What are Moving Frames?

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Moving Frames

Classical contributions:

M. Bartels ($\sim$1800), J. Serret, J. Frénet, G. Darboux, É. Cotton,

Élie Cartan

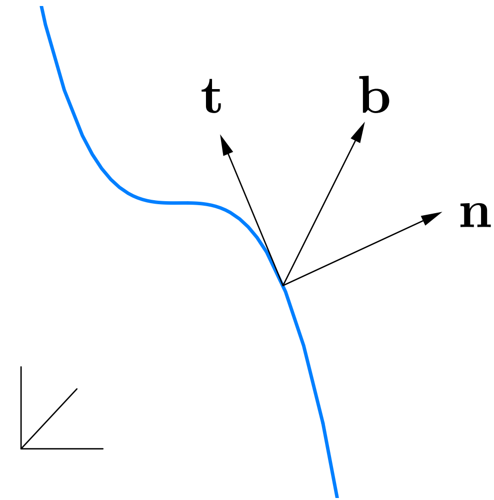
Modern developments: (1970's)

S.S. Chern, M. Green, P. Griffiths, G. Jensen, T. Ivey, J. Landsberg, . . .

The equivariant approach: (1997 –)

PJO, M. Fels, G. Marí–Beffa, I. Kogan, J. Cheh, J. Pohjanpelto, P. Kim, M. Boutin, D. Lewis, E. Mansfield, E. Hubert, O. Morozov, R. McLenaghan, R. Smirnov, J. Yue, A. Nikitin, J. Patera, . . .

Moving Frame — Space Curves



tangent	normal	binormal
$\mathbf{t} = \dfrac{dz}{ds}$	$\mathbf{n} = \dfrac{d^2 z}{ds^2}$	$\mathbf{b} = \mathbf{t} \times \mathbf{n}$

s — arc length

Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\kappa\,\mathbf{t} + \tau\,\mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\tau\,\mathbf{n}$$

κ — curvature $\qquad \tau$ — torsion

Moving Frame — Space Curves

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$\mathbf{t} = \dfrac{dz}{ds}$	$\mathbf{n} = \dfrac{d^2 z}{ds^2}$	$\mathbf{b} = \mathbf{t} \times \mathbf{n}$

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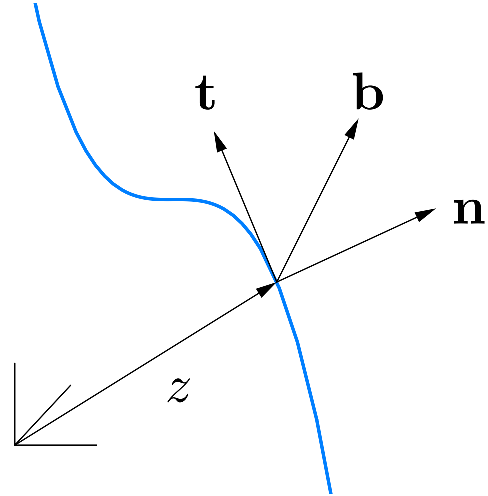


Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\,\kappa\,\mathbf{t} + \tau\,\mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\,\tau\,\mathbf{n}$$

κ — curvature $\qquad$ τ — torsion

“I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear.”

“Nevertheless, I must admit I found the book, like most of Cartan’s papers, hard reading.”

— Hermann Weyl

“Cartan on groups and differential geometry”
Bull. Amer. Math. Soc. **44** (1938) 598–601

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint Invariants and Semi-Differential Invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory

- Computer vision
  - object recognition
  - symmetry detection
- Invariant numerical methods
- Poisson geometry & solitons
- Killing tensors in relativity
- Invariants of Lie algebras in quantum mechanics
- Lie pseudogroups

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two n-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Classical Geometry — *F. Klein*

- **Euclidean group:** $G = \begin{cases} \mathrm{SE}(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n \\ \mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n \end{cases}$

  $\boxed{z \longmapsto A \cdot z + b}$ $\qquad A \in \mathrm{SO}(n)$ or $\mathrm{O}(n)$, $\quad b \in \mathbb{R}^n$, $\quad z \in \mathbb{R}^n$

  $\Rightarrow$ isometries: rotations, translations , (reflections)

- **Equi-affine group:** $G = \mathrm{SA}(n) = \mathrm{SL}(n) \ltimes \mathbb{R}^n$

  $A \in \mathrm{SL}(n)$ — volume-preserving

- **Affine group:** $G = \mathrm{A}(n) = \mathrm{GL}(n) \ltimes \mathbb{R}^n$

  $A \in \mathrm{GL}(n)$

- **Projective group:** $G = \mathrm{PSL}(n+1)$

  acting on $\mathbb{R}^n \subset \mathbb{RP}^n$

$\Longrightarrow$ Applications in computer vision

Tennis, Anyone?

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

- multiplier representation of GL(2)
- modular forms

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q} \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right)$$

Transformation group:

$$g\colon (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta} \, , \, \frac{u}{(\gamma x + \delta)^n} \right)$$

Equivalence of functions $\iff$ equivalence of graphs

$$\Gamma_Q = \{ \, (x, u) = (x, Q(x)) \, \} \subset \mathbb{C}^2$$

Moving Frames

Definition.

A moving frame is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup: $\quad G_z = \{ g \mid g \cdot z = z \} \quad$ for $\; z \in M$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity: $\quad \Longrightarrow \quad G_z = \{e\}$ for all $z \in M$.
- locally free — the orbits all have the same dimension as G:
  $\Longrightarrow \quad G_z$ is a discrete subgroup of G.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  $\not\approx$ irrational flow on the torus
- effective — the only group element which fixes *every* point in M is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

$$G_M = \bigcap_{z \in M} G_z = \{e\}$$

Proof of the Main Theorem

Necessity: Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

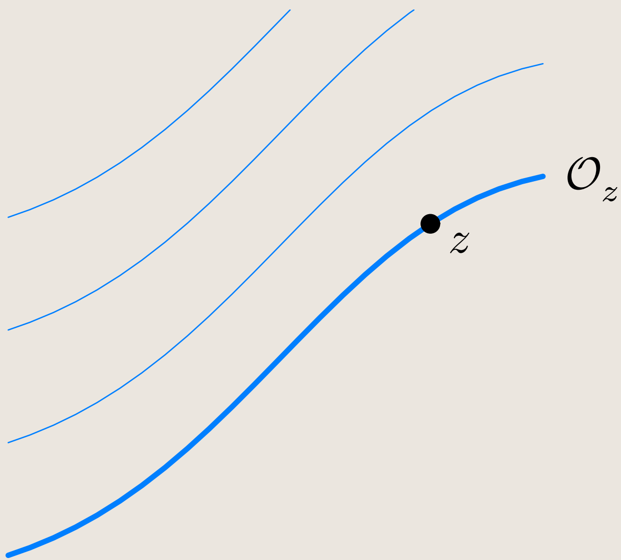
Regularity: Suppose $z_n = g_n \cdot z \longrightarrow z$ as $n \to \infty$.

By continuity, $\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$.
Hence $g_n \longrightarrow e$ in G.

Sufficiency: By direct construction — "normalization".

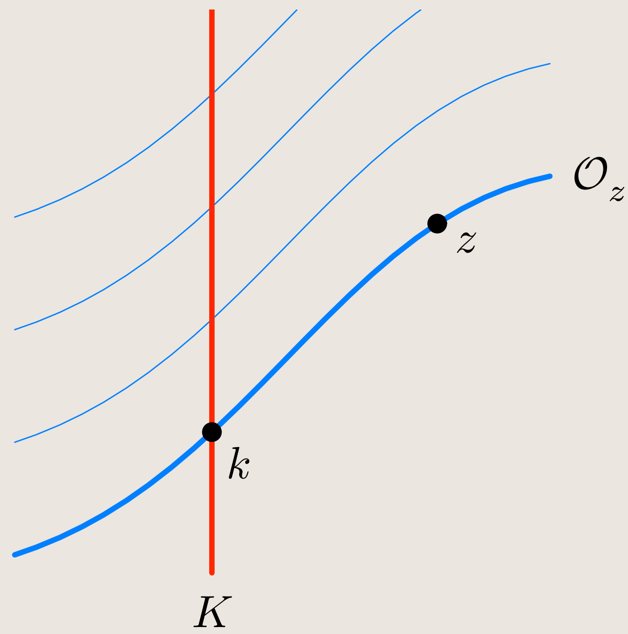
Q.E.D.

Geometric Construction



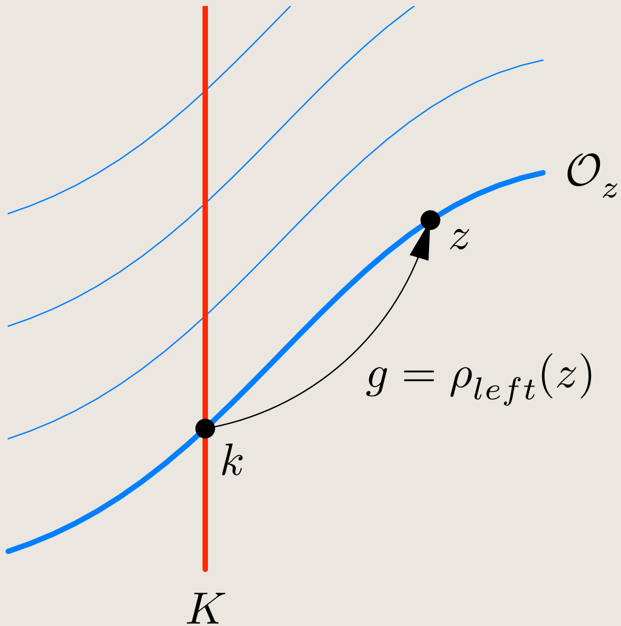
Normalization = choice of cross-section to the group orbits

Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



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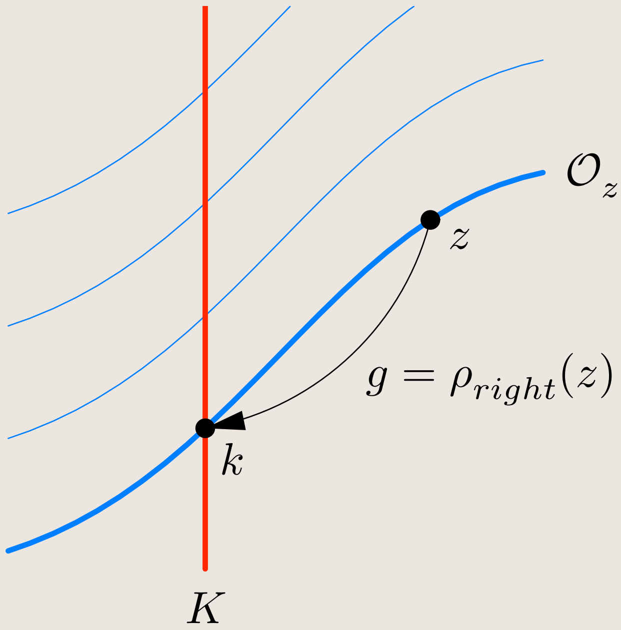

Normalization = choice of cross-section to the group orbits

K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Algebraic Construction

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \; \ldots \; , z_r = c_r \,\}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$$\Longrightarrow \quad \text{Implicit Function Theorem}$$

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Completeness of Invariants

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ \ldots \ , I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame $g = \rho(z)$ is the the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

$$\iota(z_1) = c_1, \; \ldots \; \iota(z_r) = c_r, \quad \iota(z_{r+1}) = I_1(z), \; \ldots \; \iota(z_r) = I_{m-r}(z).$$

cross-section variables

"phantom invariants"

fundamental invariants

$$\boxed{\iota\left[\,F(z_1, \ldots, z_m)\,\right] \;=\; F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))}$$

Invariantization amounts to restricting F to the cross-section

$$I \mid K = F \mid K$$

and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota \; : \; \text{functions} \quad \longmapsto \quad \text{invariants}$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation procedure.

- An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

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Euclidean Plane Curves

Special Euclidean group: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

acts on $M = \mathbb{R}^2$ via rigid motions: $w = R\,z + b$

To obtain the classical (left) moving frame we invert the group transformations:

$$\left.\begin{aligned} y &= \quad \cos\theta\,(x-a) + \sin\theta\,(u-b) \\ v &= -\sin\theta\,(x-a) + \cos\theta\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-b)$$

Assume for simplicity the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

$\Longrightarrow$ extensions to parametrized curves are straightforward

Prolong the action to J^n via implicit differentiation:

$$y = \cos\theta\,(x-a) + \sin\theta\,(u-b)$$

$$v = -\sin\theta\,(x-a) + \cos\theta\,(u-b)$$

$$v_y = \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x\sin\theta)^3}$$

$$v_{yyy} = \frac{(\cos\theta + u_x\sin\theta)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta + u_x\sin\theta)^5}$$

$$\vdots$$

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$$\vdots$$

Normalization: $\qquad r = \dim G = 3$

$$y = \cos\theta\,(x - a) + \sin\theta\,(u - b) = 0$$

$$v = -\sin\theta\,(x - a) + \cos\theta\,(u - b) = 0$$

$$v_y = \frac{-\sin\theta + u_x \cos\theta}{\cos\theta + u_x \sin\theta} = 0$$

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x \sin\theta)^3}$$

$$v_{yyy} = \frac{(\cos\theta + u_x \sin\theta)u_{xxx} - 3u_{xx}^2 \sin\theta}{(\cos\theta + u_x \sin\theta)^5}$$

$$\vdots$$

Solve for the group parameters:

$$y = \cos\theta\,(x - a) + \sin\theta\,(u - b) = 0$$

$$v = -\sin\theta\,(x - a) + \cos\theta\,(u - b) = 0$$

$$v_y = \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta} = 0$$

$\Longrightarrow$ Left moving frame $\qquad \rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x \qquad b = u \qquad \theta = \tan^{-1} u_x$$

$$a = x \qquad b = u \qquad \theta = \tan^{-1} u_x$$

Differential invariants

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x \sin\theta)^3} \longmapsto \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$v_{yyy} = \cdots \longmapsto \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}$$

$$v_{yyyy} = \cdots \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

Invariant one-form — arc length

$$dy = (\cos\theta + u_x \sin\theta)\, dx \longmapsto ds = \sqrt{1+u_x^2}\; dx$$

Dual invariant differential operator

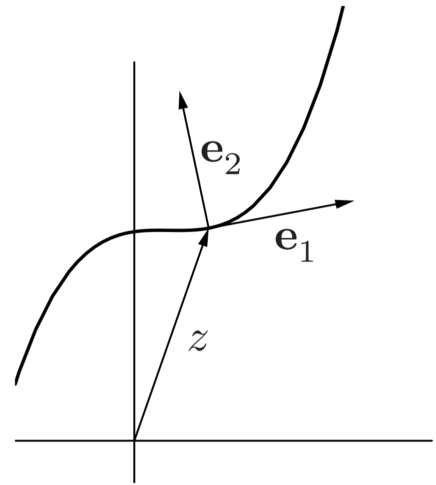
— arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\theta + u_x \sin\theta}\,\frac{d}{dx} \qquad \longmapsto \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

The Classical Picture:



Moving frame $\qquad \rho : \ (x, u, u_x) \longmapsto (R, \mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\, \mathbf{e}_1, \ \mathbf{e}_2 \,) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{e}_1 = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix} \qquad \mathbf{e}_2 = \mathbf{e}_1^\perp = \begin{pmatrix} -y_s \\ x_s \end{pmatrix}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \kappa\,\mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = -\,\kappa\,\mathbf{e}_1$$

Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\,z + \mathbf{b} \qquad\qquad A \in \mathrm{SL}(2), \qquad \mathbf{b} \in \mathbb{R}^2$$

Invert for left moving frame:

$$\left.\begin{aligned} y &= \delta\,(x-a) - \beta\,(u-b) \\ v &= -\gamma\,(x-a) + \alpha\,(u-b) \end{aligned}\right\} \qquad w = A^{-1}(z-b)$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - \beta\,u_x)\;dx \qquad\qquad D_y = \frac{1}{\delta - \beta\,u_x}\,D_x$$

Prolongation:

$$y = \delta\,(x - a) - \beta\,(u - b)$$

$$v = -\,\gamma\,(x - a) + \alpha\,(u - b)$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x}$$

$$v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta\,u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Normalization:

$$y = \delta\,(x - a) - \beta\,(u - b) = 0$$

$$v = -\gamma\,(x - a) + \alpha\,(u - b) = 0$$

$$v_y = -\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x} = 0$$

$$v_{yy} = -\frac{u_{xx}}{(\delta - \beta\,u_x)^3} = 1$$

$$v_{yyy} = -\frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5} = 0$$

$$v_{yyyy} = -\frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Equi-affine Moving Frame

$$\rho : (x, u, u_x, u_{xx}, u_{xxx}) \longmapsto (A, \mathbf{b}) \in \mathrm{SA}(2)$$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\, u_{xx}^{-5/3}\, u_{xxx} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Nondegeneracy condition: $u_{xx} \neq 0$

$\Longrightarrow$ Straight lines ($u_{xx} \equiv 0$) are "totally singular":
three-dimensional equi-affine symmetry group

Equi-affine arc length

$$dy \quad \longmapsto \quad ds = \sqrt[3]{u_{xx}}\; dx$$

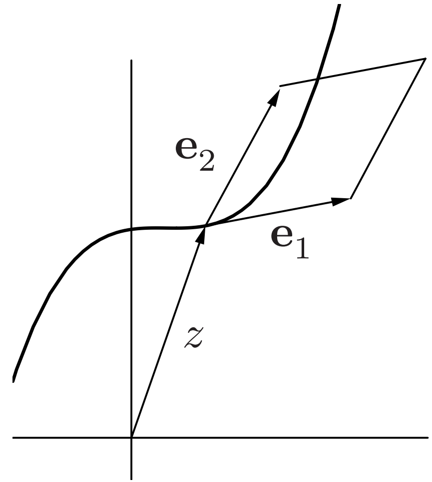
Equi-affine curvature

$$v_{yyyy} \quad \longmapsto \quad \kappa \;=\; \frac{5u_{xx}u_{xxxx} - 3u_{xxx}^2}{9u_{xx}^{8/3}}$$

$$v_{yyyyy} \quad \longmapsto \quad \frac{d\kappa}{ds}$$

$$v_{yyyyyy} \quad \longmapsto \quad \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

The Classical Picture:



$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3}\,u_{xx}^{-5/3}\,u_{xxx} \\ u_x\sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3}\,u_{xx}^{-5/3}\,u_{xxx} \end{pmatrix} = (\,\mathbf{e}_1,\,\mathbf{e}_2\,) \qquad \mathbf{b} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{e}_1 = \frac{dz}{ds} \qquad\qquad \mathbf{e}_2 = \frac{d^2 z}{ds^2}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{dz}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = \kappa\, \mathbf{e}_1$$

The Basis Theorem

Theorem. Let G be a finite-dimensional Lie group or (suitable) infinite-dimensional pseudo-group acting on p-dimensional submanifolds. Then the differential invariant algebra $\mathcal{I}(G)$ is generated by a finite number of differential invariants $I_1, \ldots, I_\ell$ and p invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$, meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ *Lie, Tresse, Ovsiannikov, Kumpera*

$\star\star$ Moving frames furnish constructive algorithms for determining the full structure of $\mathcal{I}(G)$!

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

Syzygies

However, a functional dependency or syzygy among the invariants *is* intrinsic:

e.g. $\kappa_s = \kappa^3 - 1 \iff \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Equivalence & Syzygies

Theorem. (Cartan) Two submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Proof:

Cartan's technique of the graph:

Construct the graph of the equivalence map as the solution to a (Frobenius) integrable differential system, which can be integrated by solving ordinary differential equations.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or ...) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

The Signature Map

The generating syzygies are encoded by the signature map

$$\Sigma : N \longrightarrow \mathcal{S}$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\Sigma(x) = (I_1(x), \ldots, I_m(x))$$

The image

$$\mathcal{S} = \operatorname{Im} \Sigma$$

is the signature subset (or submanifold) of N.

Equivalence & Signature

Theorem. Two submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Other Signatures

Euclidean space curves: $\mathcal{C} \subset \mathbb{R}^3$

$$\mathcal{S} = \{\ (\kappa\,,\ \kappa_s\,,\ \tau\,)\ \} \quad \subset \quad \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{ \left(H\,,\ K\,,\ H_{,1}\,,\ H_{,2}\,,\ K_{,1}\,,\ K_{,1} \right) \right\} \quad \subset \quad \mathbb{R}^3$$

- H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $\mathcal{S} \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{ \left(P\,,\ P_{,1}\,,\ P_{,2} \right) \right\} \quad \subset \quad \mathbb{R}^3$$

- P — Pick invariant

Equivalence and Signature Curves

Theorem. Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry and Signature

Theorem. The dimension of the symmetry group

$$G_N = \{ g \mid g \cdot N \subset N \}$$

of a nonsingular submanifold $N \subset M$ equals the codimension of its signature:

$$\dim G_N = \dim N - \dim \mathcal{S}$$

Corollary. For a nonsingular submanifold $N \subset M$,

$$0 \le \dim G_N \le \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point: $\dim \mathcal{S} = 0$
- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ Euclidean geometry: circles, lines, helices, spheres, cylinders, planes, . .

$\Longrightarrow$ Equi-affine plane geometry: conic sections.

$\Longrightarrow$ Projective plane geometry: W curves (*Lie & Klein*)

Discrete Symmetries

Definition. The index of a submanifold N equals the number of points in N which map to a generic point of its signature:

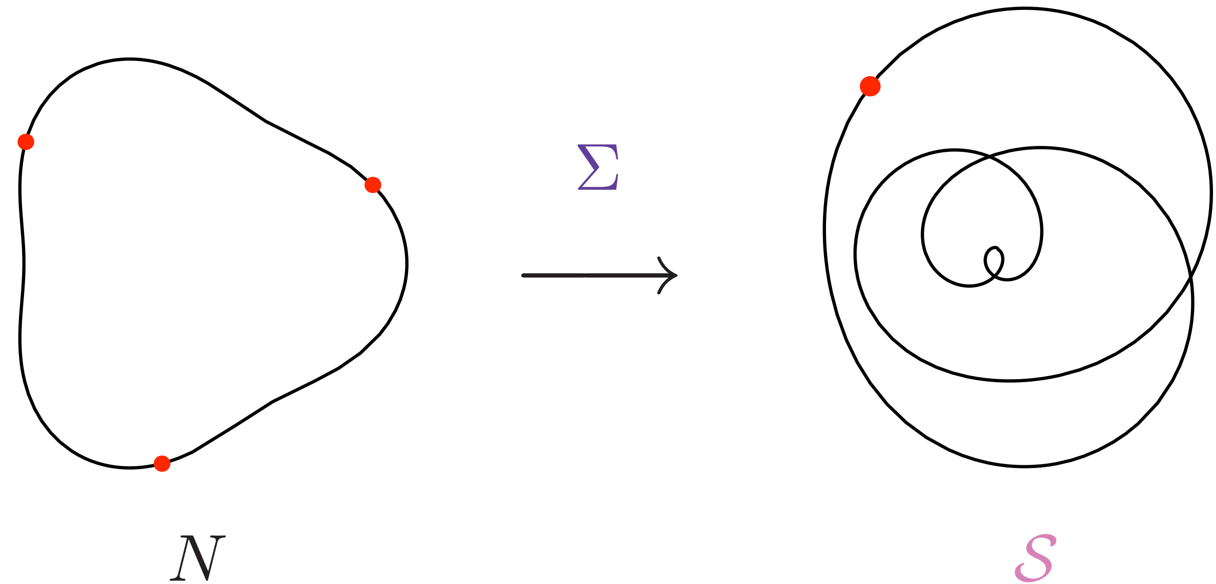
$$\iota_N = \min \left\{ \, \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \, \right\}$$

$\Longrightarrow$ Self–intersections

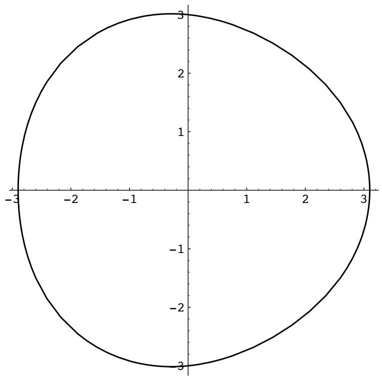
Theorem. The cardinality of the symmetry group of a submanifold N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

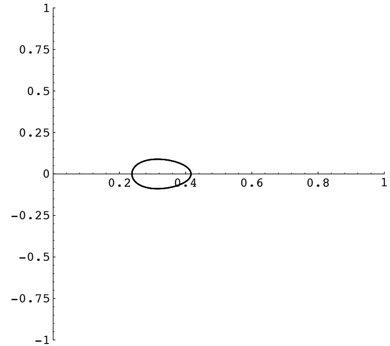
The Index



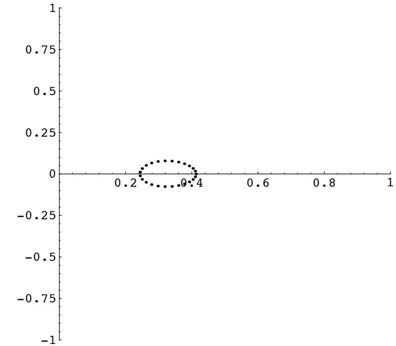
The polar curve $r = 3 + \frac{1}{10}\cos 3\theta$



The Original Curve

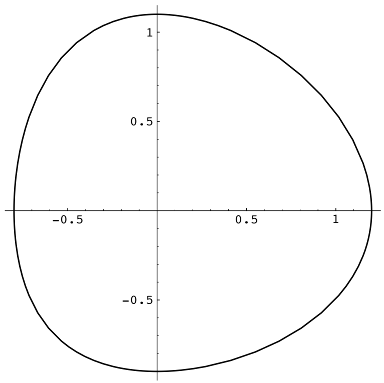


Euclidean Signature

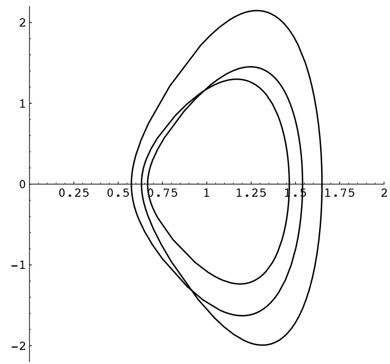


Numerical Signature

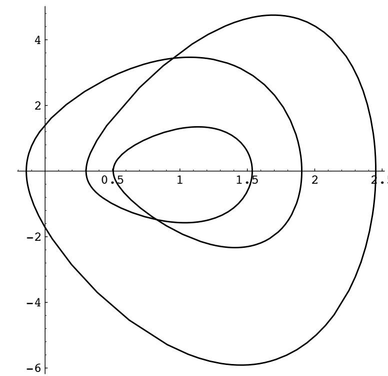
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

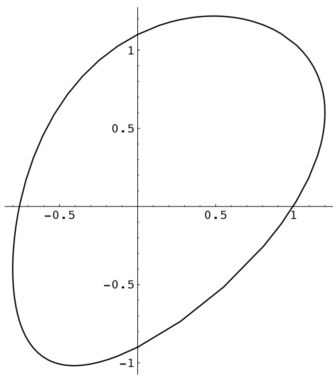


Euclidean Signature

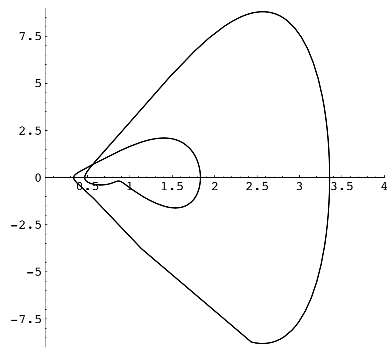


Affine Signature

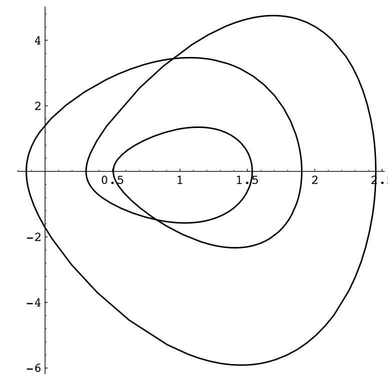
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

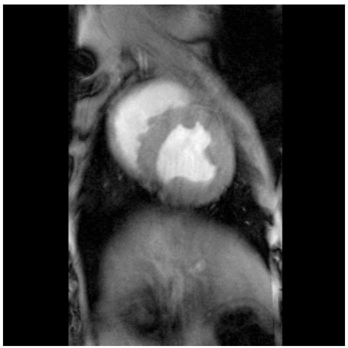


Euclidean Signature

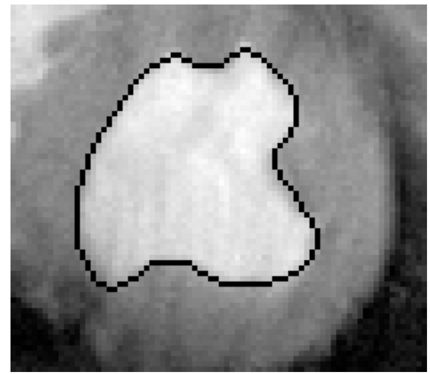


Affine Signature

Canine Left Ventricle Signature

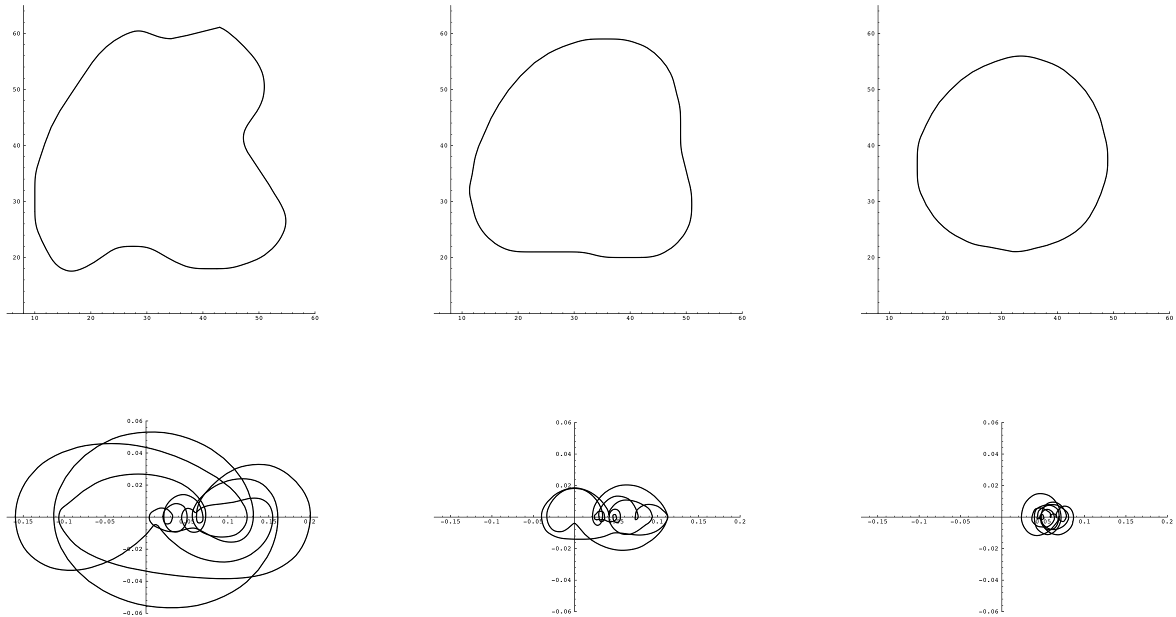


Original Canine Heart
MRI Image



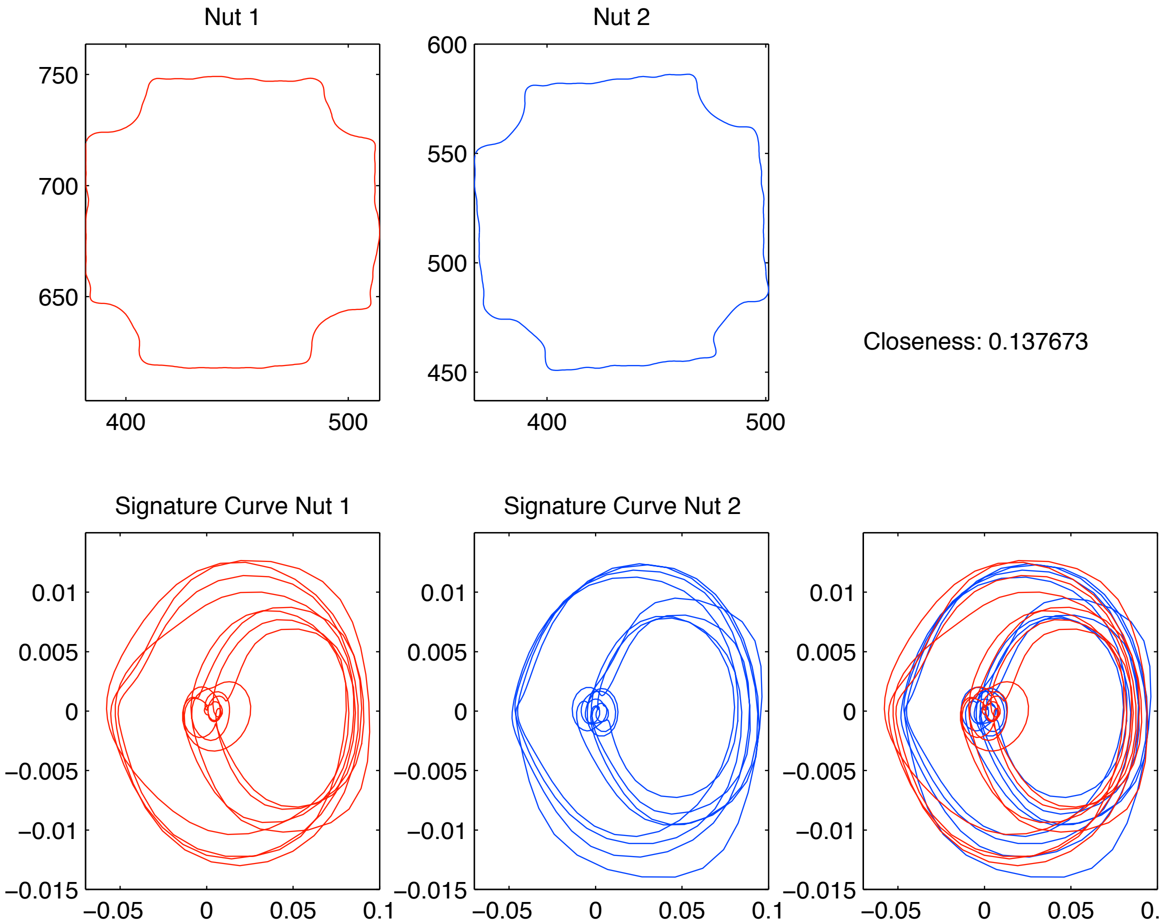
Boundary of Left Ventricle

Smoothed Ventricle Signature

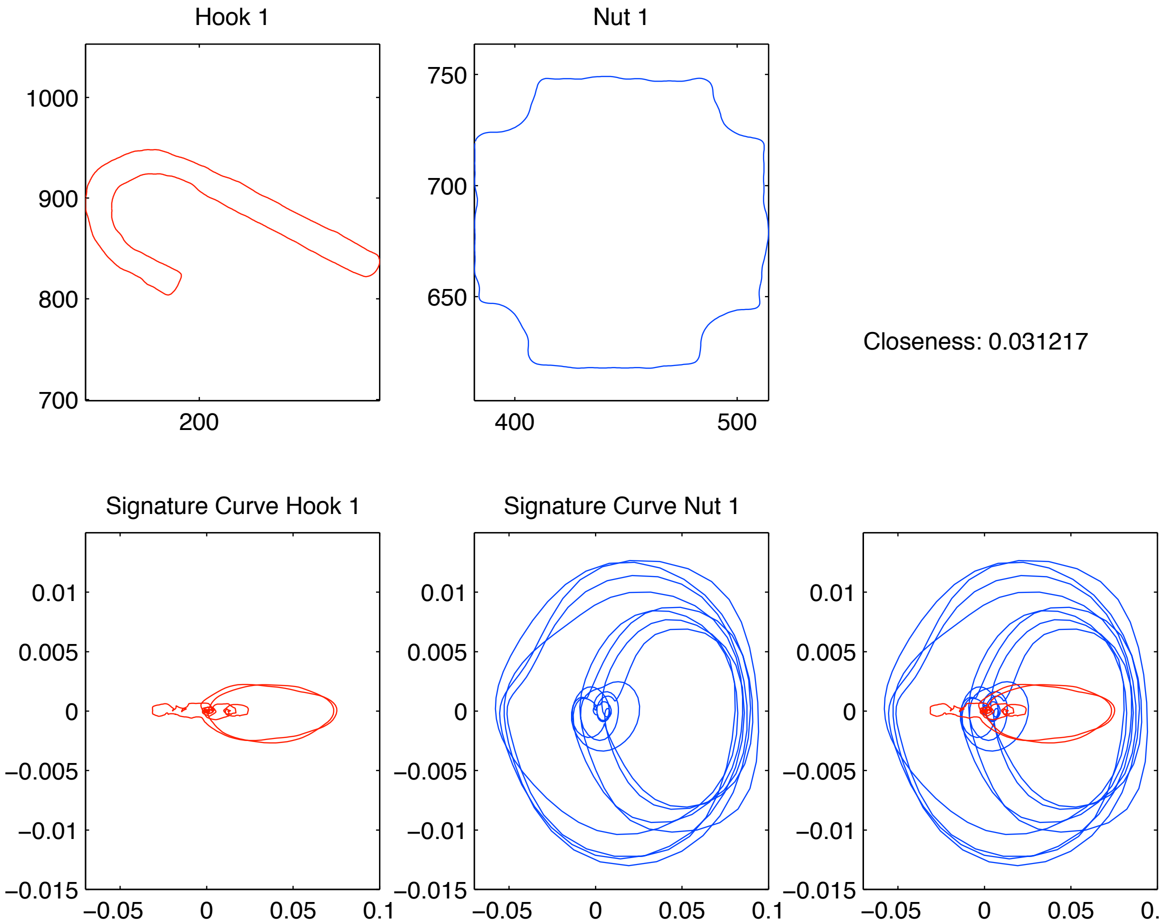


“Industrial Mathematics”

$\Longrightarrow$ Steve Haker



Nut 1
Nut 2
750
700
650
400
500
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

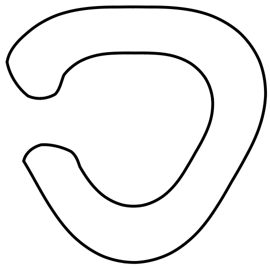


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

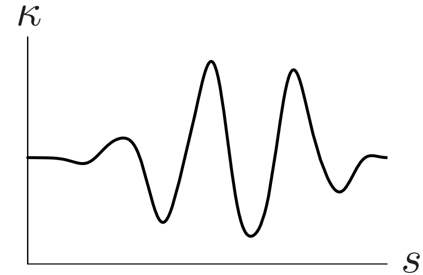
Signature Metrics

- Hausdorff
- Monge–Kantorovich transport
- Electrostatic repulsion
- Latent semantic analysis (Shakiban)
- Histograms (Kemper–Boutin)
- Diffusion metric
- Gromov–Hausdorff

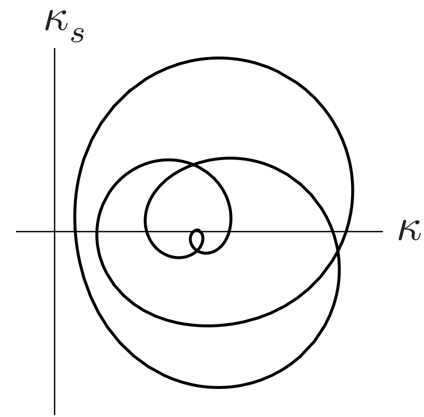
Signatures



Original curve

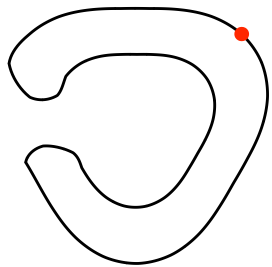


Classical Signature

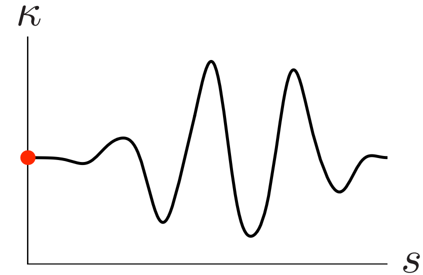


Differential invariant signature

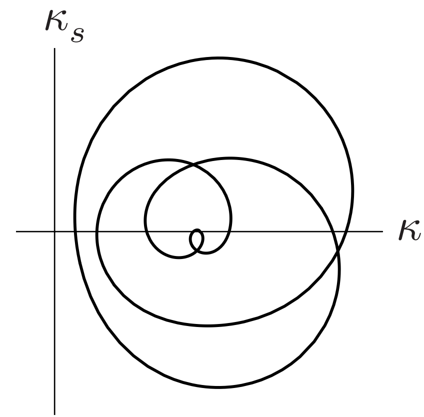
Signatures



Original curve

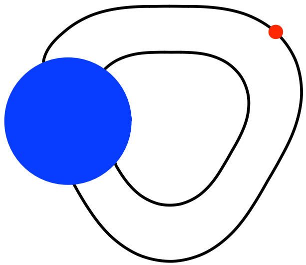


Classical Signature

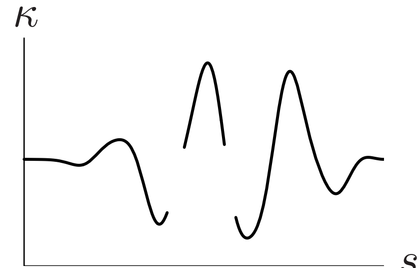


Differential invariant signature

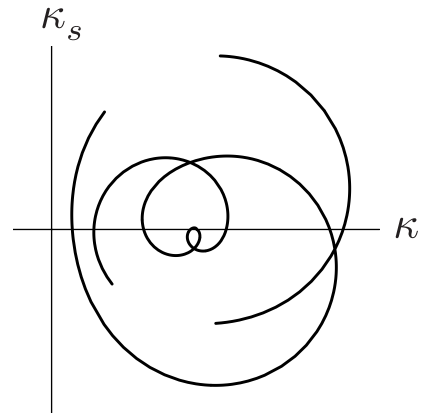
Occlusions



Original curve

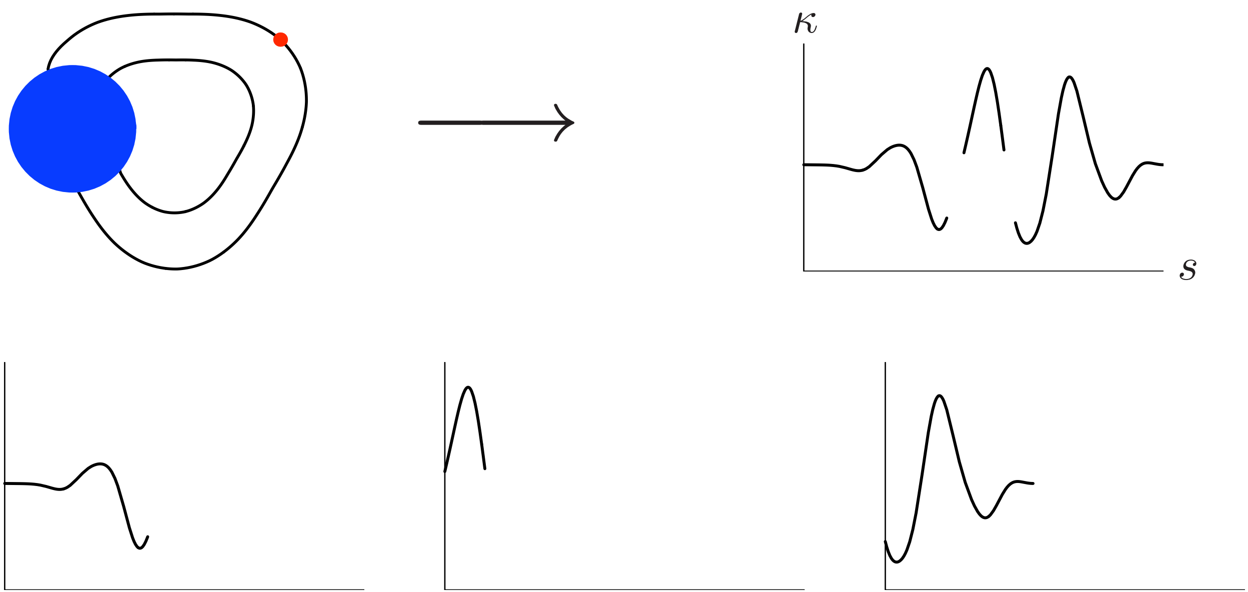


Classical Signature



Differential invariant signature

Classical Occlusions



Advantages of the Signature Curve

- Purely local — no ambiguities
- Symmetries and approximate symmetries
- Extends to surfaces and higher dimensional submanifolds
- Occlusions and reconstruction

Main disadvantage: Noise sensitivity due to dependence on high order derivatives.

Noise Reduction

Strategy #1:

Use lower order invariants to construct a signature:

- joint invariants
- joint differential invariants
- integral invariants
- topological invariants
- ...

Joint Invariants

A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \;=\; I(z_1, \ldots, z_k)}$$

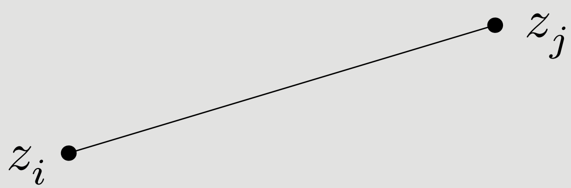
A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \;=\; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

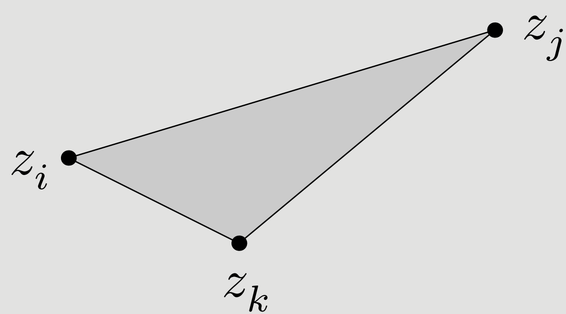
$$d(z_i, z_j) = \| \, z_i - z_j \, \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

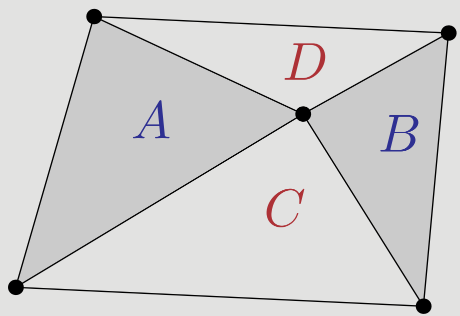
$$[\, i \ j \ k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

Theorem. Every joint projective invariant is a function of the planar cross-ratios

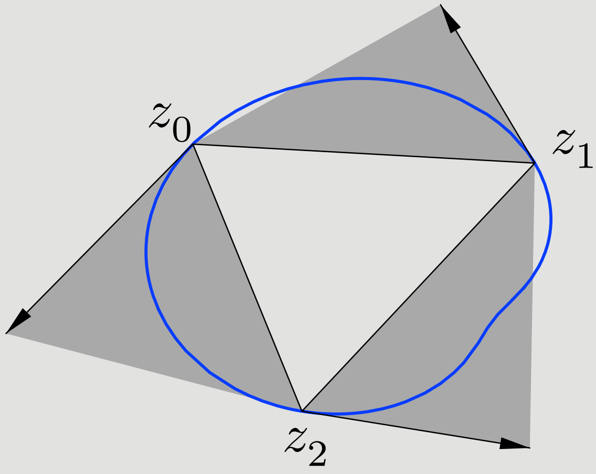
$$[\, z_i, z_j, z_k, z_l, z_m \,] \;=\; \frac{A\,B}{C\,D}$$

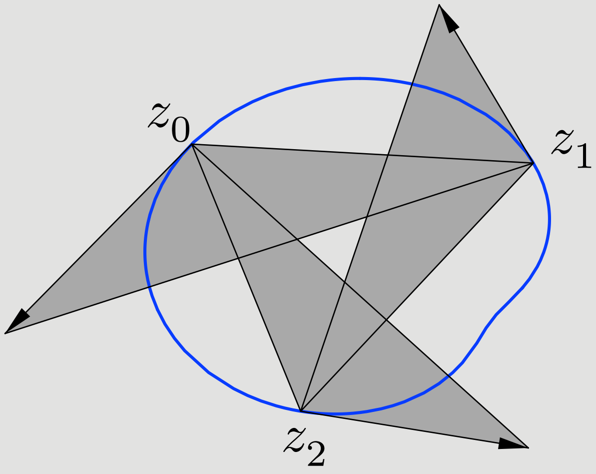


- Three–point projective joint differential invariant

  — tangent triangle ratio:

$$\frac{[\,0\ 2\ \dot{0}\,]\ [\,0\ 1\ \dot{1}\,]\ [\,1\ 2\ \dot{2}\,]}{[\,0\ 1\ \dot{0}\,]\ [\,1\ 2\ \dot{1}\,]\ [\,0\ 2\ \dot{2}\,]}$$





Joint Invariant Signatures

If the invariants depend on k points on a p-dimensional submanifold, then you need at least

$$\ell > k\,p$$

distinct invariants $I_1, \ldots, I_\ell$ in order to construct a syzygy. Typically, the number of joint invariants is

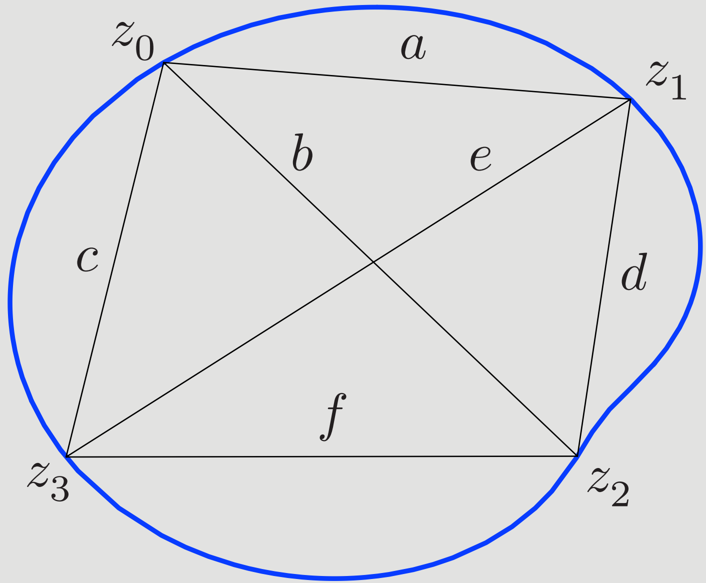
$$\ell = k\,m - r = (\#\text{points})\,(\dim M) - \dim G$$

Therefore, a purely joint invariant signature requires at least

$$k \;\geq\; \frac{r}{m-p} + 1$$

points on our p-dimensional submanifold $N \subset M$.

Joint Euclidean Signature



Joint signature map:

$$\Sigma : \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$$

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$\Longrightarrow$ six functions of four variables

Syzygies:

$$\Phi_1(a, b, c, d, e, f) = 0 \qquad \Phi_2(a, b, c, d, e, f) = 0$$

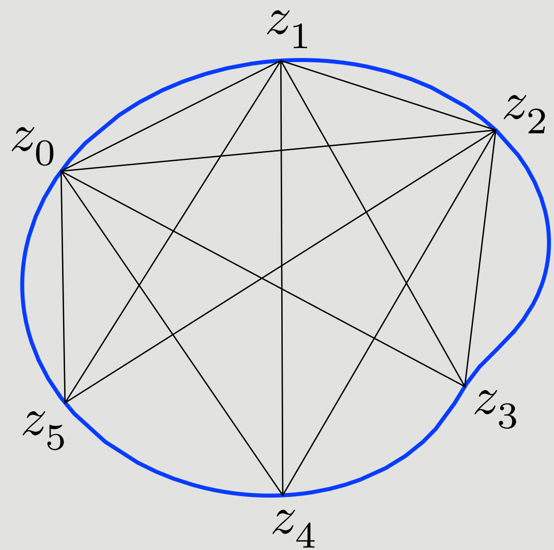
Universal Cayley–Menger syzygy $\iff$ $\mathcal{C} \subset \mathbb{R}^2$

$$\det \begin{vmatrix} 2\,a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2\,b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2\,c^2 \end{vmatrix} = 0$$

Joint Equi–Affine Signature

Requires 7 triangular areas:

$[\,0\;1\;2\,],\ [\,0\;1\;3\,],\ [\,0\;1\;4\,],\ [\,0\;1\;5\,],\ [\,0\;2\;3\,],\ [\,0\;2\;4\,],\ [\,0\;2\;5\,]$



Joint Invariant Signatures

- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies (Boutin)
- It includes the differential invariant signature and semi-differential invariant signatures as its "coalescent boundaries".
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.

Statistical Sampling

Idea: Replace high dimensional joint invariant signatures by increasingly dense point clouds obtained by multiply sampling the original submanifold.

- The equivalence problem requires direct comparison of signature point clouds.
- Continuous symmetry detection relies on determining the underlying dimension of the signature point clouds.
- Discrete symmetry detection relies on determining densities of the signature point clouds.

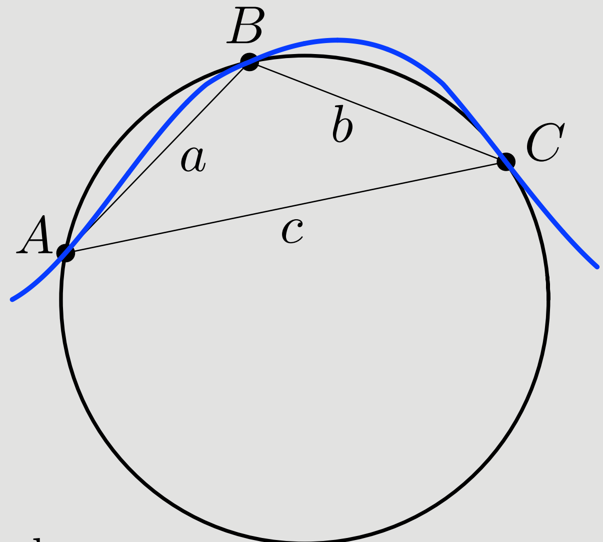
Additional Applications of Moving Frames

Symmetry–Preserving Numerical Methods

- Invariant numerical approximations to differential invariants.
- Invariantization of numerical integration methods.

$\Longrightarrow$ Structure-preserving algorithms

Numerical approximation to curvature



Heron's formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

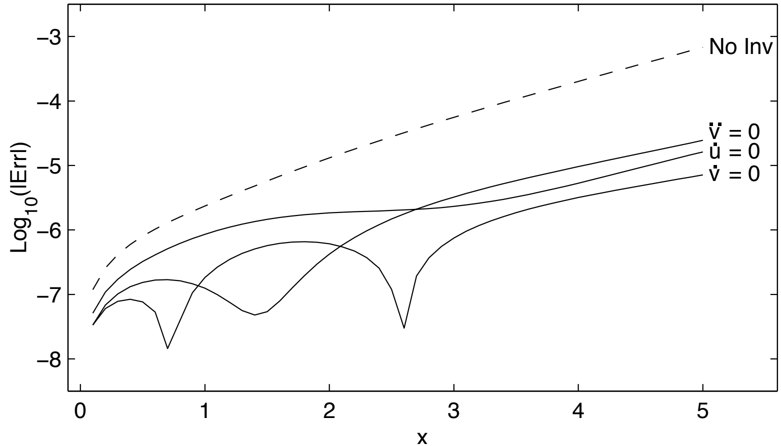
$$s = \frac{a+b+c}{2} \quad — \quad \text{semi-perimeter}$$

Invariantization of Numerical Schemes

$\Longrightarrow$ Pilwon Kim

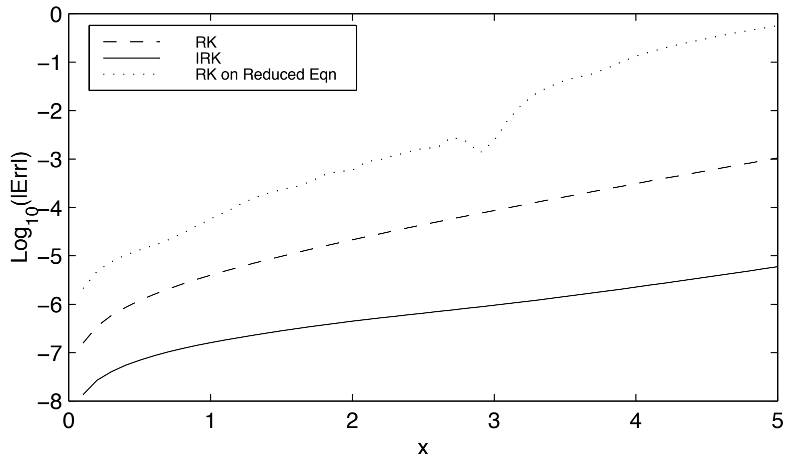
Suppose we are given a numerical scheme for integrating a differential equation, e.g., a Runge–Kutta Method for ordinary differential equations, or the Crank–Nicolson method for parabolic partial differential equations.

If G is a symmetry group of the differential equation, then one can use an appropriately chosen moving frame to invariantize the numerical scheme, leading to an invariant numerical scheme that preserves the symmetry group. In challenging regimes, the resulting invariantized numerical scheme can, with an inspired choice of moving frame, perform significantly better than its progenitor.



Invariant Runge–Kutta schemes

$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

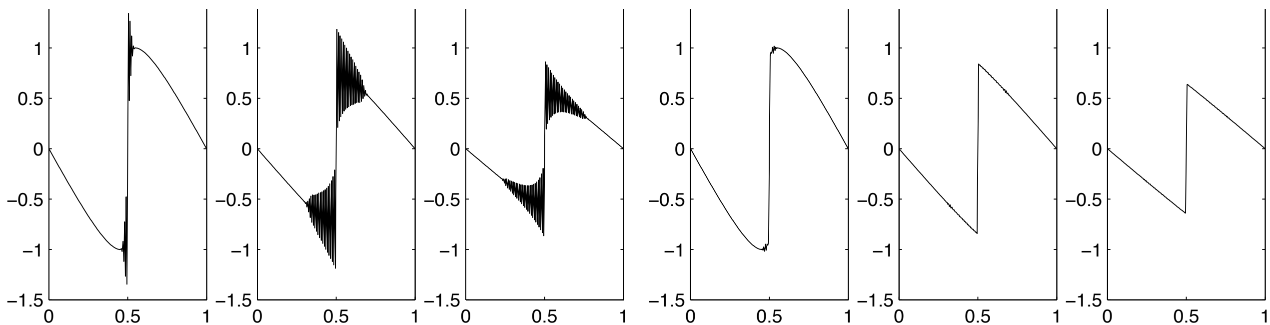


Comparison of symmetry reduction and invariantization for

$$u_{xx} + x\, u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

Invariantization of Crank–Nicolson for Burgers' Equation

$$u_t = \varepsilon\, u_{xx} + u\, u_x$$



Evolution of Invariants and Signatures

Basic question: If the submanifold evolves according to an invariant evolution equation, how do its differential invariants & signatures evolve?

Theorem. Under the curve shortening flow $C_t = -\kappa\,\mathbf{n}$, the signature curve $\kappa_s = H(t,\kappa)$ evolves according to the parabolic equation

$$\frac{\partial H}{\partial t} = H^2\, H_{\kappa\kappa} - \kappa^3 H_\kappa + 4\,\kappa^2 H$$

$\Longrightarrow$ Signature Noise Reduction Strategy #2

$\Longrightarrow$ Solitons and bi-Hamiltonian systems

Invariant Variational Problems

Problem: Given an invariant variational problem written in terms of the differential invariants, *directly* construct the invariant form of its Euler–Lagrange equations.

$\Longrightarrow$ Willmore, $\int K^2$, etc.

Example. Euclidean plane curves:

Invariant variational problem:

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(P) + \kappa\, \mathcal{H}(P).$$

$\mathcal{E}(P)$ — invariantized Euler–Lagrange expression
$\mathcal{H}(P)$ — invariantized Hamiltonian

Minimal Generating Invariants

A set of differential invariants is a generating system if all other differential invariants can be written in terms of them and their invariant derivatives.

- Euclidean curves $C \subset \mathbb{R}^3$:

  curvature κ and torsion τ.

- Equi–affine curves $C \subset \mathbb{R}^3$:

  affine curvature κ and torsion τ.

- Euclidean surfaces $S \subset \mathbb{R}^3$:

  Gauss curvature K and mean curvature H.

- Equi–affine surfaces $S \subset \mathbb{R}^3$:

  the Pick invariant P.

Classical Invariant Theory

$$M = \mathbb{R}^2 \setminus \{ u = 0 \}$$

$$G = \mathrm{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \Delta = \alpha\,\delta - \beta\,\gamma \neq 0 \right\}$$

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}\,,\ \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

Prolongation:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u \qquad \Delta = \alpha\, \delta - \beta\, \gamma$$

$$v_y = \frac{\sigma\, u_x - n\, \gamma\, u}{\Delta\, \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\gamma\, \sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

Normalization:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} = 0 \qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u = 1 \qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}} = 0$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\,\gamma\,\sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}} = \frac{1}{n(n-1)}$$

$$v_{yyy} = \cdots$$

Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\, u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\, u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\, x u^{(1-n)/n}$$

Hessian:

$$H = n(n-1)u\, u_{xx} - (n-1)^2 u_x^2 \neq 0$$

Note: $H \equiv 0$ if and only if $Q(x) = (a\, x + b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} \approx \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3(n-2)}{n^3(n-1)} \approx \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} \;= n(n-1)QQ'' - (n-1)^2 Q'^2 \;\sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = (Q,\, H)^{(1)} \;= (2n-4)Q'H - nQH' \;\sim Q_x H_y - Q_y H_x$$

$$U = (Q,\, T)^{(1)} \;= (3n-6)Q'T - nQT' \;\sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \quad \deg H = 2n-4 \quad \deg T = 3n-6 \quad \deg U = 4n-8$$

Signatures of Binary Forms

Signature curve of a nonsingular binary form $Q(x)$:

$$\mathcal{S}_Q = \left\{ (J(x)^2, K(x)) = \left(\frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $\quad H(x) \neq 0 \quad \text{and} \quad (J'(x), K'(x)) \neq 0.$

Signature map:

$$\Sigma : \Gamma_Q \longrightarrow \mathcal{S}_Q \qquad\qquad \Sigma(x) = (J(x)^2, K(x))$$

Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

Maximally Symmetric Binary Forms

Theorem. If $u = Q(x)$ is a polynomial, then the following are equivalent:

- $Q(x)$ admits a one-parameter symmetry group
- T^2 is a constant multiple of H^3
- $Q(x) \simeq x^k$ is complex-equivalent to a monomial
- the signature curve degenerates to a single point
- all the (absolute) differential invariants of Q are constant
- the graph of Q coincides with the orbit of a one-parameter subgroup

Symmetries of Binary Forms

Theorem. The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree n is:

- A two-parameter group if and only if $H \equiv 0$ if and only if Q is equivalent to a constant. $\Longrightarrow$ totally singular
- A one-parameter group if and only if $H \not\equiv 0$ and $T^2 = c\,H^3$ if and only if Q is complex-equivalent to a monomial x^k, with $k \neq 0, n$. $\Longrightarrow$ maximally symmetric
- In all other cases, a finite group whose cardinality equals the index of the signature curve, and is bounded by

$$\iota_Q \quad \leq \quad \begin{cases} 6n - 12 & \qquad U = cH^2 \\ 4n - 8 & \qquad \text{otherwise} \end{cases}$$