longmapsto\; (\tilde{x}, \tilde{y}, \tilde{t})$$

e.g. Galilean boosts (motion tracking)

$$(x, y, t) \;\longmapsto\; (x + a\,t, y + b\,t, t)$$

Complications

- Occlusion
- Ducks $\approx$ rabbits — Åström
- Outlines of 3D objects
- Bending, warping, etc.

  — pseudo-groups
- Thatcher illusion

Mathematical Setting

Ambient space:

$$M = \mathbb{R}^n, \; n = 2, 3, \ldots \quad \text{(manifold)}$$

Object:

$$N \subset M \qquad \text{submanifold}$$

Equivalences:

G finite-dimensional Lie group

acting on M

Basic equivalence problem:

$$S \approx \overline{S} \quad \Longleftrightarrow \quad \overline{S} = g \cdot S \quad \text{for} \quad g \in G$$

Symmetry (isotropy) subgroup:

$$G_S \;=\; \{\, g \in G \mid \; g \cdot S = S \,\} \;\subset\; G$$

Equivalence & Signature

Cartan's main idea:

The equivalence and symmetry properties of submanifolds are entirely prescribed by their differential invariants.

Examples of Differential Invariants

Euclidean plane curves: $C \subset \mathbb{R}^2$ $\qquad (y = u(x))$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \quad \text{— Euclidean curvature}$$

$\kappa_s, \kappa_{ss}, \ldots$ — derivatives w.r.t. arc length

$$ds = \sqrt{1+u_x^2}\; dx$$

Euclidean space curves: $C \subset \mathbb{R}^3$

$\kappa, \kappa_s, \kappa_{ss}, \ldots$ — curvature

$\tau, \tau_s, \tau_{ss}, \ldots$ — torsion

Equi-affine plane curves: $C \subset \mathbb{R}^2$

$$\kappa = \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}} \quad \text{— equi-affine curvature}$$

$\kappa_s, \kappa_{ss}, \ldots$ — derivatives w.r.t.

equi-affine arc length $ds = \sqrt[3]{u_{xx}}\; dx$

Projective plane curves: $C \subset \mathbb{RP}^2$

$\kappa = F(u^{(7)})$ — projective curvature

$\kappa_s, \kappa_{ss}, \ldots$ — derivatives w.r.t. the projective arc length $ds = P(u^{(5)})\ dx$

Euclidean surfaces: $S \subset \mathbb{R}^3$

$K,\ H$ — Gauss and mean curvature

$K_{,1}, K_{,2}, H_{,1}, H_{,2}, K_{,1,1}, \ldots$ — invariant derivatives w.r.t. the Frenet coframe ω_1, ω_2

Equi-affine surfaces: $S \subset \mathbb{R}^3$

T — Pick invariant

$K_{,1}, K_{,2}, H_{,1}, H_{,2}, K_{,1,1}, \ldots$ — invariant derivatives w.r.t. the equi-affine coframe ω_1, ω_2

The Basis Theorem

Theorem. For "any" group G acting on p-dimensional submanifolds $N \subset M$, there exists a finite generating set of differential invariants $I_1, \ldots, I_k$ and invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, so that every differential invariant

$$I = F(\,\ldots\,, \mathcal{D}_J I_\kappa,\,\ldots\,)$$

can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_i} I_\kappa$$

- Tresse
- Ovsiannikov, O $\dim < \infty$
- Kumpera, O–Pohjanpelto $\dim = \infty$

The Algebra of Differential Invariants

$\Longrightarrow$ Curves (one-dimensional submanifolds) are well understood: $k = \dim M - 1$; no syzygies. (M. Green)

For higher dimensional submanifolds (surfaces):

- The number of generating differential invariants is difficult to predict in advance.
- The invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ do not commute.
- The differentiated invariants may be subject to certain functional relations or syzygies

$$S(\,\ldots\,, \mathcal{D}_J I_\kappa,\,\ldots\,) \equiv 0.$$

Ex: the Codazzi equation relating derivatives of the Gauss and mean curvatures of a Euclidean surface.

Moving Frames

(Advertisement)

$\star\ \star$ The method of moving frames (Cartan),
especially as extended and generalized
by O–Fels–Kogan–Pohjanpelto– . . .
provides a completely constructive calculus
for finding the differential invariants,
invariant differential forms and differential
operators, commutators, recurrence formulae,
syzygies, signatures, invariant variational
problems, etc. $\star\star$

Equivalance and Invariants

- Equivalent submanifolds $N \approx \widetilde{N}$ have the same invariants: $I = \widetilde{I}$.

However, unless an invariant is constant

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \widetilde{\kappa} = 2$$

$$\Longrightarrow \quad \text{Constant curvature submanifolds}$$

it carries little information in isolation, since equivalence maps can drastically alter its dependence on the submanifold coordinates.

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \widetilde{\kappa} = \sinh x$$

However, a *syzygy*

$$I_k(x) = \Phi(I_1(x), \ldots, I_{k-1}(x))$$

among multiple invariants *is* intrinsic

$$\text{e.g.} \qquad \tau = \kappa^3 - 1 \quad \Longleftrightarrow \quad \widetilde{\tau} = \widetilde{\kappa}^3 - 1$$

Equivalence & Syzygies

Theorem. (Cartan)
Two submanifolds are (locally) equivalent if and only if they have the same syzygies among *all* their differential invariants.

- Universal syzygies — Codazzi
- Distinguishing syzygies.

Proof:

Cartan's technique of the graph:
Construct the graph of the equivalence map as the solution to a (Frobenius) integrable differential system, which can be integrated by solving ordinary differential equations.

Finiteness of Syzygies

$\star\star$ Higher order syzygies are consequences of a finite number of the lowest order syzygies.

Example. If

$$\kappa_s = H(\kappa)$$

then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

Thus, for Euclidean (or equi-affine or projective or . . .) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

The Signature Map

The generating syzygies are encoded by the *signature map*

$$\Sigma : N \quad \longrightarrow \quad \mathcal{S}$$

parametrized by the fundamental differential invariants:

$$\Sigma(x) = (I_1(x), \ldots, I_m(x)) \qquad \text{for} \qquad x \in N.$$

We call

$$\mathcal{S} = \operatorname{Im} \Sigma$$

the *signature* subset (or submanifold) of N.

The Signature Theorem

Theorem. Two submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\mathcal{S} = \overline{\mathcal{S}}$$

Differential Invariant Signatures

Plane Curves:

The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a plane curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the first two differential invariants κ and κ_s:

$$\mathcal{S} = \left\{ \left(\kappa \,,\, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Theorem. Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

More Differential Invariant Signatures

Space Curves:

The *signature curve* of a space curve $\mathcal{C} \subset \mathbb{R}^3$ is parametrized by

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \, , \, \tau \right) \right\} \subset \mathbb{R}^3$$

$\Longrightarrow$ DNA recognition (Shakiban)

Euclidean Surfaces:

The *signature surface* of a (generic) surface $N \subset \mathbb{R}^3$ under the Euclidean group is parametrized by

$$\mathcal{S} = \left\{ \left(K, \ H, \ K_{,1}, \ K_{,2} \right) \right\} \subset \mathbb{R}^4$$

$\Longrightarrow$ umbilic points

Advantages of the Signature

- Completely local
- Applies to curves, surfaces and

  higher dimensional submanifolds
- Symmetries and approximate symmetries
- Occulsions and reconstruction

Symmetry Groups

Symmetry subgroup of a submanifold:

$$G_N = \{\, g \in G \mid \ g \cdot N = N \,\} \ \subset \ G$$

Theorem. The dimension of the symmetry group of a (regular) submanifold equals the codimension of its signature:

$$\dim G_N \ = \ \dim N - \dim \mathcal{S}$$

Corollary.

$$0 \ \leq \ \dim G_N \ \leq \ p = \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point:

$$\dim \mathcal{S} = 0$$

- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ In Euclidean geometry, these are the circles, straight lines, spheres & planes.

$\Longrightarrow$ In equi-affine plane geometry, these are the conic sections.

Discrete Symmetries

Definition. The *index* of a submanifold N equals the number of points in $\mathcal{C}$ which map to a generic point of its signature $\mathcal{S}$:

$$\iota_N = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

Theorem. The number of symmetries of N equals its index:

$$\# \, G_N \; = \; \iota_N$$

$\Longrightarrow$ Approximate symmetries

Signature Metrics

- Hausdorff
- Monge–Kantorovich transport metric
- Electrostatic repulsion
- Latent semantic analysis (Shakiban)
- Histograms (Kemper–Boutin)
- Geodesic distance
- Diffusion metric
- Gromov–Hausdorff

Noise Reduction

The key objection to the differential invariant signature is its dependence on (high order) derivatives, and hence sensitivity to noise.

Noise Reduction Strategy #1: Smoothing

Apply (group-invariant) smoothing to the object.

Curvature flows:

$$C_t = -\kappa\,\mathbf{n} \qquad\qquad u_t = -\frac{u_{xx}}{1+u_x^2}$$

$\Longrightarrow$ *Hamilton–Gage–Grayson*

Noise Reduction Strategy #2: Use lower order invariants to construct a signature.

Joint Invariants

A *joint invariant* is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \;=\; I(z_1, \ldots, z_k)}$$

A *joint differential invariant* or *semi-differential invariant* is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \;=\; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

$$d(z_i, z_j) = \| \, z_i - z_j \, \|$$

Joint Equi–Affine Invariants

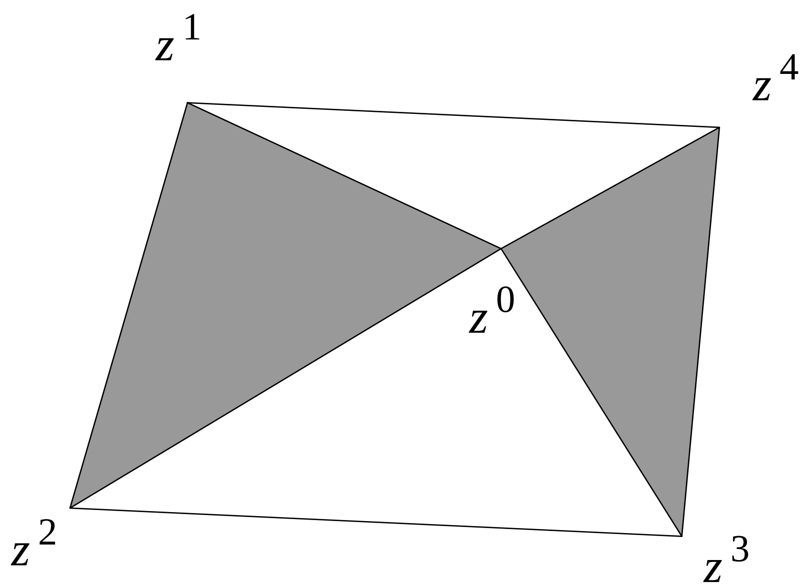
Theorem. Every joint planar equi–affine invariant is a function of the triangular areas

$$[\, i \ j \ k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$

Joint Projective Invariants

Theorem. Every joint projective invariant is a function of the planar cross-ratios

$$C(z_i, z_j, z_k, z_l, z_m) = \frac{A\,B}{C\,D}$$



Euclidean Joint Differential Invariants

— Planar Curves

- One–point

  $\Rightarrow$ curvature

$$\kappa = \frac{\dot{z} \wedge \ddot{z}}{\| \dot{z} \|^3}$$

- Two–point

  $\Rightarrow$ distances $\| z_1 - z_0 \|$

  $\Rightarrow$ tangent angles $\phi^k = \measuredangle(z_1 - z_0, \dot{z}_k)$

Equi–Affine Joint Differential Invariants — Planar Curves

- One–point

$\Rightarrow$ affine curvature

$$\kappa = \frac{(z_t \wedge z_{tttt}) + 4(z_{tt} \wedge z_{ttt})}{3(z_t \wedge z_{tt})^{5/3}} - \frac{5(z_t \wedge z_{ttt})^2}{9(z_t \wedge z_{tt})^{8/3}}$$

$$= z_s \wedge z_{ss}$$

- Two–point

$\Rightarrow$ tangent triangle area ratio

$$\frac{\dot z_0 \wedge \ddot z_0}{[\,(z_1 - z_0) \wedge \dot z_0\,]^3} = \frac{[\,\dot 0\ \ddot 0\,]}{[\,0\ 1\ \dot 0\,]^3}$$

- Three–point

$\Rightarrow$ triangle area

$$\tfrac{1}{2}\,(z_1 - z_0) \wedge (z_2 - z_0) = \tfrac{1}{2}\,[\,0\ 1\ 2\,]$$

Projective Joint Differential Invariants — Planar Curves

- One–point

  $\Rightarrow$ projective curvature

$$\kappa = \ldots$$

- Two–point

  $\Rightarrow$ tangent triangle area ratio

$$\frac{[\,0\;1\;\dot{0}\,]^3\,[\,\dot{1}\;\ddot{1}\,]}{[\,0\;1\;\dot{1}\,]^3\,[\,\dot{0}\;\ddot{0}\,]}$$

- Three–point

  $\Rightarrow$ tangent triangle ratio

$$\frac{[\,0\;2\;\dot{0}\,]\,[\,0\;1\;\dot{1}\,]\,[\,1\;2\;\dot{2}\,]}{[\,0\;1\;\dot{0}\,]\,[\,1\;2\;\dot{1}\,]\,[\,0\;2\;\dot{2}\,]}\,.$$

- Four–point

  $\Rightarrow$ area cross–ratio

$$\frac{[\,0\;1\;2\,]\,[\,0\;3\;4\,]}{[\,0\;1\;3\,]\,[\,0\;2\;4\,]}$$

Joint Euclidean Signature

For the Euclidean group $G = \mathrm{SE}(2)$ acting on curves $\mathcal{C} \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) we need at least four points

$$z_0, z_1, z_2, z_3 \in \mathcal{C}$$

Joint invariants:

$$a = \| z_1 - z_0 \| \qquad b = \| z_2 - z_0 \| \qquad c = \| z_3 - z_0 \|$$

$$d = \| z_2 - z_1 \| \qquad e = \| z_3 - z_1 \| \qquad f = \| z_3 - z_2 \|$$

$\Longrightarrow$ six functions of four variables

Joint Signature: $\Sigma : \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$

$\dim \mathcal{S} = 4 \quad \Longrightarrow \quad$ two syzygies

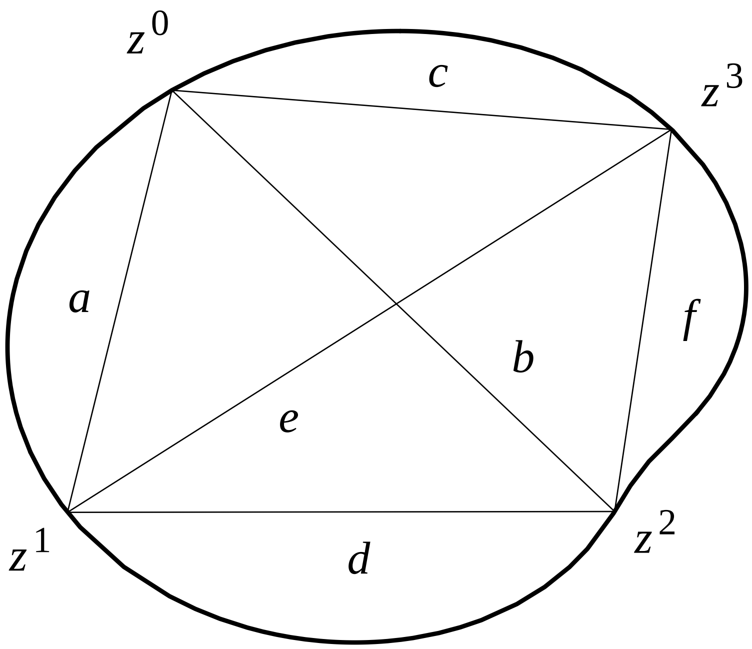
$$\Phi_1(a,b,c,d,e,f) = 0 \qquad \Phi_2(a,b,c,d,e,f) = 0$$

Universal Cayley–Menger syzygy:

$$\det \begin{vmatrix} 2a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2c^2 \end{vmatrix} = 0$$

$\Longleftrightarrow \quad \mathcal{C} \subset \mathbb{R}^2$

The Euclidean joint invariant signature encodes the distance matrix!

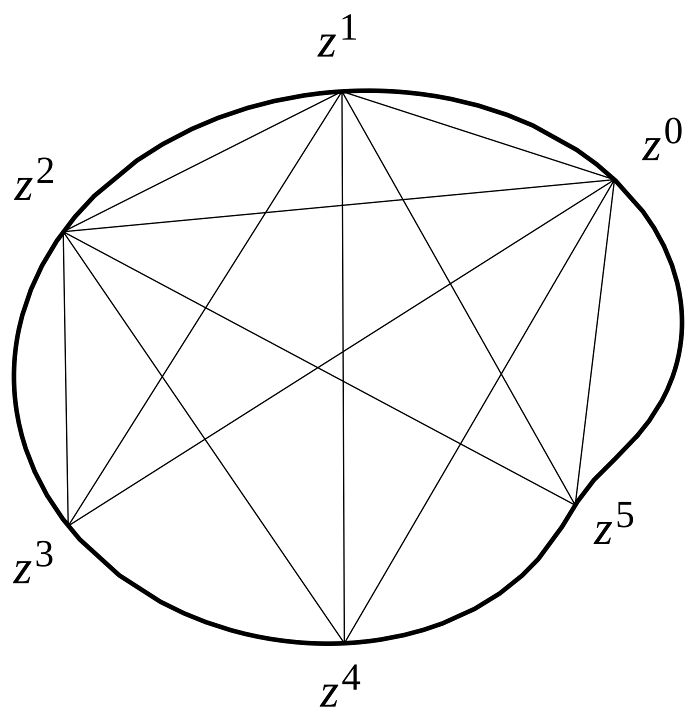


Four-Point Euclidean Joint Signature

Joint Equi–Affine Signature

Requires 7 triangular areas:

$$[\,0\;1\;2\,],[\,0\;1\;3\,],[\,0\;1\;4\,],[\,0\;1\;5\,],[\,0\;2\;3\,],[\,0\;2\;4\,],[\,0\;2\;5\,]$$



Joint Invariant Signatures

- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies (Boutin)
- It includes the differential invariant signature and semi-differential invariant signatures as its "coalescent boundaries".
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.

Histograms

Theorem. (Boutin–Kemper)
All point configurations

$$(z_1, \ldots, z_n) \in M^{\times n} \setminus V$$

lying outside a certain algebraic subvariety V are uniquely determined by their Euclidean distance histograms.

Invariant Numerical Approximations

G — Lie group acting on M

Basic Idea:

Every invariant finite difference approximation to a differential invariant must expressible in terms of the joint invariants of the transformation group.

Differential Invariant

$$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$$

Joint Invariant

$$J(g \cdot z_0, \ldots, g \cdot z_k) = J(z_0, \ldots, z_k)$$

Semi-differential invariant =

Joint differential invariant

⋆⋆ *Approximate differential invariants by joint invariants*

Euclidean Invariants

Joint Euclidean invariant:

$$\mathbf{d}(z, w) = \| \, z - w \, \|$$

Euclidean curvature:

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}$$

Euclidean arc length:

$$ds = \sqrt{1 + u_x^2} \, dx$$

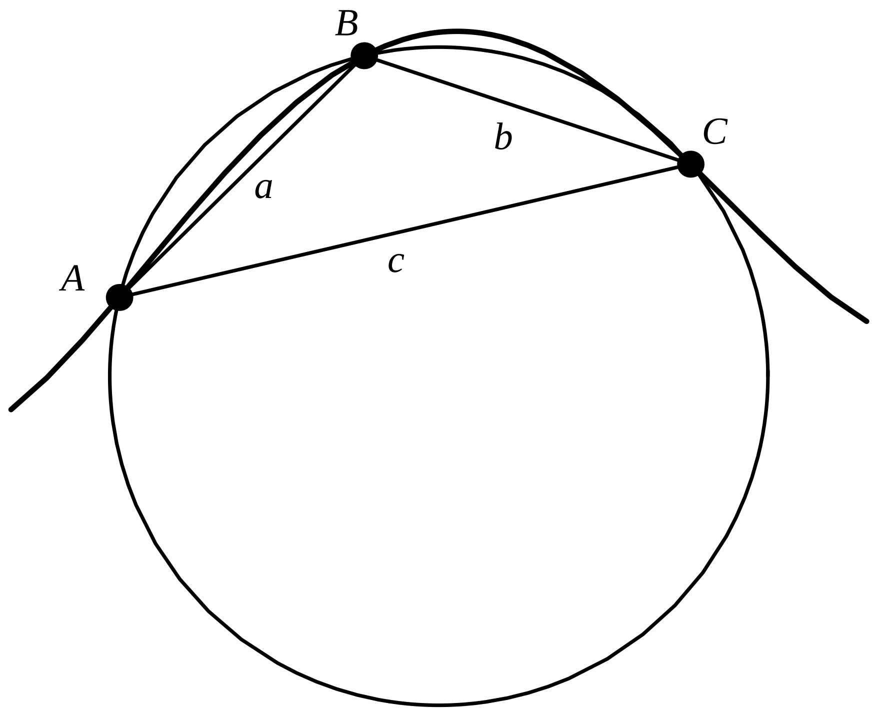
Higher order differential invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Euclidean–invariant differential equation:

$$F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0$$

Numerical approximation to curvature



Heron's formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$s = \frac{a+b+c}{2} \qquad \text{— semi-perimeter}$$

Higher order invariants

$$\kappa_s = \frac{d\kappa}{ds}$$

Invariant finite difference approximation:

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_i, P_{i-1})}$$

Unbiased centered difference:

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}) = \frac{\widetilde{\kappa}(P_i, P_{i+1}, P_{i+2}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_{i+1}, P_{i-1})}$$

Better approximation (M. Boutin):

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = 3\ \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}_{i-2} + 2\mathbf{d}_{i-1} + 2\mathbf{d}_i + \mathbf{d}_{i+1}}$$

$$\mathbf{d}_j = \mathbf{d}(P_j, P_{j+1})$$

Affine Joint Invariants

$$\mathbf{x} \to A\mathbf{x} + b \qquad \det A = 1$$

Area is the fundamental joint affine invariant

$$\begin{aligned}
[\,i\,j\,k\,] &= (P_i - P_j) \wedge (P_i - P_k) \\
&= \det \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} \\
&= \text{Area of parallelogram} \\
&= 2 \times \text{Area of triangle } \Delta(P_i, P_j, P_k)
\end{aligned}$$

Syzygies:

$$[\,i\,j\,l\,] + [\,j\,k\,l\,] = [\,i\,j\,k\,] + [\,i\,k\,l\,]$$

$$[\,i\,j\,k\,]\,[\,i\,l\,m\,] - [\,i\,j\,l\,]\,[\,i\,k\,m\,] + [\,i\,j\,m\,]\,[\,i\,k\,l\,] = 0$$

Affine Differential Invariants

Affine curvature

$$\kappa = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{9(u_{xx})^{8/3}}$$

Affine arc length

$$ds = \sqrt[3]{u_{xx}}\ dx$$

Higher order affine invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Conic Sections

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

Affine curvature:

$$\kappa = \frac{S}{T^{2/3}}$$

$$S = AC - B^2 = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

$$T = \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

Ellipse:

$$\kappa = (\pi/\mathbf{A})^{2/3}$$

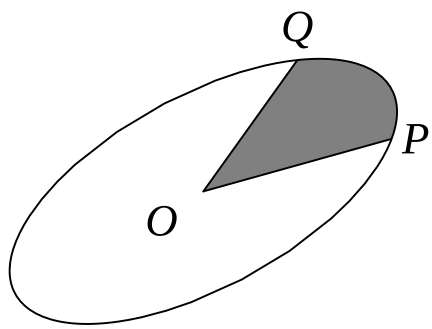
$$\mathbf{A} = \pi \frac{T}{S^{3/2}} = \text{Area}$$

Affine arc length of ellipse:

$$\int_P^Q ds = \frac{T^{1/3}}{S^{1/2}} \arcsin \sqrt{\frac{-CT}{S^2}} \left(x + \frac{CD - BE}{S} \right) \Bigg|_P^Q$$

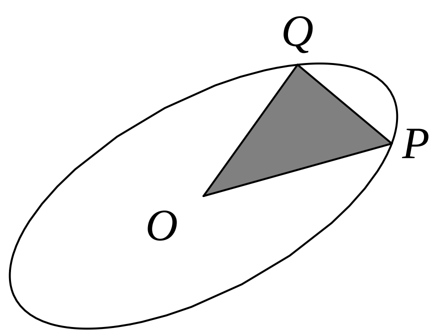
$$= 2ST^{-2/3}\mathbf{A}(P, Q)$$

$\mathbf{A}(P,Q)$:



Triangular approximation:

$\Delta(O,P,Q)$:



Total affine arc length:

$$\mathbf{L} = 2\sqrt[3]{\mathbf{A}} = -2\pi\,\frac{\sqrt[3]{T}}{\sqrt{S}}$$

Conic through five points $P_0, \ldots, P_4$:

$$[013][024][\mathbf{x}12][\mathbf{x}34] = [012][034][\mathbf{x}13][\mathbf{x}24]$$

$$\mathbf{x} = (x, y)$$

Affine curvature and arc length:

$$\kappa = \frac{S}{T^{2/3}}$$

$$ds = \text{Area } \Delta(O, P_1, P_3) = \tfrac{1}{2}[O, P_1, P_3] = \frac{N}{2S}$$

$$4T = \prod_{0 \le i < j < k \le 4} [ijk]$$

$$\begin{aligned} 4S = {} & [013]^2[024]^2([124] - [123])^2 + \\ & + [012]^2[034]^2([134] + [123])^2 - \\ & - 2[012][034][013][024]([123][234] + [124][134]) \end{aligned}$$

$$\begin{aligned} 4N = {} & - [123][134]\,\{[023]^2[014]^2([124] - [123]) + \\ & + [012]^2[034]^2([134] + [123]) + \\ & + [012][023][014][034]([134] - [123])\} \end{aligned}$$