Propagating terraces and the dynamics of front-like solutions of reaction-diffusion equations on $\mathbb{R}$

P. Poláčik*

School of Mathematics, University of Minnesota

Minneapolis, MN 55455

Abstract

We consider semilinear parabolic equations of the form

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, t > 0,$$

where f a C^1 function. Assuming that 0 and $\gamma > 0$ are constant steady states, we investigate the large-time behavior of the front-like solutions, that is, solutions u whose initial values $u(x, 0)$ are near γ for $x \approx -\infty$ and near 0 for $x \approx \infty$. If the steady states 0 and γ are both stable, our main theorem shows that at large times, the graph of $u(\cdot, t)$ is arbitrarily close to a propagating terrace (a system of stacked traveling fonts). We prove this result without requiring monotonicity of $u(\cdot, 0)$ or the nondegeneracy of zeros of f. The case when one or both of the steady states 0, γ is unstable is considered as well. As a corollary to our theorems, we show that all front-like solutions are quasiconvergent: their ω-limit sets with respect to the locally uniform convergence consist of steady states. In our proofs we employ phase plane analysis, intersection comparison (or, zero number) arguments, and a geometric method involving the spatial trajectories $\{(u(x, t), u_x(x, t)) : x \in \mathbb{R}\}$, $t > 0$, of the solutions in question.

Key words: Parabolic equations on $\mathbb{R}$, minimal propagating terraces, minimal systems of waves, global attractivity, limit sets, quasiconvergence, convergence, spatial trajectories, zero number.

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Contents

1 Introduction **3**

2 Main results **13**
2.1 Minimal systems of waves and propagating terraces 14
2.2 The case where 0 and γ are both stable 18
2.3 The case where one of the steady states 0, γ is unstable 24
2.4 The ω-limit set and quasiconvergence 29
2.5 Locally uniform convergence to a specific front and exponential convergence 31

3 Phase plane analysis **33**
3.1 Basic properties of the trajectories 34
3.2 A more detailed description of the minimal system of waves . 43
3.3 Some trajectories out of the minimal system of waves 50

4 Proofs of Propositions 2.8, 2.12 **70**

5 Preliminaries on the limit sets and zero number **71**
5.1 Properties of $\Omega(u)$ 71
5.2 Zero number 72

6 Proofs of the main theorems **74**
6.1 Some estimates: behavior at $x = \pm\infty$ and propagation 75
6.2 A key lemma: no intersection of spatial trajectories 78
6.3 The spatial trajectories of the functions in $\Omega(u)$ 82
6.4 $\Omega(u)$ contains the minimal propagating terrace 84
6.5 Ruling out other points from $K_\Omega(u)$ 86
6.6 Completion of the proofs of Theorems 2.5, 2.13, and 2.15 . . . 93
6.7 Completion of the proofs of Theorems 2.7, 2.9, and 2.17 . . . 96
6.8 Completion of the proofs of Theorems 2.11 and 2.19 101
6.9 Proof of Theorem 2.22 104

1 Introduction

Consider the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $f \in C^1(\mathbb{R})$ and $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We investigate the dynamics of solutions of (1.1) with front-like initial data. This means that given two zeros of f, say, 0 and $\gamma > 0$, we assume, roughly speaking, that $\sup u_0$ and $\liminf_{x\to-\infty} u_0(x)$ are close to γ, and $\inf u_0$ and $\limsup_{x\to\infty} u_0(x)$ are close to 0. Our general hypotheses are formulated in Section 2; for the purpose of the introduction, we assume the following more specific conditions:

$$0 \le u_0 \le \gamma, \ \lim_{x\to-\infty} u_0(x) = \gamma, \text{ and } \lim_{x\to\infty} u_0(x) = 0. \tag{1.3}$$

It is well known that in the dynamics of front-like solutions traveling waves play a prominent role. A *traveling wave* is a solution U of (1.1) of the form $U(x,t) = \phi(x - ct)$, where c, the *speed* of the wave, is a real number and ϕ, the *profile* of the wave, is a C^2 function. Clearly, the profile function must satisfy the ordinary differential equation

$$\phi_{xx} + c\phi_x + f(\phi) = 0. \tag{1.4}$$

If the profile function is monotone and bounded, then, by (1.4), the limits $w^\pm := \phi(\pm\infty) \in \mathbb{R}$ are zeros of f. In this case, we also refer to the solution U as a *traveling front* (connecting w^- and w^+). We shall mostly deal with fronts which are decreasing in x but note that the transformation $x \to -x$ allows one to study in parallel increasing fronts: if U is a traveling front, then $\tilde{U}(x,t) = U(-x,t)$ is also a traveling front with opposite speed and the profile function $\tilde{\phi} = \phi(-x)$. A traveling wave (front) with the speed $c = 0$ is also referred to as a *standing wave (front)*.

Convergence properties of solutions of (1.1) to traveling fronts have been widely studied and are well understood. The simplest situation occurs in the bistable case, that is, when the constants 0 and γ are both linearly stable steady states of the equation $\dot{\theta} = f(\theta)$ and there is no other stable steady states of this equation in the interval $[0,\gamma]$. Then it is known (see [11]) that a traveling wave connecting γ and 0 exists, is unique up to translations—in particular, its speed is uniquely determined—and it attracts all front-like solutions of (1.1) (we will be more specific about the meaning of the attractivity below). In the monostable case, when $f'(\gamma) < 0 < f'(0)$ and $f > 0$ in $(0, \gamma)$, the situation is more complicated. Several traveling fronts connecting γ and 0 then exist, their speeds forming an interval (see the pioneering papers [13, 19] or [2, 33]). In this case, for the convergence of the solution of (1.1), (1.2) to a specific front, certain asymptotics of $u_0(x)$ as $x \to \infty$ is needed (see [3, 4, 20, 32] for results on approach to traveling fronts and [16, 15, 39] for examples of other behaviors). There are many extensions of these classical results, see, for example, [2, 5, 14, 17, 21, 28, 29, 30, 32]; many more references can be found in the bibliographical notes of [33, Sect. 1.6] or in the surveys [37, 38].

It is also well-known that if there are other stable constant steady states in the interval $[0, \gamma]$, a traveling front connecting γ and 0 may not exist and a family of traveling fronts is needed to describe the behavior of the solution $u(\cdot, t)$ of (1.1), (1.2). This is the case, for instance, when $[0, \gamma]$ decomposes into closed subintervals $[0, \gamma_1]$, $[\gamma_1, \gamma]$, in each of which f is bistable and such that the unique speeds c_1 and c_2 of the traveling fronts connecting γ_1 to 0 and γ to γ_1, respectively, satisfy $0 \leq c_2 < c_1$. Then there is no traveling front connecting γ and 0 and the solution u with an initial datum as above is attracted to the two fronts: for large t, the graph of $u(\cdot, t)$ has a part resembling the front with the range in $[0, \gamma_1]$, another part resembling the front with the range in $[\gamma_1, \gamma]$, and between these two parts the graph is flat and close to the line $\{u = \gamma_1\}$. Early results on the convergence of solutions to such systems of traveling fronts (referred to as a "stacked combination of fronts") can be found in [11] and in the follow-up paper [12]. The assumptions in [11] are rather mild as the initial datum u_0 is concerned, however, the nonlinearity is assumed to satisfy additional conditions, involving in particular the nondegeneracy of the zeros 0, γ_1, γ (see also [31] for an extension of this result to monotone systems of reaction-diffusion equations). In [12], on the other hand, it is assumed that the function u_0 is monotone and the nondegeneracy condition is replaced by weaker conditions. For monotone initial data, very general results on convergence to a family of traveling fronts, under the name "minimal systems of waves," is proved in [33, Sect. I.1.5] (the original results were given in [35]; see also [23] for similar convergence results for viscous conservation laws, where some nonmonotone initial data are allowed). Finally, in the recent paper [9], which inspired our present work, the convergence to a family of traveling fronts, "propagating terraces" in [9], was proved for u_0 equal to a step function, but with $f = f(x, u)$ allowed to depend on x periodically. In this case, the notion of a traveling wave has to be replaced by that of a pulsating wave.

In the present paper, we only consider the homogeneous problem. We introduce a new technique, which allows us to prove the convergence to propagating terraces under minimal requirements on f and u_0. In particular, we do not assume any nondegeneracy conditions on the zeros of f and any monotonicity property of u_0. If the steady states 0 and γ are both stable, not necessarily asymptotically, with respect to the ordinary differential equation (ODE) $\dot{\theta} = f(\theta)$, then no conditions other than (1.3) are needed (and even these are significantly relaxed). If 0 is unstable from above, then some additional condition on u_0, such as $u_0 \equiv 0$ on an interval $[m, \infty)$ or a sufficiently fast decay of $u_0(x)$ as $x \to \infty$ is necessary, and that is the only extra condition we need (similarly, we need an extra condition if γ is unstable from below).

To give the reader a flavor of our results, we state here sample theorems on approach to a propagating terrace assuming conditions (1.3). In the first one, we assume, as in [9], the following propagation property:

(DGM) There is a compactly supported continuous function $\bar{u}_0$ with values in $[0, \gamma)$ such that the solution $\bar{u}$ of (1.1) with the initial condition $\bar{u}(\cdot, 0) = \bar{u}_0$ satisfies

$$\lim_{t \to \infty} \|\bar{u}(\cdot, t) - \gamma\|_{L^\infty_{loc}(\mathbb{R})} = 0. \tag{1.5}$$

Theorem 1.1. *Assume that* (DGM) *holds. Then there exist $k \in \mathbb{N}$, numbers $c_1 \geq \cdots \geq c_k > 0$, and functions $\phi_1, \ldots, \phi_k$, such the following statements are valid.*

(i) *For each j the function ϕ_j is a decreasing solution of equation* (1.4) *with $c = c_j$ and the limits $b_j := \phi_j(-\infty)$, $a_j := \phi_j(\infty)$ satisfy the relations*

$$a_1 = 0, \quad a_{j+1} = b_j \ \ (j = 1, \ldots, k-1), \quad b_k = \gamma.$$

(ii) *Let u_0 be any continuous function satisfying* (1.3). *If 0 is unstable from above for the equation $\dot{\theta} = f(\theta)$, assume also that $u_0 \equiv 0$ on an interval $[m, \infty)$. Then there are C^1 functions $\zeta_1, \ldots, \zeta_k$ on $(0, \infty)$ with the following properties:*

(a) $\lim_{t \to \infty} \zeta_j'(t) = 0 \quad (j = 1, \ldots, k)$*;*

(b) $\zeta_j(t) - \zeta_{j+1}(t) \to \infty$ *whenever* $j \in \{1, \ldots, k-1\}$ *is such that* $c_{j+1} = c_j$*;*

(c) *as* $t \to \infty$*, one has*

$$u(x,t) - \left(\sum_{j=1,\ldots,k} \phi_j(x - c_j t - \zeta_j(t)) - \sum_{j=1,\ldots,k-1} a_{j+1} \right) \to 0, \quad (1.6)$$

where the convergence is uniform with respect to $x \in \mathbb{R}$*.*

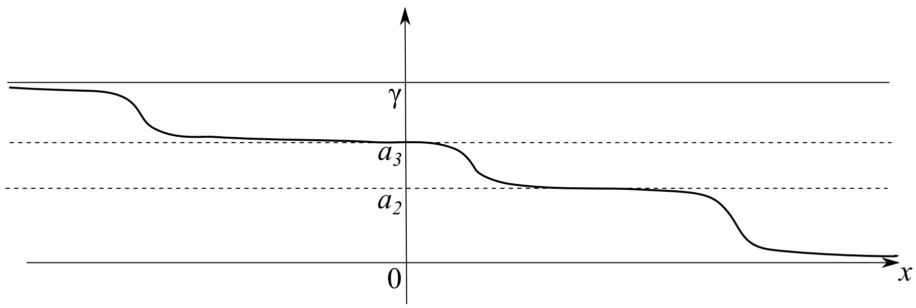

Figure 1: The graph of $u(\cdot,t) \approx \sum_{j=1,\ldots,k} \phi_j(\cdot - c_j t - \zeta_j(t)) - \sum_{j=1,\ldots,k-1} a_{j+1}$, with $k = 3$, for large t

The family of traveling fronts $U_j(x,t) = \phi_j(x - c_j t)$, $j = 1, \ldots, k$, appearing in the theorem is what we call, following [9], a minimal propagating terrace; more precisely, it is the minimal propagating terrace connecting γ and 0. The reason for the name "propagating terrace" is apparent from the graph of $u(\cdot,t)$ for large t (cp. Figure 1). The minimality property will be defined in the next section.

In the special case when u_0 is equal to a step function, Theorem 1.1 is a consequence of the main result of [9] which deals with periodically x-dependent nonlinearities $f = f(x,u)$. Note that (DGM) requires a certain structure of f. In the homogeneous case considered here, this structure can be described explicitly as follows. Set

$$F(u) := \int_0^u f(s)\, ds. \quad (1.7)$$

It is not difficult to show (see Proposition 2.12(iii) and Section 3.2 below) that (DGM) is equivalent to the following condition on f:

(F1) The point $u = \gamma$ is a unique (global) maximizer of the function F in $[0, \gamma]$ and it is an isolated zero of f in $[0, \gamma]$.

Condition (F1) implies in particular that the minimal propagating terrace consists of finitely many traveling fronts with positive speeds. The case when $u = 0$ is a unique maximizer of $F\big|_{[0,\gamma]}$ and an isolated zero of f in $[0, \gamma]$ is analogous; this time, the minimal propagating terrace consists of finitely many traveling fronts with negative speeds.

If neither (F1) nor the previous analogous case occur, the situation is more complicated in that the traveling fronts in the minimal propagating terrace may have positive, negative, as well as zero values. Nonetheless, a conclusion similar to that in Theorem 1.2 still holds if all maximizers of F are isolated critical points of F:

(F2) The function F has only finitely many maximizers in $[0, \gamma]$ and all of them are isolated zeros of f in $[0, \gamma]$.

Theorem 1.2. *Assume that* (F2) *holds. Then there exist $k \in \mathbb{N}$, numbers $c_1 \geq \cdots \geq c_k$ (not necessarily positive), and functions $\phi_1, \ldots, \phi_k$ such that the following statements are valid.*

(i) *For each j the function ϕ_j is a decreasing solution of equation* (1.4) *with $c = c_j$ and the limits $b_j := \phi_j(-\infty)$, $a_j := \phi_j(\infty)$ satisfy the relations*

$$a_1 = 0, \quad a_{j+1} = b_j \;\; (j = 1, \ldots, k-1), \quad b_k = \gamma.$$

(ii) *Let u_0 any continuous function u_0 satisfying* (1.3)*. If 0 is unstable from above for the equation $\dot{\theta} = f(\theta)$, assume also that $u_0 \equiv 0$ on an interval $[m, \infty)$; and if γ is unstable from below for this ODE, assume that $u_0 \equiv \gamma$ on an interval $(-\infty, n]$. Then there are C^1 functions $\zeta_1, \ldots, \zeta_k$ on* $(0, \infty)$ *such that conclusions* (a)-(c) *stated in Theorem 1.1 hold.*

Compared to Theorem 1.1, the only difference in the statement of Theorem 1.2, in addition to the hypotheses, is that the speeds c_j are not all positive. Note that, in view of statement (a), the c_j are the asymptotic speeds of the interfaces, or, transitions, in the graph of $u(\cdot, t)$ (cp. Figure 1). Thus, $c_j > 0$ means that the corresponding interface moves to the right and if $c_j < 0$ it moves to the left. If $c_j = 0$, the interface moves with asymptotically vanishing speed and its motion is determined by the corresponding function

$\zeta_j(t)$. Note that, although $\zeta_j'(t) \to 0$, $\zeta_j(t)$ itself may not be convergent and it may even be unbounded. In fact, if $c_j = c_{j+1}$ for some j (which happens, for example, if $f\big|_{(a_{j+1},b_{j+1})}$ is a shift of $f\big|_{(a_j,b_j)}$), then, according to statement (b), at least one of the functions ζ_j, ζ_{j+1} is unbounded. In Section 2.5, we show that under certain generic conditions (including in particular the condition that $c_j \neq c_{j+1}$ for all j), the functions ζ_j are all convergent. Moreover, in this generic case, the convergence in (1.6) is exponential. Naturally, the case when one of the steady states 0, γ is unstable for the equation $\dot{\theta} = f(\theta)$ is excluded in this convergence theorem. Even in a simple monostable case—with the minimal propagating terrace consisting of just one traveling front—the corresponding function ζ_j grows logarithmically to infinity [4].

Assuming still conditions (1.3) on u_0, let us now consider a completely general nonlinearity $f \in C^1$ with $f(0) = f(\gamma) = 0$. The minimal propagating terrace may now consist of infinitely many traveling fronts and it may also have "gaps" if there are continua of maximizers of the function F (see Section 3.2). Therefore, the approach of the solution u to the minimal propagating terrace cannot be expressed in such a simple way as in (1.6). Instead, we formulate it in terms of the Ω-limit set of u:

$$\Omega(u) := \{\varphi : u(\cdot + x_n, t_n) \to \varphi \text{ for some sequences } t_n \to \infty \text{ and } x_n \in \mathbb{R}\}. \tag{1.8}$$

Here, the convergence is in $L^\infty_{loc}(\mathbb{R})$ (the locally uniform convergence).

To explain how this limit set serves our purposes, let us first compare it to the more traditional notion of the limit set, the ω-limit set of u in $L^\infty_{loc}(\mathbb{R})$:

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to \infty\}. \tag{1.9}$$

Heuristically speaking, the choice of the locally uniform convergence in (1.8) and (1.9) means that the shape of $u(\cdot, t)$ for large times is being observed through large, but finite, "windows." In the case of $\omega(u)$, the observer is stationary and accordingly only the behavior of $u(\cdot, t)$ in fixed compact regions is captured in $\omega(u)$. On the other hand, in $\Omega(u)$, the shifts introduced by x_n mean that at any given time t_n the observer can place the window to any desired position and thereby see any finite piece of the graph of $u(\cdot, t_n)$. For example, to observe an interface moving with a constant speed c, one can choose $x_n = ct_n$, which corresponds to using the usual moving coordinate frame. When there is no preferred moving coordinate frame, as when there are infinitely many speeds in the minimal propagating terrace, it is reasonable, when examining the global shape of $u(\cdot, t)$, not to constrain the position of the windows, thus the sequence $\{x_n\}$ in the definition of $\Omega(u)$ is completely arbitrary.

Clearly, $\omega(u) \subset \Omega(u)$, but the opposite inclusion is not true in general. In this paper, $\Omega(u)$ is employed much more frequently than $\omega(u)$, as our main goal is to study the global shape of the front-like solutions at large times. However, the structure of $\omega(u)$ is of interest as well and we will get back to it below.

In the formulation of our next theorem, $\mathcal{N}$ is a (finite or countable) system of open nonoverlapping subintervals of $(0,\gamma)$ such that

$$Z_{\mathcal{N}} := [0,\gamma] \setminus \bigcup_{I\in\mathcal{N}} I \subset f^{-1}\{0\}. \tag{1.10}$$

Thus $Z_{\mathcal{N}}$ consists of zeros of f, or, constant steady states of (1.1).

Theorem 1.3. *For any function $f \in C^1$ with $f(0) = f(\gamma) = 0$, there exist a system $\mathcal{N}$ as above and families of numbers $\{c_I : I \in \mathcal{N}\}$ and functions $\{\phi_I : I \in \mathcal{N}\}$ such that the following statements hold.*

(i) *For each I the function ϕ_I is a decreasing solution of equation* (1.4) *with $c = c_I$ whose range is the interval I.*

(ii) *Let u_0 any continuous function u_0 satisfying* (1.3). *If 0 is unstable from above for the equation $\dot\theta = f(\theta)$, assume also that $u_0 \equiv 0$ on an interval $[m,\infty)$; and if γ is unstable from below for this ODE, assume that $u_0 \equiv \gamma$ on an interval $(-\infty, n]$. Then*

$$\Omega(u) = \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\} \cup Z_{\mathcal{N}}, \tag{1.11}$$

where $Z_{\mathcal{N}}$ is as in (1.10).

According to statement (i), the ϕ_I and c_I are, respectively, the speeds and profile functions of traveling fronts of (1.1). Again, they are coming from the minimal propagating terrace associated with the nonlinearity f and interval $[0,\gamma]$, as defined in the next section. Relating to our discussion of $\Omega(u)$ above, (1.11) can be interpreted as follows. Looking at any finite piece of the graph of $u(\cdot,t)$ for large t, what one sees is either a flat part given by one of the constants in $Z_{\mathcal{N}}$, or a part of the graph of the profile function ϕ_I, for some $I \in \mathcal{N}$. In Section 2.2 and 2.3, we give further information on the behavior of the solution near its interfaces for general f (see Theorems 2.11 and 2.19).

We emphasize that the minimal propagating terrace for the interval $[0,\gamma]$ is uniquely determined by the nonlinearity f. Accordingly, the set $\Omega(u)$ in Theorem 1.1 and the traveling fronts $U_j(x,t) = \phi_j(x - c_j t)$, $j = 1,\dots,k$, in Theorems 1.1, 1.2, are independent of the solution u. What distinguishes between different solutions are the functions $\zeta_j(t)$ in Theorems 1.1, 1.2 or similar functions in Theorems 2.11 and 2.19 below.

The above theorems present a substantial generalization of earlier results on convergence to propagating terraces in the homogeneous case. When (1.3) is assumed, the theorems give a fairly complete description of the large-time behavior of the solutions of (1.1), (1.2) for general nonlinearities. In the next section, we push the generality even further by considering a larger class of initial data. Our conditions on u_0 formulated in (2.1)-(2.3) are easily seen to be sharp for this kind of results.

We believe that our theorems on the structure and attractivity properties of minimal propagating terraces will be instrumental for further understanding of the dynamics of solutions of (1.1), and not just of the front-like solutions (for example, in a sequel to [22], we will employ the present results in a study of solutions with localized initial data). Here, we give one application, still in the context of front-like solutions, concerning the (small) ω-limit set of the solution u of (1.1), (1.2). We continue to assume (1.3) (for more general initial data, $\omega(u)$ is examined in Section 2.4). We say the solution u is *convergent* if $\omega(u)$ consists of a single function–necessarily a steady state of (1.1), and *quasiconvergent* if $\omega(u)$ consists entirely of steady states of (1.1). Convergence is the simpler of the two behaviors, but in some sense—numerically, for example—quasiconvergent solutions are indistinguishable from convergent solutions. Although they may not settle to any particular steady state, they move very slowly at large times, with $u_t(\cdot,t) \to 0$ in $L^\infty_{loc}(\mathbb{R})$ as $t \to \infty$. Not all bounded solutions of (1.1) are quasiconvergent. In fact, non-quasiconvergent solutions occur in equations of the form (1.1) quite frequently: they exist whenever there is an interval $[a,b]$ in which f is bistable (see [24, 26]; a broader discussion of quasiconvergence can be found in [27]). On the other hand, it was shown in [22] that the solution of (1.1), (1.2) is quasiconvergent, if bounded, provided $u_0 \ge 0$ and $u_0(-\infty) = u_0(\infty) = 0$. The following theorem says that the same is true if u_0 satisfies (1.3).

Theorem 1.4. *For any $f \in C^1$ with $f(0) = f(\gamma) = 0$ and $u_0 \in C(\mathbb{R})$ satisfying* (1.3)*, the solution u of* (1.1), (1.2) *is quasiconvergent: $\omega(u)$ consists of steady states. More precisely, $\omega(u)$ consists of constant steady states and* *decreasing standing waves of* (1.1).

Note that in this theorem there is no additional assumption on u_0 in the case where one of the basic steady states 0, γ is unstable. Therefore, Theorem 1.4 is not a direct consequence of the previous theorems. We shall derive it from more general results on propagating terraces, as given in the next section (see Subsection 2.4).

We now briefly discuss our method in the proof of the approach of solutions to propagating terraces, highlighting in particular the new technique that allows us to handle nonmonotone solutions without assuming any nondegeneracy assumptions on f. Besides common tools like intersection comparison (or zero number) arguments, our method is based on analysis of spatial trajectories of solutions of (1.1). If u is a solution, then its *spatial trajectory at time* t is the (not necessarily simple) curve $\tau(u(\cdot,t)) := \{(u(x,t), u_x(x,t)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Note that if u is a steady state, then its spatial trajectory is independent of t and it is a trajectory, in the usual sense, of the first-order system corresponding to the equation $u_{xx} + f(u) = 0$. Likewise, if u is a traveling wave, then its spatial trajectory is independent of t and it is a trajectory of the first order system corresponding to the equation $u_{xx} + cu_x + f(u) = 0$, where c is the speed of the wave. In our proofs, we identify a class of trajectories of the equations $u_{xx} + cu_x + f(u) = 0$, $c \in \mathbb{R}$, which cannot be intersected by the spatial trajectories $\tau(u(\cdot,t))$ if t is large enough. This way we show that, as $t \to \infty$, $\tau(u(\cdot,t))$ is confined to an arbitrarily small neighborhood of the spatial trajectories of traveling fronts coming from a minimal propagating terrace. From this, our theorems are derived via a unique-continuation type argument. Although, due to the generality of the results, our proofs are not short or simple, they are elementary to a large extent, with their key building blocks provided by a phase plane analysis. We remark that in our earlier work [25], we used similar techniques in a geometric proof of the convergence to traveling fronts for bistable nonlinearities. An application of the method of spatial trajectories can be seen there in a much simpler setting.

It may be instructive to compare our method with techniques used in earlier proofs of the convergence to traveling fronts and families of traveling fronts for monotone solutions. Given a solution u with $u_x < 0$, one can consider the function $p(u,t) = u_x(\sigma(u,t),t)$, where $\sigma(\cdot,t)$ is the inverse to $u(\cdot,t)$. In [12], it is shown that $p(u,t)$ satisfies a degenerate parabolic equation and convergence theorems are proved there by comparison arguments for this equation. Comparison techniques are also used in [33], although the underlying technical steps are proved in a different way, without reference to the degenerate equation for $p(u, t)$. Observe that for any t, the graph of $p(\cdot, t)$ is in fact the spatial trajectory $\tau(u(\cdot, t))$ of $u(\cdot, t)$ as defined above. Obviously, for the spatial trajectory to be such a graph, the monotonicity of $u(x, t)$ in x is necessary. In contrast, we work with spatial trajectories as curves in $\mathbb{R}^2$, not necessarily graphs, thus we do not need the monotonicity assumption.

The theorems stated above are consequences of more general results given in the next section. Specifically, Theorems 1.1, 1.2 follow from Theorem 2.11(iii) (which treats the case when 0 and γ are both stable for the equation $\dot{\theta} = f(\theta)$), Theorem 2.19(vi) (where either 0 is unstable from above or γ is unstable from below), and Proposition 2.12 (which gives the finiteness of the set of traveling fronts in the minimal propagating terrace and, under hypothesis (DGM), also the positivity of the speeds). Theorem 1.3 follows from Corollaries 2.10, 2.18; and Theorem 1.4 is contained in Theorem 2.20 and Remark 2.21(i).

The rest of this text is organized as follows. In the next section, we first define the notions of a minimal system of waves and a minimal propagating terrace, and recall their basic properties, as proved in [33]. Then we state our main theorems, first in terms of the Ω-limit set of the solutions, then in terms of minimal propagating terraces. These results are followed by a general quasiconvergence theorem for front-like solutions. Section 2 concludes with some theorems where a nondegeneracy assumption concerning zeros of f and speeds of the traveling fronts is made. The convergence results in these theorems are stronger than in the general case; in particular, they yield an exponential rate of convergence to a propagating terrace.

Sections 4 and 6 contain the proofs of our main results. In the preliminary Section 5, we recall several properties of the zero-number functional and the Ω-limit sets of bounded solutions.

Section 3 is devoted to phase plane analysis of the equations $u_{xx} + cu_x + f(u) = 0$, $c \in \mathbb{R}$. The main purpose of this analysis is to provide solutions which can be used in intersection-comparison arguments in the proofs of our main theorems. Such solutions are exhibited in Subsection 3.3, and we believe that their usefulness goes beyond the scope of this paper—they may come in handy in other studies of (1.1). Section 3 has two other parts where we recall basic properties of trajectories of the equations $u_{xx}+cu_x+f(u) = 0$, $c \in \mathbb{R}$, (Subsection 3.1) and give a more detailed description of the minimal system of waves (Subsection 3.2).

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2 Main results

Throughout the paper our *standing hypotheses on f* are as follows:

(H) f is a (globally) Lipschitz function on $\mathbb{R}$, $\gamma > 0$, $f(0) = f(\gamma) = 0$, and $f\big|_{[0,\gamma]} \in C^1[0,\gamma]$.

We assume the global Lipschitz continuity just for convenience. This is at no cost to generality: since all our results concern a bounded solution, if $f(u)$ is merely locally Lipschitz, we can always modify it outside the range of the solution to make it globally Lipschitz.

We denote by D_0 and D_γ the sets of attraction with respect to the equation $\dot\theta = f(\theta)$ of the equilibria 0 and γ. Recall that the set, or domain, of attraction of an equilibrium ζ is the set of all initial values from which the solution converges to ζ. This set trivially contains ζ, and it may or may not contain other points depending on whether ζ is asymptotically stable from above or from below. Specifically,

$$D_\zeta = \{\zeta\} \cup (\zeta, \zeta^+) \cup (\zeta^-, \zeta), \tag{2.1}$$

where (ζ, ζ^+) is the maximal interval of this form on which $f < 0$ if such an interval exists, otherwise $(\zeta, \zeta^+) = \emptyset$. The set (ζ^-, ζ) is defined in an analogous way (with $f > 0$ on (ζ^-, ζ)). The notions of stability and asymptotic stability of $\zeta \in f^{-1}\{0\}$, relative to the equation $\dot\theta = f(\theta)$, are understood in the usual Lyapunov sense (there is no need to introduce the stability of ζ relative to (1.1)). Similarly we use the notions of stability of ζ from above or from below. In terms of f, ζ is asymptotically stable from above for the equation $\dot\theta = f(\theta)$ if $(\zeta, \zeta^+) \neq \emptyset$; it is stable from above if either $(\zeta, \zeta^+) \neq \emptyset$ or there is a sequence $\zeta_n \in f^{-1}\{0\}$ such that $\zeta_n \searrow \zeta$. Analogous conditions characterize the stability from below. Unstable means not stable.

Our *standing hypotheses on initial data* $u_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ are as follows:

$$\liminf_{x\to-\infty} u_0(x) \in D_\gamma, \ \sup_{x\in\mathbb{R}} u_0(x) \in D_\gamma; \tag{2.2}$$

$$\limsup_{x\to\infty} u_0(x) \in D_0, \ \inf_{x\in\mathbb{R}} u_0(x) \in D_0. \tag{2.3}$$

Thus $\sup_{x\in\mathbb{R}} u_0(x)$ may be greater than γ, as long as it is in D_γ. We do not require the existence of the limit of $u_0(x)$ as $x \to -\infty$; only that the inferior and superior limits are both in D_γ (for the superior limit, this follows from (2.2)). Of course, if γ is unstable both from above and below, condition (2.2) reduces to the requirement that $u_0 \leq \gamma$ and $\lim_{x\to-\infty} u_0(x) = \gamma$. Similar remarks apply to the equilibrium 0.

Recall that Ω- and ω-limit sets of a bounded solution u of (1.1) were defined in (1.8) and (1.9). Since the solution u is determined uniquely by its initial value u_0, we sometimes use the symbols $\omega(u_0)$, $\Omega(u_0)$ for $\omega(u)$, $\Omega(u)$. By standard parabolic regularity estimates, the set $\{u(x+\cdot,t) : t \geq 1,\ x \in \mathbb{R}\}$ is relatively compact in $L^\infty_{loc}(\mathbb{R})$. This implies that both $\omega(u)$ and $\Omega(u)$ are nonempty, compact, and connected in $L^\infty_{loc}(\mathbb{R})$ (see Subsection 5.1 below for more details). Notice also that $\Omega(u)$ is invariant under translations: with each φ it contains $\varphi(\cdot - \xi)$ for each $\xi \in \mathbb{R}$.

2.1 Minimal systems of waves and propagating terraces

We first define, following [33], the notion of a minimal $[0,\gamma]$-system of waves. We use the notation $\tau(\phi)$ as in the introduction: if ϕ is a C^1 function on an interval J (usually $J = \mathbb{R}$), we set

$$\tau(\phi) = \{(\phi(x), \phi_x(x) : x \in J\}. \tag{2.4}$$

Definition 2.1. A $[0,\gamma]$-system of waves, or simply a *system of waves* if there is no danger of confusion, is a continuous function R on $[0,\gamma]$ with the following properties:

(i) $R(0) = R(\gamma) = 0, \quad R(u) \leq 0 \ \ (u \in [0,\gamma])$;

(ii) If $I = (a,b) \subset [0,\gamma]$ is a nodal interval of R, that is, a connected component of the set $R^{-1}(-\infty,0)$, then there is $c \in \mathbb{R}$ and a decreasing solution ϕ of (1.4) such that $\phi(-\infty) = b$, $\phi(\infty) = a$, and

$$\{(u, R(u)) : u \in (a,b)\} = \tau(\phi). \tag{2.5}$$

Thus the graph of R between its successive zeros is given by the spatial trajectory of a traveling front U (which is the same as the trajectory of its profile function ϕ).

Definition 2.2. A system of waves R_0 is said to be *minimal* if for an arbitrary system of waves R one has

$$R_0(u) \le R(u) \quad (u \in [0, \gamma]).$$

By definition, the minimal system of waves is unique. As shown in [33, Theorem 1.3.2], for any f satisfying (H), a minimal system of waves exists and can be found as follows. For each $u \in [0, \gamma]$, set

$$R_0(u) = \inf_{\phi} \phi'(0), \tag{2.6}$$

where ϕ is the decreasing profile function of a traveling front with the range in $[0, \gamma]$ such that $\phi(0) = u$. The infimum is taken over all such ϕ; if no such ϕ exists, one puts $R_0(u) = 0$.

Additional basic properties of R_0 are stated in the next theorem. We use the following notation. If ϕ is a strictly monotone C^1 function on an interval J, then p^{ϕ} is a function defined on the range of ϕ as follows:

$$p^{\phi}(u) = \phi_x(x), \text{ where } x \in J \text{ is the unique point with } \phi(x) = u. \tag{2.7}$$

In other words, $u \mapsto p^{\phi}(u)$ is the function whose graph is given by the spatial trajectory $\tau(\phi)$. Note that with this notation, relation (2.6) reads as follows:

$$R_0(u) = \inf_{\phi} p^{\phi}(u). \tag{2.8}$$

It is a simple observation, frequently used in classical studies of traveling fronts and employed in this paper as well, that ϕ is a solution of (1.4) with $\phi' < 0$ if and only if $p = p^{\phi}(u)$ is a negative solution of the first-order equation

$$p'(u) = -\frac{f(u)}{p(u)} - c. \tag{2.9}$$

Theorem 2.3. *For any f satisfying* (H)*, there exists a unique minimal system of waves R_0. Moreover, R_0 has the following properties:*

(i) $R_0^{-1}(0) \subset f^{-1}(0)$;

(ii) *If $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$ are nodal intervals of R_0 with $b_1 \le a_2$, and if c_1, c_2 are the speeds of the traveling fronts from Definition 2.1 corresponding to I_1, I_2, respectively, then $c_1 \ge c_2$.*

(iii) *If ϕ is the decreasing profile function of a traveling front with range in $[0, \gamma]$ and p^ϕ is defined as in (2.7), then $R_0(u) \le p^\phi(u)$ for all $u \in (\phi(\infty), \phi(-\infty))$.*

Statement (iii) follows directly from (2.8), but the other two statements are not so trivial. Their proofs can be found in [33, Sect. 1.3] (related results for monotone systems and multidimensional problems were proved in [34, 36]). We remark, that the existence of a minimal system of waves with finitely many zeros also follows from [11], if one assumes the existence of at least one system of waves with finitely many zeros. Also, another result of [11] can be rephrased as saying that a system of waves with finitely many zeros is necessarily minimal if its speeds have the monotonicity property as in statement (ii).

Statement (iii) in particular implies that, for any nodal interval $I = (a, b)$ of R_0, the traveling front from Definition 2.1 has the minimal profile function for all traveling fronts on that interval. More specifically, if c, ϕ are as in Definition 2.1 and ψ is the decreasing profile function of another traveling front with the same range I, then $p^\phi \le p^\psi$ on I. Of course, this remark is of relevance only in the case of nonuniqueness, disregarding spatial translations, of traveling fronts with range I. It is well known (see [33, Theorem 1.3.14]) that the nonuniqueness can occur only if either a is unstable from above and b is stable from below, or a is stable from above and b is unstable from below, where the stability is relative to the equation $\dot{\theta} = f(\theta)$. In the former case, the speed c is necessarily positive and in the latter case it is negative (see [33, Theorem 1.3.14]). Moreover, in both cases of nonuniqueness the speed c is extremal in the following sense.

Theorem 2.4. *Let R_0 be the minimal system of waves, $I = (a, b)$ a nodal interval of R_0, and let c, ϕ be as in Definition 2.1. If ψ and $\tilde{c} \neq c$ are, respectively, the decreasing profile function and speed of another traveling front with range I, then necessarily $c \neq 0$ and the following statements hold. If $c > 0$, then $\tilde{c} > c$ (that is, c is the minimal speed for the interval I); and if $c < 0$, then $\tilde{c} < c$ (c is the maximal speed for the interval I).*

This is proved in [33], see Theorem 1.3.8 and Part 2 of Theorem 1.3.14 in that monograph.

Let R_0 be the minimal system of waves. We denote by $\mathcal{N}$ the (countable) set of all nodal intervals of R_0. Since R_0 is single-valued, for each $I \in \mathcal{N}$ the speed $c = c_I$ and the solution $\phi = \phi_I$ in Definition 2.1(ii) are determined uniquely if we postulate

$$\phi(0) = \frac{a+b}{2}. \tag{2.10}$$

This way we obtain the *families of speeds and profile functions* corresponding to R_0:

$$\{c_I : I \in \mathcal{N}\},\ \{\phi_I : I \in \mathcal{N}\}. \tag{2.11}$$

We define a natural ordering on $\mathcal{N}$:

$$I_1 < I_2 \text{ if } I_1 = (a_1, b_1),\ I_2 = (a_2, b_2) \text{ and } b_1 \le a_2, \tag{2.12}$$

and write $I_1 \le I_2$ if $I_1 = I_2$ or $I_1 < I_2$. Since two different nodal intervals of R_0 cannot overlap, $\mathcal{N}$ is simply ordered by this relation. By Theorem 2.3(ii),

$$\text{if } I_1 < I_2, \text{ then } c_{I_1} \ge c_{I_2}. \tag{2.13}$$

Also, by the definition of R_0 and Theorem 2.3(i), the boundary points a, b of any interval $(a, b) \in \mathcal{N}$ are in $R_0^{-1}(0) \subset f^{-1}(0)$. However, not all elements of $R_0^{-1}(0) \setminus \{\gamma\}$ are boundary points of nodal intervals of R_0. Indeed, $R_0^{-1}(0)$ may have accumulations points and it may even contain intervals. More information on the set $R_0^{-1}(0)$ is given in Propositions 2.12 and 3.11 below (see also Figure 5 in Section 3.2).

Consider now the family of traveling fronts $U_I(x,t) = \phi_I(x - c_I t)$, $I \in \mathcal{N}$. We refer to this family as the $[0,\gamma]$-*minimal propagating terrace* or simply the *minimal propagating terrace*, as the interval $[0,\gamma]$ is fixed. Compared to [9], where the name originates, our definition is broader in that we allow the set $\mathcal{N}$ to be infinite and the family of speeds $\{c_I : I \in \mathcal{N}\}$ to include both positive, negative, as well as zero values. If the set $R_0^{-1}\{0\}$ is finite and the speeds c_I are all positive, then the minimal propagating terrace, as defined here, is essentially the same concept as in [9]. Indeed, in this case, the properties of a minimal systems of waves as given by Definition 2.2 and statements (ii), (iii) of Theorem 2.3(ii) translate to the defining properties of a minimal propagating terrace, as given in [9]. In particular, the property stated in Theorem 2.3(iii) can be interpreted as a steepness property introduced in [9]. Note, however, that in [9] propagating terraces are considered in a more general framework of spatially periodic equations and pulsating traveling fronts.

Below we use the following notation

$$\begin{aligned}\mathcal{N}^+ &:= \{I \in \mathcal{N} : c_I > 0\},\\ \mathcal{N}^- &:= \{I \in \mathcal{N} : c_I < 0\},\\ \mathcal{N}^0 &:= \{I \in \mathcal{N} : c_I = 0\}.\end{aligned} \tag{2.14}$$

2.2 The case where 0 and γ are both stable

In this subsection, we assume the following stability conditions

(S) For the equation $\dot{\theta} = f(\theta)$, the equilibrium 0 is stable from above and the equilibrium γ is stable from below.

The stability assumed here is not necessarily asymptotic. In particular, f may vanish on a neighborhood of 0 or γ in $[0, \gamma]$.

Our first results in this subsection describe the Ω-limit sets of the solution of (1.1), (1.2). In Theorem 2.7, we make an assumption on the structure of the nonlinearity, to avoid certain degeneracies, and show that $\Omega(u)$ is as in Theorem 1.3 from the introduction: it consists of the profile functions from the minimal propagating terrace and constant steady states. In this theorem, no conditions other than (2.2)-(2.3) are assumed on u_0. These conditions on u_0 are sharp; one can show easily by counterexamples that the conclusion of Theorem 2.7 does not hold in general if any of the conditions in (2.2)-(2.3) is violated. On the other hand, in Theorem 2.9 we impose slightly stronger assumptions on u_0, but there the only assumptions on the nonlinearity are (H) and (S). We do have a result, Theorem 2.5, on $\Omega(u)$ in the completely general case, when no assumptions on f or u_0 other than (H), (S) and (2.2), (2.3) are made. It is somewhat weaker, but still interesting and consequential. Moreover, one can use it in combination with other conditions on f or u_0 to obtain the stronger conclusion as in Theorems 2.7, 2.9. The last theorem of this subsection, Theorem 2.11, shows in detail how the global shape of the solutions for large times is described in terms of the minimal propagating terrace.

Throughout the subsection R_0 is the minimal system of waves (for the interval $[0, \gamma]$) and $\{c_I : I \in \mathcal{N}\}$, $\{\varphi_I : I \in \mathcal{N}\}$ are the corresponding families of speeds and profile functions, as introduced above (see (2.11)).

We start with the most general theorem assuming only the standing hypotheses and condition (S).

Theorem 2.5. *Assume that (in addition to* (H)*) hypothesis* (S) *is satisfied. Then, given any $u_0 \in C(\mathbb{R})$ satisfying* (2.2), (2.3), *one has*

$$\begin{aligned} R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\} &\subset \Omega(u_0), \\ \Omega(u_0) \subset & R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\} \cup \{\hat{\phi}_I(\cdot - \xi) : I \in \mathcal{N}^0,\, \xi \in \mathbb{R}\}, \end{aligned} \tag{2.15}$$

where $\mathcal{N}^0$ is as in (2.14) *and, for each $I \in \mathcal{N}^0$, $\hat{\phi}_I(x) = \phi_I(-x)$ $(x \in \mathbb{R})$.*

In this theorem and similar statements below, each constant from $R_0^{-1}\{0\}$ is identified with the constant function taking that value.

Recall that $I \in \mathcal{N}^0$ means that $c_I = 0$. Thus, for $I \in \mathcal{N}^0$ the function ϕ_I is a (decreasing) standing front of (1.1) and $\hat{\phi}_I$ is an increasing standing front. Compared to Theorem 1.3 in the introduction, the conclusion of Theorem 2.5 is weaker in that it allows for the possibility that $\Omega(u_0)$ contains some increasing standing fronts. Of course, this is an issue only when $\mathcal{N}^0 \neq \emptyset$, that is, when the minimal propagating terrace does include some standing fronts. We have no examples with $\hat{\phi}_I$ actually contained in $\Omega(u_0)$ for some solutions u satisfying (2.2), (2.3), but are not able to rule this possibility out in the general setting of Theorem 2.5. We will do this under additional conditions on f or u_0 (see Theorems 2.7, 2.9 below).

Remark 2.6. Clearly, by the translation invariance of $\Omega(u_0)$, if one can show that

$$\hat{\phi}_I \notin \Omega(u_0) \quad (I \in \mathcal{N}^0), \tag{2.16}$$

then (2.15) gives

$$\Omega(u_0) = R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\}. \tag{2.17}$$

Theorem 2.7. *Assume that* (S) *is satisfied. Assume further that either $\mathcal{N}^0 = \emptyset$ or the sets $\mathcal{N}^+$, $\mathcal{N}^-$ are both nonempty. Then for each $u_0 \in C(\mathbb{R})$ satisfying* (2.2), (2.3) *relation* (2.17) *holds.*

The extra conditions in this theorem concern the nonlinearity f only. They can be expressed explicitly in terms of the function

$$F(u) := \int_0^u f(s)\, ds \tag{2.18}$$

as follows.

Proposition 2.8. *One has $\mathcal{N}^0 = \emptyset$ if and only if the set of the global maximizers of F in $[0,\gamma]$ is an interval $[\gamma_*, \gamma^*]$ (reducing possibly a single point) for some $\gamma_* \le \gamma^*$. One has $\mathcal{N}^+ \neq \emptyset$ if and only if 0 is not a global maximizer of F in $[0,\gamma]$, and $\mathcal{N}^- \neq \emptyset$ if and only if γ is not a global maximizer of F in $[0,\gamma]$.*

This is proposition is proved in Section 4. Note in particular that condition $\mathcal{N}^0 = \emptyset$ is generic (it is satisfied by an open and dense set of functions f in "reasonable" topologies).

If $\mathcal{N}^0 \neq \emptyset$ and one of the sets $\mathcal{N}^+$, $\mathcal{N}^-$ is empty, then, in view Theorem 2.3(ii), the minimal propagating terrace has standing waves on its "top" or "bottom." This gives us some technical difficulties and to prove the stronger conclusion (2.17) in this case we need additional conditions on u_0. Simple sufficient conditions include conditions (1.3) from the introduction, as well as the following slightly more general conditions:

$$u_0 \ge \lim_{x\to\infty} u_0(x) \in D_0, \tag{2.19}$$

$$u_0 \le \lim_{x\to-\infty} u_0(x) \in D_\gamma \tag{2.20}$$

(see Corollary 2.10 below). Note that (2.19), (2.20) hold in particular if u_0 is a continuous monotone function satisfying (2.2), (2.3), or if u_0 is sandwiched between two shifts of such a monotone function.

We now formulate more general hypotheses on u_0, which do not require the existence of the limits $u_0(\pm\infty)$. To motivate these hypotheses, recall that we are trying to show that (2.16) holds. Now, if for some $I = (a,b) \in \mathcal{N}^0$ one has $\hat{\phi}_I \in O(u)$, then (2.15) can be used to show, as we do in Section 6.7, that for any $\alpha \in (a,b) \cap f^{-1}\{0\}$ the function $u(\cdot,t) - \alpha$ has several zeros whose mutual distances diverge to ∞ as $t \to \infty$. We can rule this scenario out, using a comparison argument, if $\limsup_{x\to\infty} u_0(x)$ and $\liminf_{x\to-\infty} u_0(x)$ bound the values of u_0 at points between zeros of $u_0 - \alpha$. For this reason, we make the following assumptions concerning elements $I = (a,b) \in \mathcal{N}^0$ (if $\mathcal{N}^0 \neq \emptyset$) (cp. Figure 2).

(Z0) There is $\alpha \in f^{-1}\{0\} \cap (a,b)$ such that

$$\limsup_{x\to\infty} u_0(x) \le \min_{x \le y_0} u_0(x), \quad \text{where } y_0 = \max\{x \in \mathbb{R} : u_0(x) = \alpha\}. \tag{2.21}$$

(Z1) There is $\beta \in f^{-1}\{0\} \cap (a,b)$ with

$$\liminf_{x\to-\infty} u_0(x) \geq \max_{x\geq y_1} u_0(x), \quad \text{where } y_1 = \min\{x \in \mathbb{R} : u_0(x) = \beta\}. \tag{2.22}$$

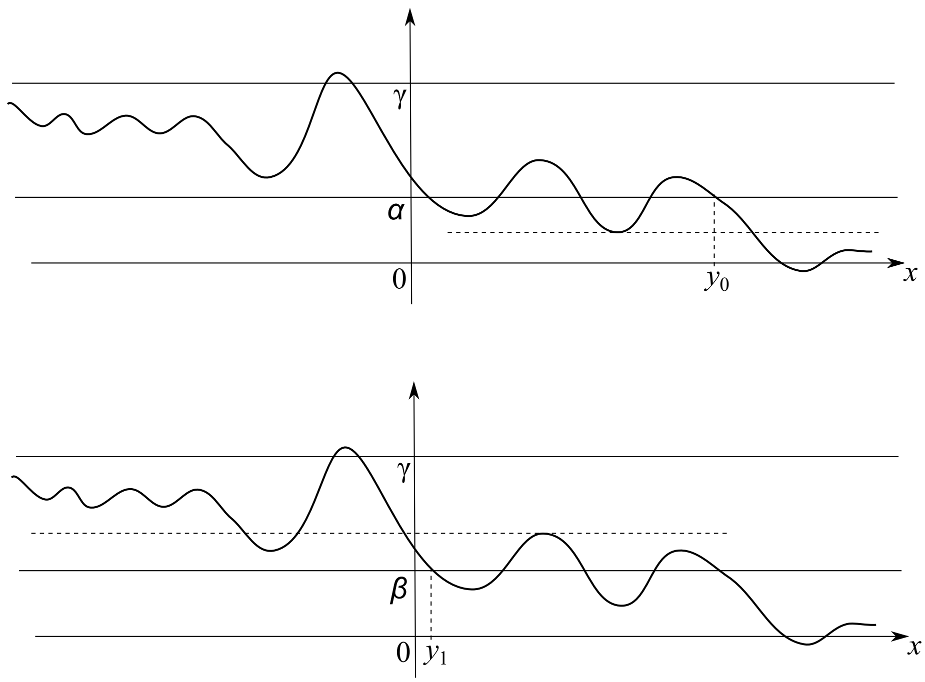

Figure 2: The figures illustrate conditions (Z0) (top) and (Z1).

We will require these conditions to be satisfied for certain $I \in \mathcal{N}^0$, as specified below. Let us make a few comments on the conditions. First of all, if $I = (a,b) \in \mathcal{N}$ and $c_I = 0$, so that there is a standing front with range (a,b), then the interval (a,b) necessarily contains zeros of f. If α is any such zero, then, by assumptions (2.2), (2.3),

$$\limsup_{x\to\infty} u_0(x) < \alpha < \liminf_{x\to-\infty} u_0(x).$$

Therefore, the value y_0 in (2.21) is well defined and so is $\min_{x\leq y_0} u_0(x)$. A simple sufficient condition for (2.21) is that the function $u_0 - \alpha$ has a unique zero. Clearly, if (2.21) holds for some $\alpha \in f^{-1}\{0\} \cap (0,\gamma)$, then it holds for any $\tilde{\alpha} \in f^{-1}\{0\} \cap (0,\gamma)$ with $\tilde{\alpha} \geq \alpha$. Thus, if $\mathcal{N}^0 \neq \emptyset$ and the set $\mathcal{N}^0$ has a minimal element I_0 with respect to the ordering (2.12), then (Z0) holds for any $I \in \mathcal{N}^0$, provided it holds for $I = I_0$. Analogous comments apply to condition (Z1).

In the next theorem, the assumptions on f are complementary to the assumption of Theorem 2.7: $\mathcal{N}^+ \neq \emptyset$ and one of the sets $\mathcal{N}^+$, $\mathcal{N}^-$ is empty.

Theorem 2.9. *Assume that* (H) *and* (S) *hold and let $u_0 \in C(\mathbb{R})$ satisfy* (2.2)*,* (2.3)*. Assume further that one of the following hypotheses* (a1)–(a3) *holds.*

(a1) $\mathcal{N}^0 \neq \emptyset$, $\mathcal{N}^+ \neq \emptyset = \mathcal{N}^-$, *and* (Z1) *holds for each $I \in \mathcal{N}^0$.*

(a2) $\mathcal{N}^0 \neq \emptyset$, $\mathcal{N}^+ = \emptyset \neq \mathcal{N}^-$, *and* (Z0) *holds for each $I \in \mathcal{N}^0$.*

(a3) $\mathcal{N}^+ = \emptyset = \mathcal{N}^-$ *(that is, $c_I = 0$ for all $I \in \mathcal{N}$) and for each $I = (a,b) \in \mathcal{N}$ conditions* (Z0), (Z1) *hold with $\alpha = \beta$.*

Then relation (2.17) *holds.*

This theorem applies in particular to initial data satisfying (2.19), (2.20):

Corollary 2.10. *The statement of Theorem 2.9 remains valid if instead of* (a1)–(a3) *one assumes that conditions* (2.19), (2.20) *are satisfied.*

Proof. If $\mathcal{N}^0 = \emptyset$ or $\mathcal{N}^+ \neq \emptyset \neq \mathcal{N}^+$, then Theorem 2.7 applies. Assume that $\mathcal{N}^0 \neq \emptyset$ and one of the sets $\mathcal{N}^+$, $\mathcal{N}^-$ is empty. Clearly, conditions (2.19), (2.20) imply that (2.21) holds for any $\alpha \in (0,\gamma)$ and (2.22) holds for any $\beta \in (0,\gamma)$. In particular, for each $I = (a,b) \in \mathcal{N}^0$ conditions (Z0), (Z1) hold with $\alpha = \beta$. Therefore Theorem 2.9 applies and we obtain the desired conclusion. $\square$

The next theorem gives a formulation of the main results of this subsection in terms the minimal propagating terrace.

Theorem 2.11. *Assume that hypothesis* (S) *holds and $u_0 \in C(\mathbb{R})$ satisfies* (2.2)*,* (2.3)*. Let u be the solution of* (1.1)*,* (1.2)*. The following statements are valid:*

(i) *For each $I = (a,b) \in \mathcal{N}$ with $c_I \neq 0$ there is a C^1 function ζ_I defined on some interval (s_I, ∞) such that the following statements hold:*

  (a) $\lim_{t\to\infty} \zeta_I'(t) = 0 \quad (I \in \mathcal{N})$;

(b) $\big((a+b)/2 - u(x + c_I t + \zeta_I(t), t)\big)x > 0 \quad (x \in \mathbb{R} \setminus \{0\},\ t > s_I)$;

(c) $\lim_{t\to\infty} \big(u(\cdot + c_I t + \zeta_I(t), t) - \phi_I\big) = 0$, *locally uniformly on* $\mathbb{R}$;

(d) *if* $I_1, I_2 \in \mathcal{N}$, $I_1 < I_2$, *and* $c_{I_1} = c_{I_2} \neq 0$, *then* $\zeta_{I_1}(t) - \zeta_{I_2}(t) \to \infty$ *as* $t \to \infty$;

(ii) *If* (2.16) *holds (as in Theorem 2.7, Theorem 2.9, or Corollary 2.10), then statement (i) is valid without the restrictions* $c_I \neq 0$, $c_{I_2} \neq 0$. *In this case, the following holds as well:*

(e) *if* $\{(x_n, t_n)\}$ *is any sequence in* $\mathbb{R}^2$ *such that* $t_n \to \infty$ *and for each* $I \in \mathcal{N}$ *one has*

$$\lim_{n\to\infty} |c_I t_n + \zeta_I(t_n) - x_n| = \infty,$$

then there exist a subsequence $\{(x_{n_k}, t_{n_k})\}$ *and* $\xi \in R_0^{-1}\{0\}$ *such that*

$$\lim_{k\to\infty} u(x_{n_k} + \cdot, t_{n_k}) = \xi,$$

locally uniformly on $\mathbb{R}$.

(iii) *If* (2.16) *holds and the set* $R_0^{-1}\{0\}$ *is finite, say*

$$R_0^{-1}\{0\} = \{a_1, \dots, a_{k+1}\}, \text{ with } 0 = a_1 < a_2 < \cdots < a_{k+1} = \gamma, \tag{2.23}$$

then $\mathcal{N} = \{I_1, \dots, I_k\}$ *with* $I_j = (a_j, a_{j+1})$, $j = 1, \dots, k$, *and, as* $t \to \infty$, *one has*

$$u(x,t) - \Big(\sum_{j=1,\dots,k} \phi_{I_j}(x - c_{I_j} t - \zeta_{I_j}(t)) - \sum_{1\le j\le k-1} a_{j+1}\Big) \to 0 \quad (x \in \mathbb{R}), \tag{2.24}$$

uniformly on $\mathbb{R}$.

Statement (d) is included here for reference; it is in fact implicitly contained in statement (c). Note that although the functions ζ_I satisfy (a), they are not convergent in general, see Remark 2.24(i) below. In Subsection 2.5, we prove the convergence of the ζ_I under some nondegeneracy conditions on the function f.

Statement (ii) can be summarized as saying that for large t the graph of $u(\cdot, t)$ has flat parts corresponding to $u(\cdot, t)$ being close to a constant on a large interval, and interfaces whose shape is roughly given by the graphs of the functions $\phi_I(\cdot - c_I t - \zeta_I(t))$, $I \in \mathcal{N}$. Note that in (e) the limit ξ may not be uniquely determined by the sequence $\{x_n, t_n\}$ even if the points x_n stay between two successive interfaces. Indeed, the minimal propagating terrace may have a "gap," or, in other words, R_0 may vanish on an interval. By Theorem 2.7, all constants ξ from that interval are elements of $\Omega(u_0)$. Therefore, a suitably chosen sequence $\{x_n, t_n\}$, will have subsequences along which different limits ξ occur in (e).

In the following proposition we give sufficient conditions for the set $R_0^{-1}\{0\}$ to be finite.

Proposition 2.12. *Let R_0 be the minimal $[0,\gamma]$-system of waves, and $\{c_I : I \in \mathcal{N}\}$, $\{\varphi_I : I \in \mathcal{N}\}$ the corresponding families of speeds and profile functions. The following statements are valid.*

(i) *If $\{I_j\}$ is a strictly monotone (infinite) sequence in $\mathcal{N}$, then $c_{I_j} \to 0$. Consequently, the set $\mathcal{N}$ is finite if for some $\epsilon > 0$ one has $|c_I| \geq \epsilon$ for all $I \in \mathcal{N}$.*

(ii) *Assume that the function $F(u) = \int_0^u f(s)\,ds$ has only finitely many maximizers in $[0,\gamma]$ and all of them are isolated zeros of f in $[0,\gamma]$. Then the set $R_0^{-1}\{0\}$ is finite. If, moreover, F has a unique maximizer ξ_{max} in $[0,\gamma]$, then $c_I \neq 0$ for each $I \in \mathcal{N}$.*

(iii) *If condition* (DGM) *from the introduction is satisfied, then the assumption of statement (ii) is satisfied with the unique maximizer $\xi_{max} = \gamma$; in this case one has $c_I > 0$ for each $I \in \mathcal{N}$.*

This proposition is proved in Section 4.

2.3 The case where one of the steady states 0, γ is unstable

We next consider the case where either 0 is unstable from above, or γ is unstable from below. Here and below, the stability is with respect to the equation $\dot{\theta} = f(\theta)$.

If 0 is unstable from above, that is, $f > 0$ on some interval $(0,\delta)$, then $R_0 < 0$ on this interval (see Theorem 2.3(i)). Hence $\mathcal{N}$ contains an interval $I = (0,b)$ with $b > 0$. Of course, I is then the minimal element of $\mathcal{N}$ in the ordering (2.12). The corresponding traveling front $U_I = \phi_I(x - c_I t)$ connects the positive steady state b to 0. Necessarily, b is stable from below, $c_I > 0$, and c_I is the minimal speed for all traveling fronts connecting b and 0 (see Theorem 2.4). We define γ_0 to be 0 if 0 is stable from above; if it is unstable from above, we define γ_0 to be the value b identified above. Similarly, if γ is unstable from below, then $\mathcal{N}$ contains an interval $I := (a, \gamma)$ with $a < \gamma$ and $c_I < 0$. We define γ_1 to be this value a if γ is unstable from below; otherwise we set $\gamma_1 = \gamma$.

Let now R_0 be the minimal $[0, \gamma]$-system of waves, and $\{c_I : I \in \mathcal{N}\}$, $\{\varphi_I : I \in \mathcal{N}\}$ the corresponding families of speeds and profile functions Using the definition of a minimal system of waves, one verifies easily that $\tilde{R}_0 := R_0 \big|_{[\gamma_0,\gamma_1]}$ is the minimal $[\gamma_0, \gamma_1]$-system of waves. Its families of speeds and profile functions are $\{c_I : I \in \tilde{\mathcal{N}}\}$, $\{\varphi_I : I \in \tilde{\mathcal{N}}\}$, where

$$\tilde{\mathcal{N}} := \{I \in \mathcal{N} : I \subset (\gamma_0, \gamma_1)\}. \tag{2.25}$$

The following theorem extends Theorem 2.5 (it gives a new result if one of the instabilities $\gamma_0 > 0$ or $\gamma_1 < \gamma$ occurs; otherwise it just recovers the statement of Theorem 2.5).

Theorem 2.13. *Assume that $u_0 \in C(\mathbb{R})$ satisfies (2.2), (2.3). Then*

$$\begin{aligned} &R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\, \xi \in \mathbb{R}\} \subset \Omega(u_0), \\ &\Omega(u_0) \subset R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\, \xi \in \mathbb{R}\} \\ &\qquad\qquad\qquad \cup \{\hat{\phi}_I(\cdot - \xi) : I \in \mathcal{N}^0,\, \xi \in \mathbb{R}\} \cup \Omega_0 \cup \Omega_1, \end{aligned} \tag{2.26}$$

where $\hat{\phi}_I(x) = \phi_I(-x)$, Ω_0 is a set of functions with range in $(0, \gamma_0)$, and Ω_1 is a set of functions with range in (γ_1, γ).

Remark 2.14. Similarly as in Remark 2.6, if (2.16) holds, that is, $\hat{\phi}_I \notin \Omega(u_0)$ for any $I \in \mathcal{N}^0$, then (2.26) is the same as

$$\Omega(u_0) = R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\, \xi \in \mathbb{R}\} \cup \Omega_0 \cup \Omega_1. \tag{2.27}$$

Sufficient conditions for (2.16) similar to Theorems 2.7, 2.9 will be given in Theorem 2.17. Here we mention a simple sufficient condition, which is sometimes convenient to use: for some $\xi > 0$ one has $u_0(x + \xi) \le u_0(x)$ ($x \in \mathbb{R}$). To see that this condition is indeed sufficient for (2.16), apply the comparison principle to obtain $u(\cdot + \xi, t) \le u(\cdot, t)$ for all $x \in \mathbb{R}$ and $t > 0$. This implies that $\varphi(\xi) \le \varphi(0)$ for each $\varphi \in \Omega(u_0)$, hence the increasing functions $\hat{\phi}_I$ cannot be contained in $\Omega(u_0)$.

If at least one of the instabilities $\gamma_0 > 0$ or $\gamma_1 < \gamma$ occurs, Theorem 2.13 and related results in Theorems 2.17 and 2.19 below only describe the part of the graph of $u(\cdot, t)$ contained in the strip $\{\gamma_0 < u < \gamma_1\}$. Without additional assumptions regarding the behavior of $u_0(x)$ as $x \to \pm\infty$, not much more can be said about about the remaining part of the graph or the sets Ω_0, Ω_1. Even in the simplest situation when $f > 0$ in $[0, \gamma]$ (in which case $\Omega(u_0)$ reduces to Ω_0), it is known that the solution does not in general approach any traveling front. It may oscillate between fronts with different speeds [15, 39] or it may even propagate faster than any traveling front [16]. As is also well known (see [32, 33], for example), in this simple case, one can guarantee that the solution approaches a traveling front by imposing a specific decay rate of $u_0(x)$ as $x \to \infty$. For the convergence to the traveling front with the minimal speed, one can assume that u_0 has a faster exponential decay rate than that traveling front, or, to have a condition independent of f, that u_0 vanishes on an interval (m, ∞). Under similar conditions, we will prove the approach of solutions to the minimal propagating terrace. For simplicity, we will assume that $u_0(x)$ vanishes identically for $x \approx \infty$ (or $\gamma - u_0(x)$ vanishes identically for $x \approx -\infty$), but this can be replaced by a condition on a sufficiently fast exponential decay of $u_0(x)$, see Remark 6.7 below. Our techniques can also be used to prove the convergence to a propagating terrace, not necessarily the minimal one, with a specific front at the bottom of the terrace (for monotone initial conditions, results to that effect are proved in [33]), but such results are not pursued here. In the next two theorems, we make the following assumptions on u_0 strengthening (2.2) or (2.3) (or both).

(U) In the case $\gamma_0 > 0$ (that is, when 0 is unstable from above for the equation $\dot{\theta} = f(\theta)$), we assume that $u_0 \geq 0$ and $u_0 \equiv 0$ on an interval $[m, \infty)$; and in the case $\gamma_1 < \gamma$ (that is, when γ is unstable from below for $\dot{\theta} = f(\theta)$), we assume that $u_0 \leq \gamma$ and $u_0 \equiv \gamma$ on an interval $(-\infty, n]$.

Theorem 2.15. *Assume that* $u_0 \in C(\mathbb{R})$ *satisfies* (2.2), (2.3) *and hypotheses* (U) *is in effect.*

(i) *If* $\gamma_0 > 0$, *then the set* Ω_0 *in Theorem 2.13 is given by*

$$\Omega_0 = \{\phi_{I_*}(\cdot - \xi) : \xi \in \mathbb{R}\},$$

where $I_* = (0, \gamma_0)$ *(*I_* *is the minimal element of* $\mathcal{N}$*).*

(ii) *If $\gamma_1 < \gamma$, then the set Ω_1 in Theorem 2.13 is given by*

$$\Omega_1 = \{\phi_{I^*}(\cdot - \xi) : \xi \in \mathbb{R}\},$$

where $I^* = (\gamma_1, \gamma)$ (I^* is the maximal element of $\mathcal{N}$).

Of course, statement (ii) is completely analogous to statement (i) (in fact, it can be deduced from (i) by a suitable transformation reversing the roles of 0 and γ). These results and Theorem 2.13 yield a description of $\Omega(u_0)$ in terms of the minimal propagating terrace, as in Theorem 2.5:

Corollary 2.16. *Assume that $u_0 \in C(\mathbb{R})$ satisfies* (2.2), (2.3) *and hypothesis* (U) *is in effect. Then*

$$\begin{aligned} R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\} &\subset \Omega(u_0), \\ \Omega(u_0) \subset R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\} &\cup \{\hat{\phi}_I(\cdot - \xi) : I \in \mathcal{N}^0,\, \xi \in \mathbb{R}\}. \end{aligned} \tag{2.28}$$

If (2.16) *holds, then* (2.28) *reduces to*

$$\Omega(u_0) = R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\}. \tag{2.29}$$

Here are sufficient conditions for the stronger conclusion (2.29).

Theorem 2.17. *Assume that either $\gamma_0 > 0$ or $\gamma_1 < \gamma$; $u_0 \in C(\mathbb{R})$ satisfies conditions* (2.2), (2.3)*; and hypothesis* (U) *is in effect. In case $\mathcal{N}^0 \neq \emptyset$, assume further that the following two conditions are satisfied:*

(b1) *If $\gamma_0 = 0$ (and $\gamma_1 < \gamma$), then either $\mathcal{N}^+ \neq \emptyset$ or* (Z0) *holds for each $I \in \mathcal{N}^0$.*

(b2) *If $\gamma_1 = \gamma$ (and $\gamma_0 > 0$), then either $\mathcal{N}^- \neq \emptyset$ or* (Z1) *holds for each $I \in \mathcal{N}^0$.*

Then (2.29) *holds.*

This result in particular applies if (2.19), (2.20) are satisfied:

Corollary 2.18. *Let the hypotheses of Corollary* (2.16) *be satisfied with the stronger hypotheses* (2.19), (2.20) *in place of* (2.2), (2.3)*. Then* (2.29) *holds.*

Proof. If $\mathcal{N}^0 = \emptyset$, then, trivially, (2.16) holds, and we obtain (2.29) from Corollary (2.16). If $\mathcal{N}^0 \neq \emptyset$, then, as already mentioned in the proof of Corollary 2.10, relations (2.19), (2.20) imply that for each $I \in \mathcal{N}^0$ conditions (Z0), (Z1) hold. Therefore, Theorem 2.17 applies and we obtain (2.29). □

We conclude this subsection with a theorem similar to Theorem 2.11 describing the shape of the solution in terms of the minimal propagating terrace.

Theorem 2.19. *Assume that* $u_0 \in C(\mathbb{R})$ *satisfies* (2.2), (2.3). *Let* u *be the solution of* (1.1), (1.2). *The following statements are valid:*

(i) *If* $\gamma_0 > 0$ *(that is,* 0 *is unstable from above), then, denoting* $I_* := (0, \gamma_0) \in \mathcal{N}$, *one has* $c_{I_*} > 0$ *and, for any* $c \in [0, c_{I_*})$ *and* $x_0 \in \mathbb{R}$,

$$\liminf_{x \leq x_0,\, t \to \infty} u(x + ct, t) \geq \gamma_0. \tag{2.30}$$

(ii) *If* $\gamma_1 < \gamma$ *(that is,* γ *is unstable from below), then, denoting* $I^* := (\gamma_1, \gamma) \in \mathcal{N}$, *one has* $c_{I^*} < 0$ *and, for any* $c \in (c_{I^*}, 0]$ *and* $x_0 \in \mathbb{R}$,

$$\limsup_{x \geq x_0,\, t \to \infty} u(x + ct, t) \leq \gamma_1. \tag{2.31}$$

(iii) *For each* $I \in \tilde{\mathcal{N}}$ *with* $c_I \neq 0$ *there is a* C^1 *function* ζ_I *defined on some interval on* (s_I, ∞) *such that the following statements hold:*

   (a) $\lim_{t\to\infty} \zeta_I'(t) = 0$;

   (b) $\big((a+b)/2 - u(x + c_I t + \zeta(t), t)\big)x > 0 \quad (x \in \mathbb{R} \setminus \{0\},\ t > s_I)$;

   (c) $\lim_{t\to\infty} \big(u(\cdot + c_I t + \zeta_I(t), t) - \phi_I\big) = 0$, *locally uniformly on* $\mathbb{R}$;

   (d) *if* $I_1, I_2 \in \tilde{\mathcal{N}}$, $I_1 < I_2$, *and* $c_{I_1} = c_{I_2} \neq 0$, *then* $\zeta_{I_1}(t) - \zeta_{I_2}(t) \to \infty$ *as* $t \to \infty$.

(iv) *If* (2.16) *holds (as in Theorem 2.17 or Corollary 2.18), then statement (iii) is valid without the restrictions* $c_I \neq 0$, $c_{I_2} \neq 0$.

(v) *If* (2.16) *holds and hypothesis* (U) *is in effect, then statement (iii) remains valid without the restrictions* $c_I \neq 0$, $c_{I_2} \neq 0$ *and with* $\tilde{\mathcal{N}}$ *replaced by* $\mathcal{N}$. *Moreover, the following statement holds as well:*

   (e) *if* $\{(x_n, t_n)\}$ *is any sequence in* $\mathbb{R}^2$ *such that* $t_n \to \infty$ *and for each* $I \in \mathcal{N}$ *one has*

$$\lim_{n\to\infty} |c_I t_n + \zeta_I(t_n) - x_n| = \infty,$$

then there exist a subsequence $\{(x_{n_k}, t_{n_k})\}$ *and* $\xi \in R_0^{-1}\{0\}$ *such that*

$$\lim_{k\to\infty} u(x_{n_k} + \cdot, t_{n_k}) - \xi = 0,$$

locally uniformly on $\mathbb{R}$.

(vi) *If the assumptions of statement (v) are satisfied and the set* $R_0^{-1}\{0\}$ *is finite, say*

$$R_0^{-1}\{0\} = \{a_1, \dots, a_{k+1}\}, \text{ with } 0 = a_1 < a_2 < \cdots < a_{k+1} = \gamma, \quad (2.32)$$

then $\mathcal{N} = \{I_1, \dots, I_k\}$ *with* $I_j = (a_j, a_{j+1})$, $j = 1, \dots, k$, *and, as* $t \to \infty$, *one has*

$$u(x,t) - \Big(\sum_{j=1,\dots,k} \phi_{I_j}(x - c_{I_j} t - \zeta_{I_j}(t)) - \sum_{1\le j\le k-1} a_{j+1} \Big) \to 0 \quad (x \in \mathbb{R}), \tag{2.33}$$

uniformly on $\mathbb{R}$.

2.4 The ω-limit set and quasiconvergence

Using the above results on propagating terraces, we now prove that the solution u (whose initial condition satisfies (2.2), (2.3)) is quasiconvergent in the sense that its ω-limit set with respect to $L^\infty_{loc}(\mathbb{R})$ consists of steady states of (1.1). For some background in quasiconvergence and examples of bounded solutions which do not have this property, we refer the reader to [24, 26, 27]. On the other hand, see [22] for a general quasiconvergence theorem for the nonnegative bounded solutions u with $u(\cdot, 0) \in C_0(\mathbb{R})$.

Theorem 2.20. *Assume that* $u_0 \in C(\mathbb{R})$ *satisfies* (2.2), (2.3). *Then the* ω-*limit set* $\omega(u_0)$ *consists of steady states of* (1.1). *More precisely, it consists of constant steady states and standing fronts.*

Remark 2.21. (i) The statement allows for the possibility that some increasing standing fronts are contained in $\omega(u_0)$. If (2.16) holds, then we can say more precisely that $\omega(u_0)$ consists of constant steady states and decreasing standing fronts. This is the case, for example, if the stronger conditions (2.19), (2.20) on u_0 are assumed (see Corollaries 2.10, 2.18).

(ii) In Section 2.5, we show that under some explicit generic conditions on f, for any u_0 satisfying (2.2), (2.3) the solution u of (1.1) is convergent: $\omega(u_0)$ consists of a single state state (see Remark 2.24(ii) for a more specific statement).

Proof of Theorem 2.20. Let $\gamma_0 \geq 0$, $\gamma_1 \leq \gamma$, and $\tilde{\mathcal{N}}$ be as in the previous subsection. Since $\omega(u_0) \subset \Omega(u_0)$, by Theorem 2.13, we have

$$\begin{aligned} \omega(u_0) \subset R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\, \xi \in \mathbb{R}\} \\ \cup \{\hat{\phi}_I(\cdot - \xi) : I \in \mathcal{N}^0,\, \xi \in \mathbb{R}\} \cup \Omega_0 \cup \Omega_1 \end{aligned} \tag{2.34}$$

(see the Theorem 2.13 for the meaning of $\hat{\phi}_I$, Ω_0, and Ω_1). Statements (i) and (ii) of Theorem 2.19 imply that all functions in $\omega(u_0)$ have the range in $[\gamma_0, \gamma_1]$. Thus, we can delete Ω_0, Ω_1 in (2.34). In other words, $\omega(u_0)$ consists of constant steady states from $R_0^{-1}\{0\}$, standing waves (which are also steady states of (1.1)), and, possibly, translates of ϕ_I, for some $I \in \tilde{\mathcal{N}}$ with $c_I \neq 0$. In order to complete the proof, we just need to show that the last possibility does not occur.

Assume that, to the contrary, $\phi_I(\cdot + \xi) \in \omega(u_0)$ for some $\xi \in \mathbb{R}$ and some $I \in \tilde{\mathcal{N}}$ with $c_I < 0$ (the case $c_I > 0$ can be ruled out in a similar way). Then, for some sequence $t_n \to \infty$ one has in particular

$$u(0, t_n) \to \phi_I(\xi). \tag{2.35}$$

Fixing any $x_0 > \xi$, we have by Theorem 2.19(iii)(c) that

$$u(x_0 + c_I t_n + \zeta_I(t_n), t_n) \to \phi_I(x_0) < \phi_I(\xi). \tag{2.36}$$

Since $c_I < 0$ and $\zeta_I(t)/t \to 0$ as $t \to \infty$ (cp. Theorem 2.19(iii)(a)), from (2.35) and (2.36) we infer that for each large enough n there is x_n such that $x_0 + c_I t_n + \zeta_I(t_n) < x_n < 0$ and

$$\phi_I(x_0) < u(x_n, t_n) < \phi_I(\xi), \quad u_x(x_n, t_n) \geq 0. \tag{2.37}$$

Passing to subsequences, we may assume that $u(x_n + \cdot, t_n) \to \psi$ in $C^1_{loc}(\mathbb{R})$ for some $\psi \in \Omega(u_0)$. By (2.37),

$$\phi_I(x_0) \leq \psi(0) \leq \phi_I(\xi), \quad \psi'(0) \geq 0. \tag{2.38}$$

However, by Theorem 2.13, each $\psi \in \Omega(u_0)$ whose range intersects the range of ϕ_I is a translate of ϕ_I (recall that the range of ϕ_I is I, a nodal interval of R_0). Thus ψ is a translate of ϕ_I, but then $\psi'(0) \geq 0$ is impossible. This contradiction completes the proof. $\square$

2.5 Locally uniform convergence to a specific front and exponential convergence

We continue to assume the standing hypothesis (H). Let R_0, $\mathcal{N}$, and $\tilde{\mathcal{N}}$ be as above. Recall, that $\mathcal{N} = \tilde{\mathcal{N}}$ if the stability hypothesis (S) is satisfied. Also recall that the order relation on $\mathcal{N}$ is defined in (2.12).

In this section, we show that under a nondegeneracy assumption on f, the functions ζ_I in Theorems 2.11, 2.19(iii) are convergent. Moreover, in a suitable moving coordinate frame, the solution converges locally uniformly to a specific front from the minimal propagating terrace with an exponential rate. Similar results have been proved in [11] for minimal propagating terraces with two bistable levels and different speeds. In [31] such a result was proved for monotone systems with finitely many bistable levels with nondegenerate equilibria and mutually distinct speeds (cp. Theorem 2.23 below).

Our first result here is somewhat different; we prove the locally uniform convergence to a shift of a specific front ϕ_I, assuming just that $a := \phi_I(\infty)$ and $b := \phi_I(-\infty)$ are nondegenerate stable zeros of f and that the speeds of the neighboring fronts of ϕ_I are different from c_I. We do not require any such conditions for the other intervals $J \in \mathcal{N}$; they do not even have to be bistable intervals for f.

Before stating our theorem, we need to introduce some notation. Let $I = (a, b) \in \tilde{\mathcal{N}}$. If $a > 0$ and $f'(a) < 0$, then a is an isolated zero of f and R_0. Therefore, the following element of $\mathcal{N}$ is well defined:

$$\underline{I} := \max\{J \in \mathcal{N} : J < I\}. \tag{2.39}$$

Similarly, if $f'(b) < 0$ and $b < \gamma$, we define

$$\bar{I} := \min\{J \in \mathcal{N} : J > I\}. \tag{2.40}$$

These are the immediate "neighbors" of I in $\mathcal{N}$.

We now fix $I = (a, b) \in \tilde{\mathcal{N}}$ and make the following assumptions:

(N) f is of class C^1 in a neighborhood of $[a, b]$ (this is an extra assumption only if $a = 0$ or $b = \gamma$), $f'(a) < 0$, $f'(b) < 0$; and if $a > 0$, then $c_{\underline{I}} \neq c_I$ (hence, $c_{\underline{I}} > c_I$); and if $b < \gamma$, then $c_{\bar{I}} \neq c_I$ (hence, $c_{\bar{I}} < c_I$).

Define c_I^- and c_I^+ as follows:

$$
\begin{aligned}
c_I^- &= \begin{cases} -\infty & \text{if } b = \gamma, \\ \dfrac{c_{\bar{I}} + c_I}{2} & \text{if } b < \gamma, \end{cases} \\
c_I^+ &= \begin{cases} \infty & \text{if } a = 0, \\ \dfrac{c_{\underline{I}} + c_I}{2} & \text{if } a > 0. \end{cases}
\end{aligned} \tag{2.41}
$$

Obviously, $c_I^- < c_I < c_I^+$.

Theorem 2.22. *Assume that (in addition to* (H)*) hypothesis* (N) *is satisfied for some* $I \in \tilde{\mathcal{N}}$ *and let* c_I^-, c_I^+ *be as above. Let* u *be the solution of* (1.1), (1.2), *where* $u_0 \in C(\mathbb{R})$ *satisfies* (2.2), (2.3). *Then there are constants* $\eta \in \mathbb{R}$, $\kappa > 0$, *and* $\vartheta > 0$ *such that for all sufficiently large* $t > 0$ *one has*

$$u(x,t) \geq b - \kappa e^{-\vartheta t} \quad (x < c_I^- t), \tag{2.42}$$

$$|u(x,t) - \phi_I(x - c_I t - \eta)| \leq \kappa e^{-\vartheta t} \quad (c_I^- t < x < c_I^+ t), \tag{2.43}$$

$$u(x,t) \leq a + \kappa e^{-\vartheta t} \quad (x > c_I^+ t). \tag{2.44}$$

The proof of Theorem 2.22 is given in Section 6.9. Similarly as in [31] (see also [8]), we derive the *locally uniform* convergence property (2.43) from a *uniform* convergence property for solutions of an auxiliary asymptotically autonomous problem.

We conclude the section with an exponential convergence result under the following global hypothesis.

(G) f is of class C^1 in a neighborhood of $[\gamma_0, \gamma_1]$, $f'(\xi) < 0$ for each $\xi \in R_0^{-1}\{0\} \cap [\gamma_0, \gamma_1]$, and the speeds c_I, $I \in \mathcal{N}$, are mutually distinct.

For the meaning of $\gamma_0 \geq 0$ and $\gamma_1 \leq \gamma$ see Subsection 2.3. Recall in particular that

$$\tilde{\mathcal{N}} := \{I \in \mathcal{N} : I \subset (\gamma_0, \gamma_1)\}.$$

Note that the nondegeneracy assumption in (G) is satisfied if all left and right global maximizers of the function F in $[0, \gamma]$ are nondegenerate zeros of f (cp. Section 3.2). Hypotheses (G) implies that the set $R_0^{-1}\{0\}$ is finite, hence, the sets $\tilde{\mathcal{N}}$ and $\mathcal{N}$ are finite. We can write

$$R_0^{-1}\{0\} \cap [\gamma_0, \gamma_1] = \{b_1, \dots, b_{k+1}\}, \text{ with } \gamma_0 = b_1 < b_2 < \cdots < b_{k+1} = \gamma_1, \tag{2.45}$$

and then $\tilde{\mathcal{N}} = \{I_1, \dots, I_k\}$ with $I_j = (b_j, b_{j+1})$, $j = 1, \dots, k$.

Theorem 2.23. *Assume that hypotheses* (H), (G) *are satisfied and let b_j, $j = 1, \dots, k+1$, and I_j, $j = 1, \dots, k$, be as above. Then, for each $u_0 \in C(\mathbb{R})$ satisfying* (2.2), (2.3), *there are constants η_j, $j = 1, \dots, k$, and $\vartheta > 0$ such that*

$$\lim_{t\to\infty} e^{\vartheta t} \sup_{c^-_{I_k} t < x < c^+_{I_1} t} \Big| u(x,t) - \Big(\sum_{j=1,\dots,k} \phi_{I_j}(x - c_{I_j} t - \eta_j) - \sum_{1\le j\le k-1} b_{j+1} \Big) \Big| = 0. \tag{2.46}$$

This theorem is a direct consequence of Theorem 2.22. Notice that if $\gamma_0 = 0$ and $\gamma_1 = \gamma$, that is, the equilibria 0, γ are stable for the equation $\dot\theta = f(\theta)$, then $\mathcal{N} = \tilde{\mathcal{N}}$, $c^-_{I_k} = -\infty$, and $c^+_{I_1} = \infty$. Thus (2.46) gives the uniform convergence on $\mathbb{R}$ with the exponential rate. In this case, Theorem 2.23 is a special case of [31, Theorem 2.2] for monotone systems, save for the minor detail that it is assumed in [31] that $0 \le u_0 \le \gamma$ (it is rather straightforward to modify the proof in [31] so that it also covers the more general assumptions (2.2), (2.3)).

Remark 2.24. (i) The assumption that the speeds c_I are mutually distinct is necessary in Theorem 2.23. Indeed, relation (2.46) in particular implies that the functions $\zeta_I(t)$ in Theorem 2.19(iii) are convergent. However, if $c_{I_1} = c_{I_2}$ for some $I_1, I_2 \in \tilde{\mathcal{N}}$, $I_1 < I_2$, then, by Theorem 2.19(iii)(d), $\zeta_{I_1}(t) - \zeta_{I_2}(t) \to \infty$. Hence, in this case, ζ_{I_1}, ζ_{I_2} cannot both be convergent.

(ii) Under the assumptions of Theorem 2.23, the solution u is convergent in $L^\infty_{loc}(\mathbb{R})$: $\omega(u)$ consists of a single function φ. Specifically, if $c_I = 0$ for some (unique) $I \in \mathcal{N}$, then φ is a shift of ϕ_I, and if $c_I \neq 0$ for all $I \in \mathcal{N}$, then φ is a constant in $R_0^{-1}\{0\}$. This follows directly from (2.46).

3 Phase plane analysis

In this section we are concerned with solutions of the equation

$$v_{xx} + cv_x + f(v) = 0, \tag{3.1}$$

and trajectories of the corresponding planar system

$$v_x = w, \quad w_x = -cw - f(v). \tag{3.2}$$

Unless specified otherwise, by a solution of (3.1) or (3.2) we always mean a maximally defined solution. Such solutions are all global (defined on $\mathbb{R}$), by the global Lipschitz continuity of f, as assumed in the standing hypothesis (H).

For the sake of convenience, throughout the section we assume the following slightly stronger standing hypotheses

(H') $f \in C^1(\mathbb{R})$, f' is bounded on $\mathbb{R}$, $\gamma > 0$, $f(0) = f(\gamma) = 0$.

Thus, we assume that f is of class C^1 on $\mathbb{R}$, not just on $[0, \gamma]$ as assumed in (H). This will spare us some cumbersome formulations when considering the flow of (3.2) near the equilibria $(0, 0)$, $(\gamma, 0)$ if only one-sided derivatives of f at 0 and γ are defined. Obviously, if (H) holds, one can achieve (H') by a modification of f outside the interval $[0, \gamma]$. It is important to note, however, that all results of this section which will be used in the proofs of our theorems concern the behavior of solutions of (3.1) while they stay in $[0, \gamma]$. Of course, such results are unaffected by any modification of f outside $[0, \gamma]$.

We start with basic results concerning trajectories of (3.2) (Subsection 3.1). Then, in Subsection 3.2, we give a more detailed description of the minimal system of waves. Finally, in Subsection 3.3, we exhibit a class of trajectories not intersecting the graph of R_0. These trajectories will be used in the intersection-comparison arguments in the proofs of our theorems; they are key ingredients of our method.

3.1 Basic properties of the trajectories

For later reference, we recall several elementary properties of solutions of (3.1). Most of these results are available in the literature in some form, but for the sake of completeness or perspective we often include the proofs, or brief sketches and references.

For a solution ψ of (3.1), $\tau(\psi)$ denotes its trajectory in $\mathbb{R}^2$ (cp. (2.4)). We also use the notation p^ψ introduced in (2.7). As mentioned in the introduction, for strictly monotone solutions, $\tau(\psi)$ is the graph of the function p^ψ.

In the following lemma (see also Figure 3), we consider the solutions of (3.1) satisfying the initial conditions

$$\psi(0) = \xi, \quad \psi'(0) = \eta. \tag{3.3}$$

Lemma 3.1. *Assuming that* $\xi \in [0, \gamma]$, $\eta \neq 0$, *and* $c_1 \leq c_2$, *let* ψ_1, ψ_2 *be the solutions of* (3.1) (3.3) *with* $c = c_1$, $c = c_2$, *respectively. Then there is* $\epsilon > 0$ *such that*

$$\begin{aligned} p^{\psi_1}(u) &< p^{\psi_2}(u) \quad (u \in (\xi - \epsilon, \xi)), \\ p^{\psi_1}(u) &> p^{\psi_2}(u) \quad (u \in (\xi, \xi + \epsilon)), \end{aligned} \tag{3.4}$$

(if $c_1 < c_2$*) or* $p^{\psi_1} \equiv p^{\psi_2}$ *(if* $c_1 = c_2$*).*

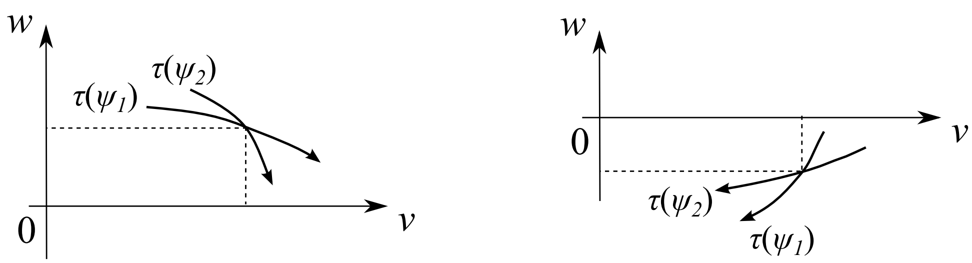

Figure 3: The intersecting trajectories $\tau(\psi_1)$, $\tau(\psi_2)$ correspond to speeds $c_1 < c_2$, respectively.

Proof. Since $\eta \neq 0$, the functions p^{ψ_1}, p^{ψ_2} are defined in a neighborhood of $u = \xi$ and satisfy there equations (2.9) with $c = c_1$, $c = c_2$, respectively. This implies (3.4) if $c_1 < c_2$. If $c_1 = c_2$, the uniqueness for the Cauchy problem gives $p^{\psi_1} \equiv p^{\psi_2}$. $\square$

If ζ is a zero of f, we set

$$\lambda^{\pm}(c) := \frac{-c \pm \sqrt{c^2 - 4f'(\zeta)}}{2}. \tag{3.5}$$

These are the eigenvalues of the linearization of the right hand side of (3.2) at the equilibrium $(\zeta, 0)$.

We next examine solutions ψ of (3.1) such that

$$\psi(x) \to \zeta \text{ as } x \to \infty, \quad \psi'(x) < 0 \text{ for } x \approx \infty, \tag{3.6}$$

or

$$\psi(x) \to \zeta \text{ as } x \to -\infty, \quad \psi'(x) < 0 \text{ for } x \approx -\infty. \tag{3.7}$$

It is a well known elementary observation that (3.6) implies that the corresponding solution (ψ, ψ') of (3.2) converges to the equilibrium $(\zeta, 0)$; similarly for (3.7).

Solutions of (3.1) satisfying (3.6) or (3.7) can exist only if the eigenvalues $\lambda^{\pm}(c)$ are real, that is, if either $f'(\zeta) \leq 0$, or $f'(\zeta) > 0$ and $|c| \geq \sqrt{2f'(\zeta)}$ (otherwise $(\zeta, 0)$ is a focus and we could not have the required monotonicity). In the next two lemmas, we assume $f'(\zeta) \leq 0$. Note that, in this case

$$\lambda^-(c) < 0 \leq \lambda^+(c) \quad (c > 0), \qquad \lambda^-(c) \leq 0 < \lambda^+(c) \quad (c < 0). \tag{3.8}$$

Lemma 3.2. *Let ζ be a zero of f with $f'(\zeta) \leq 0$, and let $\lambda^{\pm}(c)$ be the eigenvalues as in* (3.5)*. If $c > 0$, then the following statements are valid:*

(pi) *There is a solution ψ of* (3.1)*, unique up to translations, such that* (3.6) *together with the following relation are satisfied*

$$\lim_{x\to\infty} \frac{\psi'(x)}{\psi(x)-\zeta} \to \lambda^-(c). \tag{3.9}$$

(pii) *If ψ is a solution of* (3.1) *satisfying* (3.6) *but not* (3.9)*, then*

$$\lim_{x\to\infty} \frac{\psi'(x)}{\psi(x)-\zeta} \to 0. \tag{3.10}$$

A necessary and sufficient condition (in the case $f'(\zeta) \leq 0$ considered here) for the existence of such a solution is that $f'(\zeta) = 0$ (that is, $\lambda^+(c) = 0$) and $f(u) > 0$ for all $u > \zeta$ sufficiently close to ζ (that is, ζ is unstable from above for the equation $\dot{\theta} = f(\theta)$).

(piii) *If ψ is a solution of* (3.1) *satisfying* (3.7)*, then*

$$\lim_{x\to-\infty} \frac{\psi'(x)}{\psi(x)-\zeta} \to \lambda^+(c). \tag{3.11}$$

Necessarily, such a solution ψ is unique up to translations; a sufficient condition for its existence is $f'(\zeta) < 0$.

Proof. (Alternative arguments can be found in [2, 33].) It is well known (see [33, Theorem 1.3.3], for example) that if $\psi(x) \neq \zeta$ for large x and $\psi(x) \to \zeta$ as $x \to \infty$, then the limit on the left-hand side of (3.9) exists and is equal to one of the eigenvalues $\lambda^{\pm}(c)$. This can also be interpreted as the convergence of the solution $(\psi(x), \psi'(x))$ to the equilibrium $(\zeta, 0)$ in an eigendirection:

$$\frac{(\psi(x)-\zeta, \psi'(x))}{|(\psi(x)-\zeta, \psi'(x))|} \to \pm\frac{(1,\lambda)}{|(1,\lambda)|}, \tag{3.12}$$

where $\lambda = \lambda^-(c)$ or $\lambda = \lambda^+(c)$ (note that $(1, \lambda^\pm(c))$ is an eigenvector corresponding to the eigenvalue $\lambda^\pm(c)$).

The relations $\lambda^-(c) < 0 \leq \lambda^+(c)$ imply that the stable manifold W^s of the equilibrium $(\zeta, 0)$ of (3.2) is one-dimensional and it is tangent at $(\zeta, 0)$ to $(1, \lambda^-(c))$. The set $W^s \setminus \{(\zeta, 0)\}$ consists of two trajectories, one of them gives a solution ψ satisfying (3.6), (3.9) (the latter follows from the tangency of W^s to $(1, \lambda^-(c))$). The other trajectory in $W^s \setminus \{(\zeta, 0)\}$ yields a solution which is increasing to ζ, and there are no other trajectories having the asymptotics (3.12) with $\lambda = \lambda^-(c)$. By this we have proved statement (pi). Also, we see that if a solution ψ satisfies (3.6) but not (3.9), then (3.12) holds with $\lambda = \lambda^+(c)$ (this implies (3.10) in the case $\lambda^+(c) = 0$ considered below).

Now, if $\lambda^+(c) > 0$, then $(\zeta, 0)$ is a hyperbolic saddle and there are no solutions satisfying (3.6) other than those in W^s. Thus condition $\lambda^+(c) = 0$ is necessary for the existence of such a solution. Assuming $\lambda^+(c) = 0$, the equilibrium $(\zeta, 0)$ has a one-dimensional local center manifold W^c. It is tangent at $(\zeta, 0)$ to the eigenvector $(1, \lambda^+(c)) = (1, 0)$, hence it can be parameterized by the v-coordinate, which also defines a natural ordering on W^c. A necessary and sufficient condition for the existence of a solution approaching $(\zeta, 0)$ in the direction $(-1, 0)$ is that the equilibrium $(\zeta, 0)$ is asymptotically stable from above on W^c. This condition in particular requires that $(\zeta, 0)$ be an isolated equilibrium from above in W^c; in other words f must be of one sign in an interval $(\zeta, \zeta + \epsilon)$ with $\epsilon > 0$. Considering the direction of the flow on the v axis, one shows easily that if $f < 0$ in $(\zeta, \zeta + \epsilon)$, then $(\zeta, 0)$ is unstable from above on W^c, whereas if $f > 0$ in $(\zeta, \zeta + \epsilon)$ then it is asymptotically stable from above. This completes the proof of statement (pii).

For the proof of (piii), we note that the equilibrium $(\zeta, 0)$ has a one-dimensional local center-unstable manifold W^{cu} (W^{cu} is the center manifold or the unstable manifold, according to whether $f'(\zeta) = 0$ or $f'(\zeta) < 0$). This manifold is tangent to $(1, \lambda^+(c))$, and it has the property that if ψ is a solution of (3.1) such that $\psi(x) \to \zeta$ as $x \to -\infty$, then $(\psi(x), \psi'(x)) \in W^{cu}$ for all large negative x. Applying this to the solution in (piii) and using $\psi' < 0$, we obtain (3.11). The fact that the solution is unique up to translations follows from the one-dimensionality of W^{cu} (alternatively, one can consult [33, Lemma 1.3.1] for a simple uniqueness argument). If $f'(\zeta) < 0$, then $(\zeta, 0)$ is a hyperbolic saddle and one of the trajectories on $W^u \setminus \{(\zeta, 0)\}$ gives a solution satisfying (3.7), (3.11). $\square$

Here are analogous statements for $c < 0$, which we include without proofs.

Lemma 3.3. *Let ζ be a zero of f with $f'(\zeta) \leq 0$, and let $\lambda^{\pm}(c)$ be the eigenvalues as in* (3.5)*. If $c < 0$, then the following statements are valid:*

(ni) *There is a solution ψ of* (3.1)*, unique up to translations, such that* (3.7) *together with the following relation are satisfied*

$$\lim_{x \to -\infty} \frac{\psi'(x)}{\psi(x) - \zeta} \to \lambda^{+}(c). \tag{3.13}$$

(nii) *If ψ is a solution of* (3.1) *satisfying* (3.7) *but not* (3.13)*, then*

$$\lim_{x \to -\infty} \frac{\psi'(x)}{\psi(x) - \zeta} \to 0. \tag{3.14}$$

A necessary and sufficient condition (in the case $f'(\zeta) \leq 0$) for the existence of such a solution is that $f'(\zeta) = 0$ (that is, $\lambda^{-}(c) = 0$) and $f(u) < 0$ for all $u < \zeta$ sufficiently close to ζ (that is, ζ is unstable from below for the equation $\dot{\theta} = f(\theta)$).

(niii) *If ψ is a solution of* (3.1) *satisfying* (3.6)*, then*

$$\lim_{x \to \infty} \frac{\psi'(x)}{\psi(x) - \zeta} \to \lambda^{-}(c). \tag{3.15}$$

Necessarily, such a solution ψ is unique up to translations; a sufficient condition for its existence is $f'(\zeta) < 0$.

In the next lemma, we consider the case with real negative eigenvalues $\lambda^{\pm}(c)$.

Lemma 3.4. *Let ζ be a zero of f with $f'(\zeta) > 0$, and let $c > \sqrt{2f'(\zeta)}$. Then the following statements are valid:*

(i) *There is a solution ψ of* (3.1)*, unique up to translations, such that* (3.6) *and* (3.9) *are satisfied.*

(ii) *There is also a solution ψ of* (3.1) *satisfying* (3.6) *but not* (3.9)*. Each such solution satisfies*

$$\lim_{x \to \infty} \frac{\psi'(x)}{\psi(x) - \zeta} \to \lambda^{+}(c). \tag{3.16}$$

Proof. Under the given assumptions, the equilibrium $(\zeta, 0)$ is a stable node of (3.2) with two distinct eigenvalues: $\lambda^-(c) < \lambda^+(c) < 0$. The argument for the existence and uniqueness of the solution in (i) is similar as in the proof of Lemma 3.2(pi); just this time, one uses the strong stable manifold, which is a one-dimensional invariant manifold tangent to the eigenvector $(1, \lambda^-(c))$, in place of the stable manifold. (For alternative arguments see [33, Theorem 1.3.4], for example). The solutions of (3.2) which stay outside the strong stable manifold and approach $(\zeta, 0)$, do so in the direction of either $(1, \lambda^+(c))$ or $-(1, \lambda^+(c))$. Since the stable manifold of $(\zeta, 0)$ is two-dimensional, we can find solutions converging in either of these directions, which implies statement (ii). $\square$

We include without proof analogous results for $c < 0$. Note that in this case $\lambda^+(c) > \lambda^-(c) > 0$.

Lemma 3.5. *Let ζ be a zero of f with $f'(\zeta) > 0$, and let $c < -\sqrt{2f'(\zeta)}$. Then following statements are valid:*

(i) *There is a solution ψ of* (3.1)*, unique up to translations, such that* (3.7) *and* (3.13) *are satisfied.*

(ii) *There is also a solution ψ of* (3.1) *satisfying* (3.7) *but not* (3.13)*. Each such solution satisfies*

$$\lim_{x \to -\infty} \frac{\psi'(x)}{\psi(x) - \zeta} \to \lambda^-(c). \tag{3.17}$$

Remark 3.6. (i) The results in the previous lemmas can be equivalently formulated in terms of the function p^ψ. For example, for the solution ψ satisfying (3.9) the function p^ψ satisfies

$$p^\psi(u) \nearrow 0, \quad (p^\psi)'(u) \to \lambda^-(c), \tag{3.18}$$

as $u \to \zeta$, whereas for a solution ψ satisfying (3.10), one has

$$p^\psi(u) \nearrow 0, \quad (p^\psi)'(u) \to 0. \tag{3.19}$$

This follows from the definition of p^ψ (see (2.7)): for $u = \psi(x)$, one has

$$\frac{p^\psi(u)}{u - \zeta} = \frac{\psi'(x)}{\psi(x) - \zeta}.$$

(ii) The stable manifold of the equilibrium $(\zeta, 0)$, as used in the proof of Lemma 3.2(i), depends continuously on $c > 0$. This can be stated in terms of the function p^ψ as follows. Stressing the dependence on $c > 0$, let $\hat{p}(u, c) = p^\psi(u)$, where $\psi = \psi^c$ is as in Lemma 3.2(i) (note that p^ψ is independent of the choice of the translation of ψ). Then $\hat{p}(u, c)$ is continuous in c: if $c_0 > 0$ and $\hat{p}(\cdot, c_0)$ is defined on an interval $[\zeta, d]$ (with $\hat{p}(\zeta, c_0) = 0$), then, for $c \approx c_0$, $\hat{p}(u, c)$ is also defined on $[\zeta, d]$ and $\hat{p}(u, c) \to \hat{p}(u, c_0)$ as $c \to c_0$, uniformly on $[\zeta, d]$. An analogous remark applies to the solutions in Lemma 3.3.

(iii) From the existence of the limit of the function $\psi'(x)/(\psi(x) - \zeta)$, one can make conclusions about the asymptotics of the functions $\psi(x) - \zeta$ and ψ'. For example, if ψ is as in (3.16), then one can show easily (see [33, Section 1.5.3]) that the following holds:

$$\lim_{x \to \infty} \frac{\log |\psi(x) - \zeta|}{x} = \lambda^+(c).$$

Using this and (3.16), one obtains in particular that for each $\epsilon > 0$ there is $x_1 \in \mathbb{R}$ such that

$$e^{(\lambda^+(c) - \epsilon)x} < |\psi(x) - \zeta|, \ |\psi_x(x)| < e^{(\lambda^+(c) + \epsilon)x} \quad (x \geq x_1). \tag{3.20}$$

We next consider solutions of (3.1) satisfying

$$\lim_{x \to -\infty} v(x) = b \quad \lim_{x \to \infty} v(x) = a, \quad v' < 0. \tag{3.21}$$

Recall that

$$F(u) = \int_0^u f(s)\, ds. \tag{3.22}$$

Lemma 3.7. *Assume that* $0 \leq a < b \leq \gamma$, $f(a) = f(b) = 0$.

(i) *If* $c > 0$ *and a solution of* (3.1), (3.21) *exists, then* $f(u) > 0$ *for all* $u < b$ *sufficiently close to* b *and*

$$F(u) < F(b) \quad (u \in [a, b)). \tag{3.23}$$

(ii) *If* $c < 0$ *and a solution of* (3.1), (3.21) *exists, then* $f(u) < 0$ *for all* $u > a$ *sufficiently close to* a *and*

$$F(u) < F(a) \quad (u \in (a, b]). \tag{3.24}$$

(iii) *If* $c = 0$, *then a solution of* (3.1), (3.21) *exists if and only if*

$$F(u) < F(a) = F(b) \quad (u \in (a, b)). \tag{3.25}$$

These results are contained in Theorems 1.3.9-1.3.11 of [33]. We just remark that the relations (3.23)-(3.25) also follow from the monotonicity properties of the function $u_x^2/2 + F(u)$, as discussed below.

In the next lemma, we consider perturbations of the solution v as in (3.21) (see Figure 4).

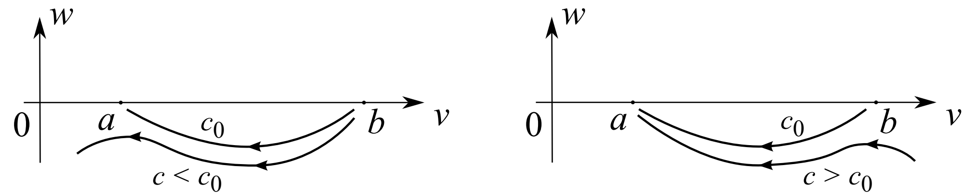

Figure 4: . The heteroclinic trajectory for $c = c_0$ corresponds to the solution v as in (3.21). The trajectories for $c \neq c_0$ correspond to the solutions in Lemma 3.8, assuming that the speed c_0 extremal as in Remark 3.9(i).

Lemma 3.8. *Let* a, b *be as in Lemma 3.7. Assume that for some* $c_0 \in \mathbb{R}$ *a solution* v *of* (3.1), (3.21) *with* $c = c_0$ *exists.*

(i) *For each* $c < c_0$ *there is a solution* ψ *of* (3.1) *such that* $\psi(x) \to b$ *as* $x \to -\infty$, $\psi'(x) < 0$ *for* $x \approx -\infty$, *and*

$$p^{\psi}(u) < p^{v}(u) \quad (u \in (a, b)). \tag{3.26}$$

(ii) *For each* $c > c_0$ *there is a solution* ψ *of* (3.1) *such that* $\psi(x) \to a$ *as* $x \to \infty$, $\psi'(x) < 0$ *for* $x \approx \infty$, *and*

$$p^{\psi}(u) < p^{v}(u) \quad (u \in (a, b)). \tag{3.27}$$

Proof. We only prove statement (i); the proof of (ii) is analogous. Given any $c < c_0$, one can construct ψ as follows. Take a sequence $\nu_n \searrow 0$, and let ψ_n be the solution of (3.1) with $\psi_n(0) = b$, $\psi_n'(0) = -\nu_n$. By Lemma 3.1, $p^{\psi_n}(u) < p^{v}(u)$ for all $u \in (a, b)$. It can be proved that, as $n \to \infty$, the limit of the functions p^{ψ_n}, is the function p^{ψ} corresponding to a solution ψ with the stated properties. The details of this elementary construction can be found in [33, Sect 1.2.2] or [2]. $\square$

Remark 3.9. (i) In Lemma 3.8(i), if c_0 is the minimal (or unique) speed for the interval (a, b) in the sense that there is no traveling front with range (a, b) and speed $c < c_0$, then relation (3.26) obviously holds on the semiclosed interval $[a, b)$ when we define $p^v(a) = 0$. This in particular applies if $(a, b) = I$ for some $I \in \mathcal{N}^+ \cup \mathcal{N}^0$ (see Theorem 2.4). Similarly, if c_0 is the maximal speed for the interval (a, b) (in particular, if $(a, b) = I$ for some $I \in \mathcal{N}^- \cup \mathcal{N}^0$) then (3.27) holds on $(a, b]$ ($p^v(b) = 0$).

(ii) If $c_0 > 0$, then Lemma 3.7(i) implies that $f'(b) \le 0$. Therefore, for each $c \in (0, c_0)$, the solution ψ in Lemma 3.8(i) is a solution, determined uniquely up to translations, as in Lemma 3.2(piii) with $\zeta = a$. Similarly, if $c_0 < 0$ and $c \in (c_0, 0)$, the solution in Lemma 3.8(ii) is a solution as in Lemma 3.3(niii)

Finally, we include some results concerning the role of the functional

$$H(v, w) := \frac{w^2}{2} + F(v). \tag{3.28}$$

If (v, v_x) is a solution of (3.2), then

$$\frac{dH(v(x), v_x(x))}{dx} = -cv_x^2. \tag{3.29}$$

Thus, if $c > 0$, then H is decreasing along nonstationary solutions; and if $c < 0$, then $-H$ is decreasing. In other words, if $c \neq 0$, then either H or $-H$ is Lyapunov functional of (3.2). This implies that each bounded nonstationary orbit of (3.2) is a heteroclinic orbit between two (distinct) equilibria (cp. [33, Theorem 1.3.2]).

For $c = 0$, (3.2) is a Hamiltonian system with the Hamiltonian H. Each trajectory of (3.2) is contained in a level set of H. Note that the level sets of H are symmetric about the v-axis.

System (3.2) with $c = 0$ has only four types of bounded trajectories (or, orbits): equilibria—all of them on the u-axis, nonstationary periodic orbits (or, closed orbits), homoclinic orbits, and heteroclinic orbits. This follows easily from the symmetry of the level sets of H and the fact that in the half-plane $\{w > 0\}$ the v component of the solutions is increasing, whereas in $w < 0$ it is decreasing. Thus, orbits with more than one intersection with the v-axis are periodic orbits. Any nonstationary bounded orbit with exactly one intersection with the v-axis is a homoclinic orbit and a bounded orbit which does not intersect the v-axis at all is a heteroclinic orbit. We view orbits as subsets of $\mathbb{R}^2$, although our descriptive terminology, like periodic solutions, reflects properties of the corresponding solutions of (3.2). This should cause no confusion.

Lemma 3.10. *Let $c = 0$. Assume that $\Sigma \subset [0,\gamma]\times\mathbb{R}$ is a compact, connected set, which is invariant under the flow of* (3.2)*.*

(i) *If Σ does not contain any nonstationary periodic orbit of* (3.2) *(so it consists of equilibria, homoclinic orbits, and heteroclinic orbits), then H is constant on Σ.*

(ii) *If Σ is a heteroclinic or homoclinic loop (that is, a Jordan curve consisting of a homoclinic orbit and its limit equilibrium, or of two homoclinic orbits and their limit equilibria), then there is a sequence of periodic orbits $\mathcal{O}_n$ contained inside the loop such that*

$$\operatorname{dist}((\xi,\eta),\mathcal{O}_n) \to 0 \text{ as } n \to \infty \quad ((\xi,\eta) \in \Sigma). \tag{3.30}$$

Statement (i) is a part of [22, Lemma 3.1]. We just remark that, with no effect on the validity of the statement, one can modify $f(u)$ outside $[0,\gamma]$ so as to achieve that it has the same shape near $u = \pm\infty$ as assumed in [22]. Statement (ii) is a part of Lemma 3.2 in [22].

3.2 A more detailed description of the minimal system of waves

Throughout the section, R_0 is the minimal system of waves (for the interval $[0,\gamma]$). The existence of R_0 together with its basic properties are stated in Theorem 2.3. Here we give a more detailed description of R_0.

Let $\{c_I : I \in \mathcal{N}\}$, $\{\phi_I : I \in \mathcal{N}\}$ be the families of speeds and profile functions corresponding to R_0, as defined in Section 2. As in (2.14),

$$\begin{aligned}\mathcal{N}^+ &= \{I \in \mathcal{N} : c_I > 0\},\\ \mathcal{N}^- &= \{I \in \mathcal{N} : c_I < 0\},\\ \mathcal{N}^0 &= \{I \in \mathcal{N} : c_I = 0\}.\end{aligned}$$

Of course, some of these sets may be empty.

If $\mathcal{N}^+ \neq \emptyset$, we further define

$$\begin{aligned}\gamma_* &:= \sup \bigcup_{I\in\mathcal{N}^+} I \qquad (3.31)\\ &= \sup\{b \in (0,\gamma] : \mathcal{N}^+ \text{ contains the interval } (a,b) \text{ for some } a \in [0,b)\}.\end{aligned}$$

If $\mathcal{N}^+ = \emptyset$, we set $\gamma_* = 0$. Similarly, we set $\gamma^* = \gamma$ if $\mathcal{N}^- = \emptyset$. If $\mathcal{N}^- \neq \emptyset$, we define

$$\begin{aligned}\gamma^* &:= \inf \bigcup_{I\in\mathcal{N}^-} I \qquad (3.32)\\ &= \inf\{a \in [0,\gamma) : \mathcal{N}^- \text{ contains the interval } (a,b) \text{ for some } b \in (a,\gamma]\}.\end{aligned}$$

By the continuity of R_0, we have $R_0(\gamma_*) = R_0(\gamma^*) = 0$. Consequently, γ_*, γ^* are zeros of the function $f = F'$ (cp. Theorem 2.3).

We say that a critical point $\xi \in [0,\gamma]$ of F is a *left-global maximizer* of F in $[0,\gamma]$ (or, simply, a left-global maximizer) if

$$F(v) \leq F(\xi) \quad (0 \leq v < \xi). \qquad (3.33)$$

If the first inequality in (3.33) is strict, we say ξ is a *strict left-global maximizer*. Similarly we define (strict) right-global maximizers. Note that we count 0 as a strict left-global maximizer and γ as a strict right-global maximizer.

We denote by Γ^-, Γ^+, Γ^0 the sets of strict left-global, strict right-global, and global maximizers, respectively. Obviously, $\Gamma^- \cup \Gamma^0$, $\Gamma^+ \cup \Gamma^0$ consist of left-global and right-global maximizers, respectively.

Proposition 3.11. *The following statements are valid.*

(i) $0 \leq \gamma_* \leq \gamma^* \leq \gamma$*;*

(ii) *one has*

$$R_0^{-1}\{0\} \cap [\gamma_*, \gamma^*] \subset \Gamma^0, \qquad (3.34)$$
$$R_0^{-1}\{0\} \cap [0, \gamma_*] \subset \Gamma^-, \qquad (3.35)$$
$$R_0^{-1}\{0\} \cap [\gamma^*, \gamma] \subset \Gamma^+; \qquad (3.36)$$

(iii) *for each $I = (a,b) \in \mathcal{N}$ one has*

$$c_I > 0 \text{ if and only if } b \leq \gamma_*, \qquad (3.37)$$
$$c_I < 0 \text{ if and only if } \gamma^* \leq a, \qquad (3.38)$$
$$c_I = 0 \text{ if and only if } \gamma_* \leq a < b \leq \gamma^*; \qquad (3.39)$$

(iv) *each of the sets* $R_0^{-1}\{0\} \cap [0, \gamma_*]$, $R_0^{-1}\{0\} \cap [\gamma^*, \gamma]$ *is finite or countable,* γ_* *is the only possible accumulation point of the set* $R_0^{-1}\{0\} \cap [0, \gamma_*]$, *and* γ^* *is the only possible accumulation point of* $R_0^{-1}\{0\} \cap [\gamma^*, \gamma]$;

(v) *statement (i) of Proposition 2.12 holds: If* $\{I_j\}$ *is a strictly monotone (infinite) sequence in* $\mathcal{N}$, *then* $c_{I_j} \to 0$.

The proof of Proposition 3.11 is given below, following some preliminary results. Figure 5 illustrates some possibilities of how R_0 can look like.

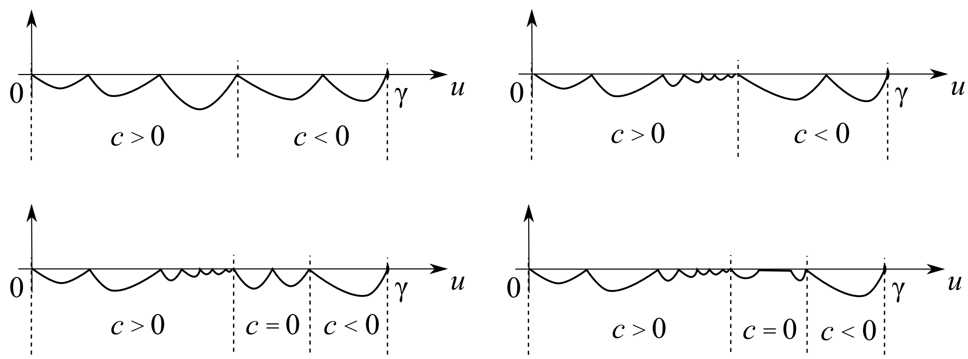

Figure 5: Possible graphs of R_0, with the signs of the speeds of the corresponding traveling fronts indicated. The first figure corresponds to a generic case—finitely many fronts with nonzero speeds; the other figures depict some "degenerate" cases.

Remark 3.12. Obviously, $F''(\xi) \leq 0$ if $\xi \in \Gamma^- \cup \Gamma^0 \cup \Gamma^+$ and $\xi \in (0, \gamma)$. Hence statement (ii) of Proposition 3.11 implies that

$$f'(\xi) = F''(\xi) \leq 0 \quad (\xi \in R_0^{-1}\{0\} \cap (0, \gamma)). \tag{3.40}$$

Lemma 3.13. *For any* $I = (a, b) \in \mathcal{N}$ *one has*

$$|c_I| \leq 2\sqrt{\|f'\|_{L^\infty(I)}}; \tag{3.41}$$

$$\textit{if } c_I \geq 0, \textit{ then } \; F(u) < F(b) \quad (u \in (a, b)); \tag{3.42}$$

$$\textit{if } c_I \leq 0, \textit{ then } \; F(u) < F(a) \quad (u \in (a, b)). \tag{3.43}$$

Proof. Recall that c_I is the speed of a traveling front, namely, ϕ_I, whose range is the interval I. Moreover, either it is the unique speed for the interval I

or it is extremal as specified in Theorem 2.4. Estimate (3.41) now becomes a well-known classical result (see [33, Theorem 1.3.14] and formula (2.22) in [33, Theorem 1.2.3]). Relations (3.42), (3.43) follow from Lemma 3.7. □

Recall that we have defined the ordering on $\mathcal{N}$ by (2.12).

Lemma 3.14. *Let $I_0 = (a_0, b_0) \in \mathcal{N}$.*

(i) *If $I_0 \in \mathcal{N}^+$ (i.e., $c_{I_0} > 0$), then the set $\{I \in \mathcal{N} : I < I_0\}$ is finite and R_0 does not vanish identically on any (nonempty) open subinterval of $[0, a_0]$.*

(ii) *If $I_0 \in \mathcal{N}^-$ (i.e., $c_{I_0} < 0$), then the set $\{I \in \mathcal{N} : I > I_0\}$ is finite and R_0 does not vanish identically on any open subinterval of $[b_0, \gamma]$.*

Proof. We only prove (i), the proof of (ii) being completely analogous.

Assuming that the set $\mathcal{M} := \{I \in \mathcal{N} : I < I_0\}$ is infinite, we find a sequence $\{I_j\}$ in $\mathcal{M}$ with $|I_j| \to 0$, where $|I|$ stands for the length of the interval I. By (3.41) and the fact that f vanishes at the end-points of I_j (see Theorem 2.3(i)), one has $c_{I_j} \to 0$. Hence, for some j, $c_{I_j} < c_{I_0}$, in contradiction to (2.13). This contradiction shows that $\mathcal{M}$ is finite.

We prove the second statement in (i). In view of the finiteness of $\mathcal{M}$, it is sufficient to prove that if $I = (a, b)$, $I \in \mathcal{M} \cup \{I_0\}$, then R_0 cannot vanish identically on any interval of the form $(a - \epsilon, a)$ with $\epsilon > 0$. Suppose it does. Then also $f \equiv f' \equiv 0$ on $J := (a - \epsilon, a)$. Fix any $\zeta \in J$. For $c \in (0, c_I]$ consider the solution $\psi = \psi^c$ of (3.1) as in Lemma 3.2(pi). Take first $c = c_I$. Since the trajectories $\tau(\psi)$ and $\tau(\phi_I)$ cannot intersect (these are trajectories of the same autonomous system (3.2)), the minimality of R_0, as stated in Theorem 2.3(iii), implies that there is x_0 such $\psi' < 0$ on $[x_0, \infty)$ and $\psi(x_0) = b$. For the function p^ψ defined in (2.7) this means that $p^\psi < 0$ on $(\zeta, b]$ (and $p^\psi(\zeta) = 0$). Using the continuous dependence on c (cp. Remark 3.6(ii)), we find $c < c_I$, $c \approx c_I$, such that the corresponding function p^{ψ^c} is negative on $(\zeta, b]$ as well. Take now $c_0 := c_I$, $v := \phi_I$ in Lemma 3.8(i) and let $\tilde{\psi}$ be the solution in the conclusion of Lemma 3.8(i) with the same $c < c_I$ as above. The trajectory of this solution cannot intersect the trajectory of ψ^c (again, these are trajectories of the same equation). Therefore, in view of (3.26) and Remark 3.9(i), we necessarily have $\tilde{\psi}' < 0$ on $\mathbb{R}$ and $\tilde{\psi}(x) \to \xi$ as $x \to \infty$ for some $\xi \in [\zeta, a)$. This is a contradiction to the minimality property of R_0. The proof is now complete. □

Proof of Proposition 3.11. If $\mathcal{N}^+ \neq \emptyset$, take any $I_0 = (a_0, b_0) \in \mathcal{N}^+$. By Lemma 3.14, there is $k \in \mathbb{N}$ such that

$$\{I \in \mathcal{N} : I < I_0\} = \{I_1, \dots, I_k\}.$$

We choose the labels so that $I_k < \cdots < I_1 < I_0$. Obviously, these interval comprise all nodal intervals of $R_0\big|_{[0,b_0]}$. In other words, R_0 vanishes identically on $[0, b_0] \setminus \cup_{j=0,\dots k} I_j$. By Lemma 3.14, $R_0\big|_{[0,b_0]}$ does not vanish on any open subinterval of $(0, b_0)$, therefore,

$$0 = a_k \ \text{ and } \ b_j = a_{j-1} \ \ (j = 1, \dots, k), \tag{3.44}$$

where the a_j and b_j are defined by $I_j = (a_j, b_j)$, $j = 1, \dots, k$.

Since $c_{I_j} \geq c_{I_0} > 0$ (by (2.13)), (3.42) gives

$$F(u) < F(b_j) \quad (u \in (a_j, b_j);\, j = 0, \dots, k). \tag{3.45}$$

Using these relations and (3.44), we obtain that the points b_j are strict left-global maximizers, hence

$$R_0^{-1}\{0\} \cap [0, b_0] = \{0, b_k, \dots, b_0\} \subset \Gamma^-. \tag{3.46}$$

Since the above results are valid for an arbitrary $I_0 \in \mathcal{N}^+$, it is clear that the set $R_0^{-1}\{0\} \cap [0, \gamma_*]$ is finite or countable, γ_* is its only possible accumulation point (it is the accumulation point precisely if the set $\mathcal{N}^+$ is infinite), and (3.35) holds. Moreover, γ_* is a strict left- global maximizer itself: $\gamma_* \in \Gamma^-$. Analogous arguments show that $\gamma^* \in \Gamma^+$, it is the only possible accumulation point of the set $R_0^{-1}\{0\} \cap [\gamma^*, \gamma]$, this set is finite or countable, and (3.36) holds. We have hereby proved that statement (iv) of Proposition 3.11 holds. From $\gamma_* \in \Gamma^-$ and $\gamma^* \in \Gamma^+$, we have $\gamma_* \leq \gamma^*$, hence the relations (i) are valid. Statement (iii) follows directly from the definition of γ_*, γ^*.

We next show that (3.34) holds, thus completing the proof of statement (ii). It follows from (3.39) that the graph of $R_0\big|_{[\gamma_*,\gamma^*]}$ in the (v, w)-plane consists of trajectories of system (3.2) with $c = 0$ (some of these trajectories are equilibria). Since R_0 is continuous, the graph is a connected compact set in $[0, \gamma] \times \mathbb{R}$. Therefore, by Lemma 3.10(i), the Hamiltonian $w^2/2 + F(v)$ is constant on the graph. Consequently, the function F is constant on the set $R_0^{-1}\{0\} \cap [\gamma_*, \gamma^*]$. Since γ_* is a left-global maximizer and γ^* is a right-global maximizer, this set consists of global maximizers, that is, (3.34) holds.

Statement (i) follows from (3.41), as already noted above. The proof is now complete. $\square$

In the next lemma, we recall a result on the asymptotics of the profile functions ϕ_I.

Lemma 3.15. *Given $I = (a, b) \in \mathcal{N}^+ \cup \mathcal{N}^-$; set $\zeta := a$ if $c_I > 0$ and $\zeta := b$ if $c_I < 0$. Then the profile function $\psi := \phi_I$ satisfies* (3.9) *or* (3.13)*, according to whether $c_I > 0$ or $c_I < 0$, respectively ($\lambda^{\pm}(c)$ is as in* (3.5) *with $c = c_I$).*

Proof. (Alternative arguments can also be found in [2, 33]). Assume for definiteness that $c_I > 0$; the case $c_I < 0$ is analogous. Note that, due to the monotonicity of ϕ_I, the eigenvalues $\lambda^{\pm}(c_I)$ are necessarily real.

Assume first that $f'(a) \leq 0$ (note that, by (3.40) this holds, except possibly for the case $a = 0$).

If (3.9) is not satisfied, then (3.10) must hold. We show that this leads to a contradiction. For $\tilde{c} \in (0, c_I]$, let $\tilde{\psi}$ be the solution of (3.1) with the asymptotics (3.9), where c is replaced with $\tilde{c}$. For $u > a$, $u \approx a$, we then have

$$p^{\psi}(u) > p^{\tilde{\psi}}(u)$$

(cp. Remark 3.6(i)). If $\tilde{c} = c_I$, this relation has to remain valid for all $u \in (0, b]$, since the trajectories $\tau(\psi)$, $\tau(\tilde{\psi})$ cannot intersect and $\tilde{\psi}$ cannot converge to b as $x \to -\infty$, by the minimality property of R_0. The same relation is then true for $\tilde{c} < c_I$, $\tilde{c} \approx c_I$. Fix such a $\tilde{c}$ and take any $c \in (\tilde{c}, c_I)$. Using Lemma 3.8(i) and Lemma 3.1, we find a solution ψ_0 of (3.1) such that

$$p^{\psi}(u) > p^{\psi_0}(u) > p^{\tilde{\psi}}(u) \quad (u \in (a, b)), \tag{3.47}$$

$\psi_0(-\infty) = b$. By (3.47), this solution also satisfies $\psi_0(\infty) = a$ and we have a contradiction to the minimality property of R_0.

Assume next that $f'(a) > 0$, hence, necessarily $a = 0$. Since the eigenvalues $\lambda^{\pm}(c_I)$ are real (by the monotonicity of φ_I), we have $c_I \geq 2\sqrt{f'(0)}$. Now, the limit on the left-hand side of (3.9) (with $\zeta = 0$) exists and equals one of the eigenvalues $\lambda^{\pm}(c_I)$. If $c_I = 2\sqrt{f'(0)}$, then $\lambda^-(c_I) = \lambda^+(c_I)$, hence (3.9) (with $c = c_I$) holds trivially. If $c_I > 2\sqrt{f'(0)}$, then, assuming that (3.9) does not hold, one can take $\tilde{c} \in (2\sqrt{f'(0)}, c_I)$ and repeat the arguments above to get a contradiction. □

Using the previous lemma, we get the following uniqueness result.

Lemma 3.16. *Let $I = (a, b) \in \mathcal{N}^+ \cup \mathcal{N}^-$. Then the following statements are valid.*

(i) *Let* ψ *be a solution of* (3.1) *with* $c \leq c_I$ *such that for some* $\bar{b} \in (a, b)$ *one has*

$$p^{\psi}(u) \leq p^{\phi_I}(u) \quad (u \in (a, \bar{b})), \tag{3.48}$$

and $\psi(x) \to a$ *as* $x \to \infty$ *(i.e.* $p^{\psi}(u) \to 0$ *as* $u \searrow a$*). Then necessarily* $c = c_I$*. Moreover, one has* $p^{\psi} \equiv p^{\phi_I}$ $(a, \bar{b})$*, except possibly in the case when* $a = 0$*,* $f'(0) > 0$*, and* $c_I = 2\sqrt{f'(0)}$*.*

(ii) *Let* ψ *be a solution of* (3.1) *with* $c \geq c_I$ *such that for some* $\bar{a} \in (a, b)$ *one has*

$$p^{\psi}(u) \leq p^{\phi_I}(u) \quad (u \in (\bar{a}, b)), \tag{3.49}$$

and $\psi(x) \to b$ *as* $x \to -\infty$ *(i.e.* $p^{\psi}(u) \to 0$ *as* $u \nearrow b$*). Then necessarily* $c = c_I$*. Moreover, one has* $p^{\psi} \equiv p^{\phi_I}$ *on* $(\bar{a}, b)$*, except possibly in the case when* $b = \gamma$*,* $f'(\gamma) > 0$*, and* $c_I = 2\sqrt{f'(\gamma)}$*.*

Note that the exceptional case $f'(0) > 0$ and $c_I = 2\sqrt{f'(0)}$, is precisely the case when the eigenvalues $\lambda^{\pm}(c)$ in (3.5) with $c = c_I$, $\zeta = 0$ are both equal to $2\sqrt{f'(0)}$. In this case, the conclusion $p^{\psi} \equiv p^{\phi_I}$ may fail, depending on the nonlinearity.

Proof. We only prove statement (i); the proof of (ii) is analogous.

Consider first the exceptional case: $a = 0$, $f'(0) > 0$, and $c_I = 2\sqrt{f'(0)}$. Then for $c < c_I$, the eigenvalues $\lambda^{\pm}(c)$ in (3.5) (with $\zeta = 0$) are imaginary, hence no solution ψ with the assumed properties can exist.

We proceed assuming that if $a = 0$, then either $f'(0) \leq 0$, or $f'(0) > 0$ and $c_I > 2\sqrt{f'(0)}$. Note that, by (3.40), we have $f'(a) \leq 0$ if $a > 0$.

Take first $c = c_I$. We show that $p^{\psi} \equiv p^{\phi_I}$ on $(a, \bar{b})$. If $f'(a) \leq 0$ and $c_I < 0$, this identity follows directly from the uniqueness statement of Lemma 3.3(niii) (with $\zeta = a$). If $f'(a) \leq 0$ and $c_I > 0$, then the identity follows from Lemma 3.15 (which gives the asymptotics of ϕ_I), Lemma 3.2(pi) (which gives the uniqueness of the solution which such asymptotics), and Lemma 3.2(pii) (which shows that no other solution $\psi(x)$ approaching a as $x \to \infty$ can satisfy (3.48), see also Remark 3.6). If $a = 0$ and $f'(0) > 0$ (note that in this case $c_I > 0$, see the remarks preceding Theorem 2.4), then the assumption $c_I > 2\sqrt{f'(0)}$ guarantees that $\lambda^-(c_I) < \lambda^+(c_I) < 0$ (again the eigenvalues correspond to $\zeta = 0$). The previous argument applies when one uses statements (i), (ii) of Lemma 3.4 in place of statements (pi), (pii) of Lemma 3.2.

Suppose now that $c < c_I$. We need to show that no solution ψ with the assumed properties exists. Assume it does. Then, by Lemma 3.1, the inequality (3.48) has to be strict. Pick any $u_0 \in (a, \bar{b})$ and take a solution $\tilde{\psi}$ of (3.1) with $c = c_I$ such that

$$p^{\psi}(u_0) < p^{\tilde{\psi}}(u_0) < p^{\phi_I}(u_0).$$

By Lemma 3.1, we necessarily have

$$p^{\psi}(u) < p^{\tilde{\psi}}(u) < p^{\phi_I}(u) \quad (u \in (a, u_0)),$$

which implies that $\psi(x) \to a$ as $x \to \infty$. This contradicts the validity of the conclusion for $c = c_I$ proved above. $\square$

3.3 Some trajectories out of the minimal system of waves

As in the previous subsection, R_0 is the minimal $[0, \gamma]$-system of waves, and $\{c_I : I \in \mathcal{N}\}$, $\{\phi_I : I \in \mathcal{N}\}$ are the families of speeds and profile functions corresponding to R_0. We also use the notation $\mathcal{N}^{\pm}$, $\mathcal{N}^0$ introduced in (2.14). In accord with (2.4), we denote by $\tau(R_0)$ the graph of the function $w = R_0(v)$ in the (v, w)-plane.

This subsection contains key technical ingredients of the proofs of our main results. We identify here a class of solutions of (3.1) which will be used in intersection comparison arguments with a solution u of (1.1). As we will show in Subsection 6.2, the spatial trajectories of these solutions have to become disjoint from the spatial trajectories $\tau(u(\cdot, t))$ as $t \to \infty$. Thus, we want to find a "large" set of such solutions of (3.1), thereby narrowing the possibilities where $\tau(u(\cdot, t))$ can be located for large times. The pairs (c, ψ), ψ being a solution of equation (3.1) (on $\mathbb{R}$), that will work for us are of the following types (see Figure 6).

(A1) ψ is a nonconstant periodic solution with $0 < \psi < \gamma$ (hence, necessarily, $c = 0$).

(A2) There are $x_1 < x_2$ such that

$$\begin{aligned} &\psi(x) \in (0, \gamma) \quad (x \in (x_1, x_2)), \\ &\psi(x_i) \in \{0, \gamma\}, \quad (i = 1, 2), \\ &|\psi'(x)| > 0 \quad (x \in [x_1, x_2]). \end{aligned} \tag{3.50}$$

(A3p) For some $I_0 = (a,b) \in \mathcal{N}^+$, $x_0 < \hat{x}$, and $\zeta \in [a,b) \cap (0,b)$, one has $c < c_{I_0}$ and

$$\begin{aligned} &\psi(x_0) = 0, \quad \psi'(x_0) > 0, \\ 0 < \psi(x) < b, \ &\psi'(x) > 0 \quad (x \in (x_0, \hat{x})), \\ a < \psi(x) &< b \quad (x \in (\hat{x}, \infty)), \\ \lim_{x\to\infty} &(\psi(x), \psi'(x)) = (\zeta, 0). \end{aligned} \tag{3.51}$$

(A3n) For some $I_0 = (a,b) \in \mathcal{N}^-$, $\check{x} < x_0$, and $\zeta \in (a,b] \cap (a,\gamma)$, one has $c > c_{I_0}$ and

$$\begin{aligned} &\psi(x_0) = \gamma, \quad \psi'(x_0) > 0, \\ a < \psi(x) < \gamma, \ &\psi'(x) > 0 \quad (x \in (\check{x}, x_0)), \\ a < \psi(x) &< b \quad (x \in (-\infty, \check{x})), \\ \lim_{x\to-\infty} &(\psi(x), \psi'(x)) = (\zeta, 0). \end{aligned} \tag{3.52}$$

For the case when $\zeta = 0$ is unstable from above for the equation $\dot{\theta} = f(\theta)$ (hence, $f'(0) \geq 0$), we will also need pairs (c, ψ) of type (A3p0), as described below. Recall that for $c \geq 2\sqrt{f'(0)}$

$$\lambda^{\pm}(c) = \frac{-c \pm \sqrt{c^2 - 4f'(0)}}{2} \tag{3.53}$$

(this is formula (3.5) with $\zeta = 0$). The eigenvalues are real if $c \geq 2\sqrt{f'(0)}$. This always applies to $c = c_{I_0}$, if $I_0 = (0,b) \in \mathcal{N}^+$ for some $b > 0$, since there is then the decreasing solution ϕ_{I_0} of (3.1) with $\phi_{I_0}(\infty) = 0$.

(A3p0) For some $b > 0$ and $x_0 \in \mathbb{R}$, one has $I_0 = (0,b) \in \mathcal{N}^+$,

$$c > 2\sqrt{f'(0)}, \quad \lambda^+(c) > \lambda^-(c_{I_0}), \tag{3.54}$$

and the following relations are satisfied

$$\begin{aligned} &0 < \psi(x) < b \quad (x \in (x_0, \infty)), \\ &\psi(x_0) = 0, \ \psi'(x_0) > 0, \\ &\lim_{x\to\infty} (\psi(x), \psi'(x)) = (0,0), \\ &\psi \text{ has the asymptotics (3.16) (with } \zeta = 0). \end{aligned} \tag{3.55}$$

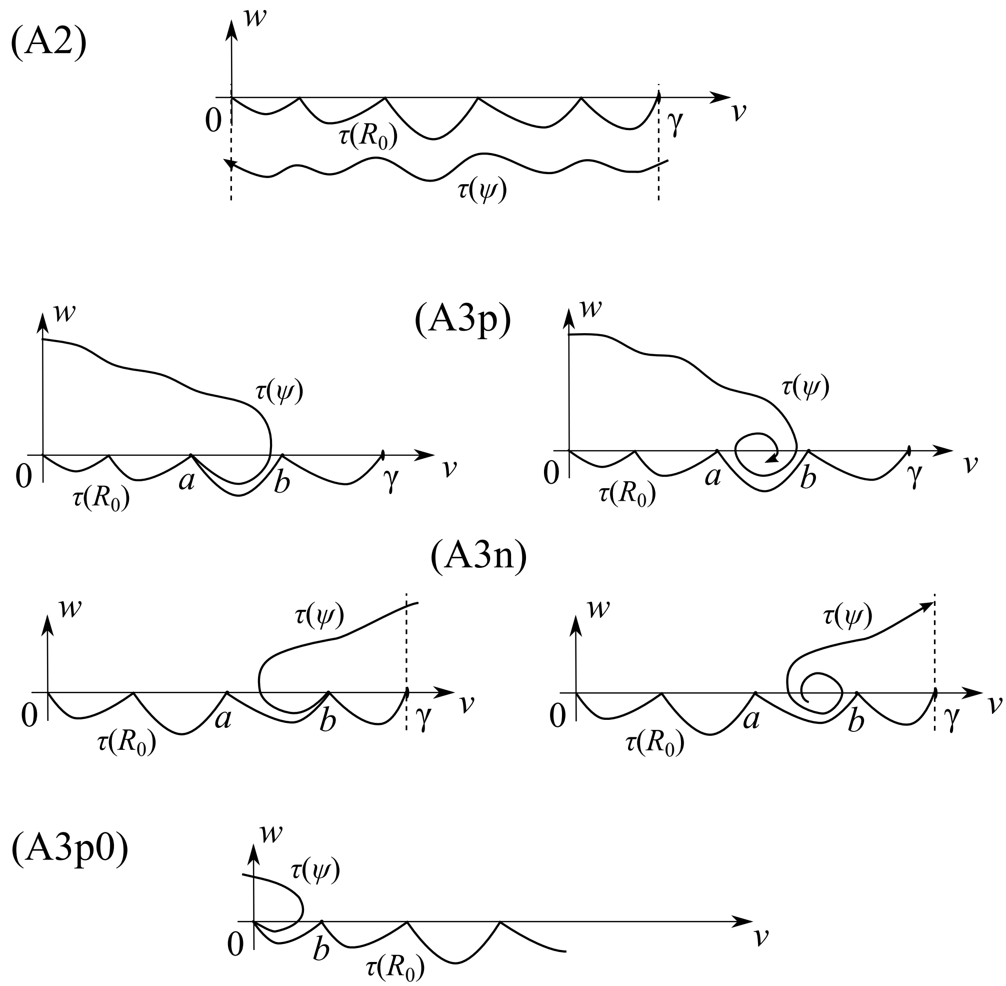

Figure 6: The graph of R_0 and the trajectories of solutions ψ of type (A2), (A3p), (A3n), (A3p0). In the (A3p) figure, the example on the left—with the limit equilibrium $(a, 0)$—is valid only if $a > 0$. Similarly, the example on the left in (A3n) is valid only if $b < \gamma$.

Lemma 3.10(ii) will be our main source of solutions of type (A1). Solutions of type (A2) will be found using the following result. Recall, that γ_0, γ_1 were defined in Section 2.3. We have $\gamma_0 > 0$ if 0 is unstable from above for the equation $\dot{\theta} = f(\theta)$. In that case $I_* = (0, \gamma_0)$ is the minimal element of $\mathcal{N}$ and we have

$$c_{I_*} \geq c_I \quad (I \in \mathcal{N}), \quad c_{I_*} > 0 \tag{3.56}$$

(see (2.13) and the remarks preceding Theorem 2.4). Similarly, $\gamma_1 < \gamma$ if γ is unstable from below for this ODE and then $I^* = (\gamma_1, \gamma)$ is the maximal element of $\mathcal{N}$, and

$$c_{I^*} \leq c_I \quad (I \in \mathcal{N}) \quad c_{I^*} < 0. \tag{3.57}$$

Lemma 3.17. *Let $\xi \in [0, \gamma]$, $\eta < R_0(\xi)$, and let ψ be the solution of* (3.1), (3.3) *for some $c \in \mathbb{R}$. Then the following statements are valid.*

(i) *Assume that $\xi \in [a, b]$ for some $I := (a, b) \in \mathcal{N}$ and $c \geq c_I$. If $f'(\gamma) > 0$ assume also that $c > c_{I^*}$ (by* (3.57)*, this is satisfied in particular if $c > c_I$ or $c = 0$). Then there is $x_1 \leq 0$ such that*

$$\psi(x_1) = \gamma, \text{ and } \psi'(x) < 0 \quad (x \in [x_1, 0]). \tag{3.58}$$

(ii) *Assume that $\xi \in [a, b]$ for some $I := (a, b) \in \mathcal{N}$ and $c \leq c_I$. If $f'(0) > 0$ assume also that $c < c_{I_*}$ (by* (3.56)*, this is satisfied in particular if $c < c_I$ or $c = 0$). Then there is $x_2 \geq 0$ such that*

$$\psi(x_2) = 0, \text{ and } \psi'(x) < 0 \quad (x \in [0, x_2]). \tag{3.59}$$

(iii) *Assume that $\xi \in [0, \gamma] \setminus \cup_{I \in \mathbb{N}} \bar{I}$ and $c = 0$. Then there are $x_1 < x_2$ such that* (3.58) *and* (3.59) *hold.*

Proof. Out of statements (i), (ii), we only prove (ii); the proof of (i) is analogous. Assume the hypotheses of (ii) are satisfied.

Consider the following region the (v, w)-plane:

$$G^- := \{(v, w) : 0 < v < \gamma, \, w < R_0(v)\}. \tag{3.60}$$

The boundary of G^- consists of the graph $\tau(R_0)$ and the half-lines $\{(0, w) : w < 0\}$, $\{(\gamma, w) : w < 0\}$. By assumption, one has $(\xi, \eta) \in G^-$ or (ξ, η) is contained in one of the two half-lines (in the latter case one has $a = 0$ or $b = 0$). Replacing (ξ, η) by a nearby point on the trajectory $\tau(\psi)$, we may assume without loss of generality that $(\xi, \eta) \in G^-$. Since $R_0 \leq 0$ and $R_0 = 0$ only at zeros of f, one verifies easily that if the conclusion of statement (ii) is not true, then one of the following two possibilities has to occur:

(a) There is $x_0 > 0$ such that $(\psi(x), \psi'(x)) \in G^-$ for $x \in [0, x_0)$ and

$$(\psi(x_0), \psi'(x_0)) \in \tau(R_0) \cap \{(v, w) : w < 0\}.$$

(b) One has $(\psi(x), \psi'(x)) \in G^-$ for $x \in [0, \infty)$ and

$$(\psi(x), \psi'(x)) \to (\zeta, 0) \in \tau(R_0) \text{ as } x \to \infty. \tag{3.61}$$

To complete the proof, we need to rule both these possibilities out.

We start with (a). Assume that it holds. Then, in particular, $\psi' < 0$ in $[0, x_0]$ and $(\psi(x_0), \psi'(x_0)) = (u_0, R_0(u_0))$ for some u_0 with $R_0(u_0) < 0$. Thus there is a nodal interval $J \in \mathcal{N}$ of R_0 such that $u_0 \in J$. Near $(u_0, R_0(u_0))$, the graph of $\tau(R_0)$ then coincides with the trajectory $\tau(\phi^J)$ of the corresponding profile function. Since $u_0 = \psi(x_0) < \psi(0) = \xi \in [a, b] = \bar{I}$, we have $J \leq I$ in the ordering of $\mathcal{N}$. Hence, by (2.13), $c_J \geq c_I$. Now, in terms of the functions p^ψ, p^{ϕ^J}, possibility (a) means that there is $\epsilon > 0$ such that

$$p^\psi(u_0) = p^{\phi^J}(u_0), \quad p^\psi(u) < p^{\phi^J}(u), \ \ (u \in (u_0, u_0 + \epsilon)).$$

Since ψ is the solution of (3.1) with $c \leq c_I$, these relations yield a contradiction to Lemma 3.1. Possibility (a) is thus ruled out.

Assume that (b) holds. Obviously, $\zeta \in R_0^{-1}\{0\}$ and $\zeta \leq a < b$. Therefore, if ζ is the left end point of some interval $J \in \mathcal{N}^+ \cup \mathcal{N}^-$, then $c_J \geq c_I \geq c$. In this situation, we immediately get a contradiction from Lemma 3.16(i) (in case $\zeta = 0$ and $f'(0) > 0$, the lemma applies due to the extra assumption $c < c_{I_*}$). Now, Proposition 3.11(iii),(iv) implies that ζ is necessarily the left end point of some $J \in \mathcal{N}^+ \cup \mathcal{N}^-$ if $\zeta < \gamma_*$, or $\zeta > \gamma^*$, or $\zeta = \gamma^*$ and the set $\mathcal{N}^-$ is finite. Hence, in these cases we are done.

We next consider the case of $\zeta = \gamma^*$ with $\mathcal{N}^-$ infinite (in particular, a cannot be equal to γ^*, so $a > \gamma^*$). Proposition 3.11(iv) guarantees that there is an interval $J \in \mathcal{N}^-$ (in fact, infinitely many of them) such that $J < I$. Therefore, $c \leq c_I \leq c_J < 0$. If we now replace η with a slightly larger value in $(-\infty, R_0(\xi))$, then the solution from the new initial condition, with the same c, can no longer converge to γ^* by the uniqueness result in Lemma 3.3(niii) (the lemma applies to $\zeta = \gamma^*$ because γ^*, being a maximizer of $F\big|_{[0,\gamma]}$, satisfies $f'(\zeta) \leq 0$). Thus, to the new initial condition we can apply the arguments above and get a contradiction.

It remains to consider the case $\gamma_* \le \zeta < \gamma^*$ (which can occur only if $\gamma_* < \gamma^*$). Since $\gamma_* \le \zeta \le a < b$, we have $c \le c_I \le 0$ (cp. Proposition 3.11(iii)). Therefore, by (3.29), the function

$$H(\psi(x), \psi'(x)) = \frac{(\psi'(x))^2}{2} + F(\psi(x)) \tag{3.62}$$

is monotone nondecreasing. If $a \ge \gamma^*$, then there is $x_0 \ge 0$ such that $\psi(x_0) = \gamma^*$, $\psi'(x_0) < 0$ (the latter follows from $(\psi(x_0), \psi'(x_0)) \in G^-$). Consequently,

$$F(\gamma^*) < H(\psi(x_0), \psi'(x_0)) \le H(\zeta, 0) = F(\zeta), \tag{3.63}$$

contradicting the fact that γ^* is a global maximizer of F. If $a < \gamma^*$, then, since $\gamma_* \le \zeta \le a$, we have $I = (a, b) \subset (\gamma_*, \gamma^*)$ (cp. Proposition 3.11(iii)) and b itself is a global maximizer of $F\Big|_{[0,\gamma]}$ (Proposition 3.11(ii)). Also, $c_I = 0$, which implies that H is constant on the closure of the trajectory $\tau(\phi_I)$. Therefore,

$$F(b) = H(b, 0) \le H(\xi, R_0(\xi)),$$

since $(\xi, R_0(\xi))$ is on the trajectory $\tau(\phi_I)$. Similarly as in (3.63), using the nondecrease of H along the trajectory of ψ and the relation $\eta < R_0(\xi)$, we obtain

$$H(\xi, R_0(\xi)) < H(\xi, \eta) = H(\psi(0), \psi'(0)) \le H(\zeta, 0) = F(\zeta).$$

Combining the above inequalities, we have contradicted the maximality of $F(b)$.

Thus, possibility (b) leads to a contradiction in all cases. The proof of statement (ii) in now complete.

Assume now that $c = 0$ and ξ is as in statement (iii). We prove that (3.59) holds for some x_2; the proof of the existence of x_1 satisfying (3.58) is analogous and is omitted.

By Proposition 3.11, we necessarily have $\xi \in [\gamma_*, \gamma^*]$ and one of the following possibilities has to occur:

(c) There is a sequence of intervals $I_n = (a_n, b_n)$ in $\mathcal{N}^+ \cup \mathcal{N}^0$ such that $b_n \nearrow \xi$.

(d) There is $\xi_0 \in [0, \xi)$ such that $[\xi_0, \xi] \subset R_0^{-1}\{0\}$.

If (c) holds and n is large enough, then, clearly, the trajectory $\tau(\psi)$ contains the point $(b_n, \tilde{\eta})$ for some $\tilde{\eta} < 0$. Replacing (ξ, η) by $(b_n, \tilde{\eta})$ and applying statement (ii) (which is legitimate since $c = 0$ and $c_{I_n} \geq 0$), we find $x_2 \geq 0$ such that (3.59) holds.

Assume now that (d) holds and take the minimal ξ_0 with the indicated property. Since $R_0^{-1}\{0\} \subset f^{-1}\{0\}$, we have $f \equiv 0$ on $[\xi_0, \xi]$. Since also $c = 0$, equation (3.1) tells us that $\psi' \equiv \mathit{const} = \eta < 0$, as long as $\psi(x)$ remains in $[\xi_0, \xi]$. If $\xi_0 = 0$, this implies, trivially, that (3.59) holds for some $x_2 \geq 0$. If $\xi_0 > 0$, we simply replace (ξ, η) by $(\xi_0, \eta) \in \tau(\psi)$ and note that then either (c) or statement (ii) applies due to the minimality of ξ_0. This completes the proof of statement (iii). $\square$

Corollary 3.18. *Let $\xi \in [0, \gamma]$ and $\eta < R_0(\xi)$. Then for some $c \in \mathbb{R}$ the solution ψ of* (3.1), (3.3) *has the following property. There are $x_1 < x_2$ such that*

$$\psi(x_1) = \gamma, \ \psi(x_2) = 0, \ \textit{and } \psi'(x) < 0 \quad (x \in [x_1, x_2]). \tag{3.64}$$

Hence, ψ is a solution of type (A2).

Proof. If ξ is as in Lemma 3.17(iii), the conclusion follows immediately upon taking $c = 0$. Otherwise, $\xi \in [a, b]$ for some $I := (a, b) \in \mathcal{N}$.

Take first $c = c_I$ and observe that the hypotheses of at least one of the statements (i), (ii) of Lemma 3.17 are satisfied. Indeed, if that were not the case, we would have $f'(0) > 0$, $f'(\gamma) > 0$, and $c_{I_*} \leq c \leq c_{I^*}$. This is impossible, however, because $c_{I_*} > 0 > c_{I^*}$ (see (3.56), (3.57)).

Assume for definiteness that the hypotheses of Lemma 3.17(i) are satisfied (the other case is analogous). Then (3.58) holds for some $x_1 \leq 0$. Clearly, this remains valid (with a possibly different $x_1 \leq 0$) if we perturb c slightly. We can thus take $c < c_I$, $c \approx c_I$, such that (3.58) holds. With this new choice of c, the hypotheses of statement (ii) of Lemma 3.17 are satisfied. Hence, (3.59) holds as well, and consequently (3.64) holds for some $x_2 > x_1$. $\square$

Remark 3.19. With ψ and c as in Corollary 3.18, the function $\hat{\psi}(x) := \psi(-x)$ is a solution of (3.1) with c replaced by $-c$ and

$$\hat{\psi}(-x_2) = 0, \ \hat{\psi}(-x_1) = \gamma, \text{ and } \hat{\psi}'(x) > 0 \quad (x \in [-x_2, -x_1]). \tag{3.65}$$

Hence, $\hat{\psi}$ is also a solution of type (A2).

Solutions of types (A3p), (A3n), and (A3p0) will be found using the following lemma (and the analogous results of Lemma 3.22). In its statement, we consider a solution ψ^c of (3.1) whose existence is guaranteed by Lemma 3.2(pi) (the solution is unique up to translations).

Lemma 3.20. *Assume that $I = (a, b) \in \mathcal{N}^+$; in case $a = 0$ assume also that $f'(0) \leq 0$. For any $c > 0$ let $\psi^c(x)$ be a solution of* (3.1) *satisfying* (3.6), (3.9) *with $\zeta := a$. If $\delta > 0$ is sufficiently small, then for each $c \in (c_I - \delta, c_I)$ the solution ψ^c satisfies* (3.51) *with some $x_0 = x_0(c)$, $\hat{x} = \hat{x}(c)$ (and with $\zeta = a$). Moreover, after replacing ψ^c by a translation such that its critical point $\hat{x}$ is put at $x = 0$ (and $x_0 < 0$), the following statements are valid:*

(i) *The function $c \mapsto \psi^c(0)$ is continuous, increasing, and $\psi^c(0) \to b$ as $c \nearrow c_I$.*

(ii) *For any $c_1, c_2 \in (c_I - \delta, c_I)$ with $c_1 < c_2$, the trajectories*

$$\{(\psi^{c_i}(x), \psi_x^{c_i}(x)) : x \in (x_0(c_i), \infty)\}, \quad i = 1, 2,$$

are disjoint, and each of them is disjoint from the trajectory $\tau(\phi_I)$.

(iii) *If φ^c denotes the restriction of ψ^c to the interval $[0, \infty)$ (where ψ^c is decreasing), then, given any $u_0 \in (a, b)$, the function p^{φ^c} is defined on $(a, u_0]$ for all $c < c_I$ sufficiently close to c_I and one has*

$$p^{\varphi^c}(u) \to p^{\phi_I}(u) \text{ as } c \nearrow c_I \tag{3.66}$$

uniformly on $(a, u_0]$.

Remark 3.21. Although not used below, this remark will further clarify what happens as $c \nearrow c_I$. By elementary considerations similar to those given in the proof of Lemma 3.28 below, one can show that, as $c \nearrow c_I$, the trajectories $\{(\psi^c(x), \psi_x^c(x)) : x \in (x_0(c), \infty)\}$ approach in the Hausdorff distance the set composed of the trajectory $\tau(\phi_I)$, the point $(b, 0)$, and a trajectory of (3.2) with $c = c_I$ in the center-stable manifold of $(b, 0)$ (cp. Figure 7).

The proof of Lemma 3.20 is given below after some preliminary results. Here are analogous statements for $\mathcal{N}^-$, which we include without proof.

Lemma 3.22. *Assume that $I = (a, b) \in \mathcal{N}^-$; in case $b = \gamma$ assume also that $f'(\gamma) \leq 0$. For any $c \in (c_I, 0)$ let $\psi^c(x)$ be a solution of satisfying* (3.7) *and* (3.13) *with $\zeta := b$ (see Lemma 3.3(ni)). If $\delta > 0$ is sufficiently small, then*

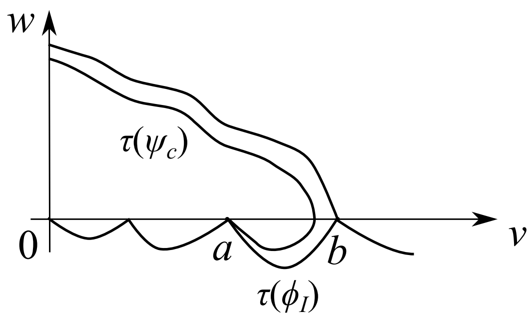

Figure 7: The trajectory of ψ_c for $c < c_I$, $c \approx c_I > 0$.

for each $c \in (c_I, c_I + \delta)$ *the solution* ψ^c *satisfies* (3.52) *with some* $x_0 = x_0(c)$, $\check{x} = \check{x}(c)$ *(and with* $\zeta = b$*). Moreover, after replacing* ψ^c *by a translation such that its critical point* $\check{x}$ *is put at* $x = 0$ *(and* $x_0 > 0$*), the following statements are valid:*

(i) *The function* $c \mapsto \psi^c(0)$ *is continuous, decreasing, and* $\psi^c(0) \to a$ *as* $c \searrow c_I$.

(ii) *For any* $c_1, c_2 \in (c_I, c_I + \delta)$ *with* $c_1 < c_2$, *the trajectories*

$$\{(\psi^{c_i}(x), \psi_2^{c_i}(x)) : x \in (-\infty, x_0(c_i))\}, \quad i = 1, 2,$$

are disjoint, and each of them is disjoint from the trajectory $\tau(\phi_I)$.

(iii) *If* φ^c *denotes the restriction of* ψ^c *to the interval* $(-\infty, 0]$ *(where* ψ^c *is decreasing), then, given any* $u_0 \in (a, b)$, *the function function* p^{φ^c} *is defined on* $[u_0, b)$ *for all* $c > c_I$ *sufficiently close to* c_I *and one has*

$$p^{\varphi^c}(u) \to p^{\phi_I}(u) \text{ as } c \searrow c_I \tag{3.67}$$

uniformly on $[u_0, b)$.

The following relations will come in handy in the proof of Lemma 3.20 and other places below.

Lemma 3.23. *Assume that* $\zeta \in f^{-1}\{0\}$ *and* $\lambda^\pm(c)$ *are as in* (3.5). *If* $f'(\zeta) \leq 0$, *then*

$$\lambda^-(c_2) < \lambda^-(c_1) \text{ and } \lambda^+(c_2) \leq \lambda^+(c_1) \quad (c_2 > c_1 \geq 0). \tag{3.68}$$

If $f'(\zeta) > 0$, *then*

$$\lambda^-(c_2) < \lambda^-(c_1), \quad \lambda^+(c_2) > \lambda^+(c_1) \quad (c_2 > c_1 \geq 2\sqrt{f'(\zeta)}). \tag{3.69}$$

These relations follow by simple estimates of the derivatives of the eigenvalues $\lambda^{\pm}(c)$:

$$\frac{d\lambda^{\pm}(c)}{dc} = \frac{1}{2}\left(-1 \pm \frac{c}{\sqrt{c^2 - 4f'(\zeta)}}\right)$$

The following result will also be used in the proof of Lemma 3.20.

Lemma 3.24. (i) *For each $I = (a, b) \in \mathcal{N}^+$ there is $\nu > 0$ with the following property. If $\xi \in (b - \nu, b)$, $\eta > R_0(\xi)$, and ψ is the solution of* (3.1), (3.3) *for some $c \in [0, c_I]$, then there is $x_0 < 0$ such that*

$$\psi(x_0) = 0, \;\; \psi'(x_0) > 0, \;\; \textit{and } \psi(x) \in (0, b) \;\; (x \in (x_0, 0]); \tag{3.70}$$

moreover, if $\eta > 0$, then $\psi' > 0$ in $[x_0, 0]$; and if $R_0(\xi) < \eta \le 0$, then ψ has a unique critical point $\hat{x}$ in $[x_0, 0]$ and $\psi''(\hat{x}) < 0$.

(ii) *For each $I = (a, b) \in \mathcal{N}^-$ there is $\nu > 0$ with the following property. If $\xi \in (a, a + \nu)]$, $\eta > R_0(\xi)$, and ψ is the solution of* (3.1), (3.3) *for some $c \in [c_I, 0]$, then there is $x_0 > 0$ such that*

$$\psi(x_0) = \gamma, \;\; \psi'(x_0) > 0, \;\; \textit{and } \psi(x) \in (a, \gamma) \;\; (x \in (0, x_0]); \tag{3.71}$$

moreover, if $\eta > 0$, then $\psi' > 0$ in $(0, x_0]$; and if $R_0(\xi) < \eta \le 0$, then ψ has unique critical point $\check{x}$ in $[0, x_0]$ and one has $\psi''(\check{x}) > 0$.

Proof. We only prove statement (i); the proof of (ii) is analogous.

By Lemma 3.7(i) there is $\nu > 0$ such that

$$f(u) > 0 \quad (u \in [b - \nu, b)). \tag{3.72}$$

Since b is a strict left-global maximizer of F (cp. Proposition 3.11(ii),(iii)), we can make ν smaller so that, moreover,

$$F(v) < F(u) \quad (u \in [b - \nu, b], \; v \in [0, u)). \tag{3.73}$$

To show that statement (i) holds with this ν, assume that ξ, η, c and ψ, satisfy the stated conditions.

First, we consider the case $\eta > 0$. We show that there is $x_0 = x_0(c) < 0$ such that $\psi(x_0) = 0$ and $\psi_x > 0$ in $[x_0, 0]$ (in particular, relations (3.70) hold). In other words, we want to show that as $-x$ increases, $(\psi(x), \psi_x(x))$

leaves the first quadrant $Q_+ := \{(v,w) : v > 0, w > 0\}$ through the positive w-axis. If this is not true, then $(\psi(x), \psi_x(x))$ either hits the v-axis at some finite $\bar{x} < 0$ or it stays in Q_+ and converges as $x \to -\infty$ to some equilibrium $(\theta, 0)$ with $\theta \in [0, b)$. Consider the former: there is $\bar{x} < 0$ such that $\psi_x(\bar{x}) = 0$ and $(\psi(x), \psi_x(x)) \in Q_+$ for $x \in (\bar{x}, 0]$. Then, using the fact that the function H in (3.62) is nonincreasing (since $c \geq 0$, cp. (3.29)), we obtain

$$F(\psi(\bar{x})) = H(\psi(\bar{x}), 0) \geq H(\psi(0), \psi_x(0)) = \frac{\eta^2}{2} + F(\xi) > F(\xi), \tag{3.74}$$

a contradiction to (3.73). A similar argument shows that $\psi(x)$ cannot converge to any $\theta \in [0, b)$ as $x \to -\infty$ (this would give $F(\theta) > F(\xi)$). Hence, the conclusion of statement (i) holds if $\eta > 0$.

Consider now the case $\eta \leq 0$. We claim that there is $x \leq 0$ such that $\psi_x(x) = 0$. Indeed, if not, then $\psi_x < 0$ for all $x < 0$ and $\psi(x)$ converges as $x \to -\infty$ to some $\theta \in f^{-1}\{0\}$. Since $c \in [0, c_I]$, necessarily, $\theta \leq b$ (by Lemma 3.1, $\tau(\psi)$ stays above $\tau(\phi_I)$). The case $\theta = b$ is excluded by the uniqueness statement in Lemma 3.2(piii) and the existence result in Lemma 3.8(i). Thus, $\theta < b$ and this contradicts (3.72). This proves our claim. Obviously, if $\hat{x} \in (-\infty, 0)$ is the critical point of ψ_x with minimal absolute value, then $\psi(x) \in (\xi, b)$ for each $x \in [\hat{x}, 0)$. Moreover, by (3.72), $f(\psi(\hat{x})) > 0$, hence, by equation (3.1), $\psi''(\hat{x}) < 0$. In particular, $\hat{x}$ is an isolated zero of ψ_x. Thus for any $y > \hat{x}$ sufficiently close to $\hat{x}$, the solution $\tilde{\psi} := \psi(\cdot - y)$ satisfies the conditions in statement (i) with $\eta > 0$. Using what we have already proved for the case $\eta > 0$, we find $x_0 < 0$ such that the relations (3.70) are satisfied and $\hat{x}$ is the unique critical point of ψ. □

Proof of Lemma 3.20. We have $f'(a) \leq 0$ (this is an assumption if $a = 0$; if $a > 0$, it holds automatically by (3.40)).

Recall that ψ^c identified in the assumptions of the Lemma 3.20 is the solution, unique up to translations, which is decreasing for large x and has the asymptotics (3.9) (with $\zeta = a$). Also, its trajectory $\tau(\psi^c)$ is contained in the stable manifold W^s of the equilibrium $(a, 0)$ (cp. the proof of Lemma 3.2(i)). For $c = c_I$, the function ψ^c coincides, after a suitable translation, with ϕ_I (see Lemma 3.15); thus $\tau(\psi^{c_I})$ is a heteroclinic orbit from $(b, 0)$ to $(a, 0)$. Using the continuity of W^s with respect to c, by taking $c < c_I$, $c \approx c_I$, we can find points on $\tau(\psi^c)$ arbitrarily close to $(b, 0)$. More specifically, for each $c < c_I$, $c \approx c_I$, there is $y_c \in \mathbb{R}$ such that the following relations are valid:

$$\psi^c_x(x) < 0 \quad (x \geq y_c), \quad \psi^c(y_c) = b - \nu/2, \tag{3.75}$$

where ν is as Lemma 3.24(i). Moreover, for any $c_1, c_2 \approx c_I$ with $c_1 < c_2 < c_I$ we have

$$p^{\psi^{c_1}}(u) > p^{\psi^{c_2}}(u) > p^{\phi_I}(u) \tag{3.76}$$

on any interval $(a, \bar{b}] \subset (a, b)$ on which $p^{\psi^{c_1}}$, $p^{\psi^{c_2}}$ are both defined. This follows from the relations $\lambda^-(c_2) < \lambda^-(c_1) < \lambda^-(c_I)$ (see (3.68)), the asymptotics (3.9) (cp. Remark 3.6(i)), and Lemma 3.1. In particular, taking $\bar{b} = b - \nu/2$, we see from (3.75), (3.76) that Lemma 3.24(i) applies to the solution $\psi := \psi^c(\cdot - y_c)$. This shows that for some $x_0 = x_0(c) < y_c$, ψ^c satisfies (3.51) with $\zeta = a$. Clearly, ψ^c has a critical point somewhere in $(x_0(c), y_c)$ and by Lemma 3.24(i) this critical point is unique. We now replace ψ^c by a translation, if necessary, so that $x = 0$ is the critical point.

The continuity of the stable manifold W^s with respect to c guarantees that $\psi^c(0)$ depends continuously on c. Since $\tau(\psi^c)$ contains points arbitrarily close to $(b, 0)$, we have that $\psi^c(0) \to b$ as $c \nearrow c_I$. This and (3.76) prove statement (i) of Lemma 3.20.

Statement (ii) follows from (3.76) and Lemma 3.1.

For statement (iii), we note that the function p^{ψ^c} is defined on the interval $(a, \psi^c(0)]$. By statement (i), this intervals includes $(a, u_0] \subset (a, b)$ if c is close enough to c_I. Therefore, the uniform convergence (3.66) is just an interpretation of the continuous dependence of the stable manifold W^s on c (cp. Remark 3.6(ii)). The proof is complete. $\square$

In the following results, we identify a set of points $(\xi, \eta) \in \mathbb{R}^2$ which lie on trajectories of solutions of types (A2), (A3p), or (A3p0). To elucidate the significance of the regions formed by such point, recall that our goal is to constraint possible locations of the spatial trajectories $\tau(u(\cdot, t))$, $t \approx \infty$, of the solution of (1.1), (1.2). When we show that $\tau(u(\cdot, t))$ cannot intersect the trajectory of any solution which is of type (A2), (A3p), or (A3p0), then $\tau(u(\cdot, t))$ has to be disjoint from the identified regions.

For any interval $J \subset [0, \gamma]$, we denote

$$S_J := \{(v, w) \in \mathbb{R}^2 : v \in J\}. \tag{3.77}$$

Proposition 3.25. (i) *Assume that $I = (a, b) \in \mathcal{N}^+$. In case $a = 0$, assume also that $\theta = 0$ is stable from above for the equation $\dot{\theta} = f(\theta)$. Then there is $\epsilon > 0$ with the following property. For each $(\xi, \eta) \in S_{(b-\epsilon, b]} \setminus \tau(R_0)$ one can find $c \in \mathbb{R}$ and a solution ψ of* (3.1) *such that $(\xi, \eta) \in \tau(\psi)$ and either* (A2) *holds or* (A3p) *holds with $I_0 = I$.*

(ii) *Assume that* $I = (a,b) \in \mathcal{N}^-$. *In case* $b = \gamma$, *assume also that* $\theta = \gamma$ *is stable from below for the equation* $\dot{\theta} = f(\theta)$. *Then there is* $\epsilon > 0$ *with the following property. For each* $(\xi, \eta) \in S_{[a,a+\epsilon)} \setminus \tau(R_0)$ *one can find* $c \in \mathbb{R}$ *and a solution* ψ *of* (3.1) *such that* $(\xi,\eta) \in \tau(\psi)$ *and either* (A2) *holds or* (A3n) *holds with* $I_0 = I$.

(iii) *Assume that* $I = (0,b) \in \mathcal{N}^+$ *for some* $b \in (0,\gamma]$ *and that* $\theta = 0$ *is unstable from above for the equation* $\dot{\theta} = f(\theta)$. *Then there is* $\epsilon > 0$ *with the following property. For each* $(\xi,\eta) \in S_{(b-\epsilon,b]} \setminus \tau(R_0)$ *one can find* $c \in \mathbb{R}$ *and a solution* ψ *of* (3.1) *such that* $(\xi,\eta) \in \tau(\psi)$ *and either* (A2) *holds or* (A3p0) *holds with* $I_0 = I$.

We will only prove statements (i), (iii); the proof of (ii) is analogous to the proof of (i) and is omitted. We carry out some steps of the proofs of these results in the following lemmas.

Lemma 3.26. *Assume that* $I = (a,b) \in \mathcal{N}^+$ *and let*

$$B := \{(v,w) : v \in (a,b),\ R_0(v) \le w \le -R_0(v)\}. \tag{3.78}$$

If $(\xi,\eta) \in S_{[0,\gamma]} \setminus B$, *then there is* $c \in \mathbb{R}$ *such that the solution of* ψ *of* (3.1), (3.3) *satisfies* (A2).

Proof. This follows directly from Corollary 3.18 and Remark 3.19. □

Lemma 3.27. *Assume that* $I = (a,b) \in \mathcal{N}^+$, $c \ge c_I$, *and let* B *be as in* (3.78). *If* $(\xi,\eta) \in B \setminus \tau(\phi_I)$ *and* ψ *is the solution of* (3.1), *then* $(\psi(x), \psi'(x)) \in \operatorname{Int} B$ *for all* $x > 0$. *Moreover, there is* $\mu > 0$ *such that if* $(\xi,\eta) \in S_{(b-\mu,b)} \cap B \setminus \tau(\phi_I)$, *then* $\psi'(x) \le 0$ *for some* $x \ge 0$.

Proof. The first statement (the invariance of B) follows directly from Lemma 3.1, upon noting that the graph of $R_0\big|_{(a,b)}$ coincides with $\tau(\phi_I)$, and the graph of $-R_0\big|_{(a,b)}$ coincides with $\tau(\hat{\phi}_I)$, where $\hat{\phi}_I(x) := \phi_I(-x)$ is a solution of (3.1) with $c = -c_I < 0 < c_I$.

Let now $\mu \in (0,\nu]$, where ν is as in Lemma 3.24(i) and such that also (3.72) is satisfied. We prove that the second statement is satisfied for this μ. Consider the solution ψ with (ξ,η) satisfying the given conditions. If $\psi'(x) > 0$ for all $x > 0$, then $\psi(x) \to \zeta$, as $x \to \infty$, for some $\zeta \in (b-\mu, b] \cap f^{-1}\{0\}$. Necessarily, $\zeta = b$, for there are no zeros of f in $(b-\mu,b)$ (see (3.72)). To rule this possibility out, we employ the functional H (see (3.28), (3.29)). It decreases along (ψ, ψ_x), as $c \geq c_I > 0$, whereas it increases along $(\hat{\phi}_I, \hat{\phi}_I')$. Now, by Lemma 3.24(i), the trajectory $\tau(\psi)$ intersects $\tau(\hat{\phi}_I)$. Therefore, since $(b, 0)$ is the limit of $(\hat{\phi}_I(x), \hat{\phi}_I'(x))$, as $x \to \infty$, it cannot at the same time be the limit of $(\psi(x), \psi_x(x))$ as $x \to \infty$. $\square$

Lemma 3.28. *Assume that the hypotheses of Lemma 3.20 are satisfied and let a sufficiently small $\delta > 0$ and ψ^c, $c \in (c_I - \delta, c_I)$, be as in that lemma. Then there is $\epsilon > 0$ such that, with B as in* (3.78)*, one has*

$$B \cap S_{(b-\epsilon, b]} \setminus \tau(R_0) \subset \bigcup_{c \in (c_I - \delta, c_I)} \tau(\psi^c). \tag{3.79}$$

Proof. It follows directly from Lemma 3.20 that (3.79) holds for some $\epsilon > 0$ if B is replaced by

$$B_0^- := \{(v, w) : v \in (a, b),\ R_0(v) \leq w \leq 0\}. \tag{3.80}$$

Thus, we only need to prove that (3.79) holds if B is replaced by $B^+ := B \setminus B_0^-$. This follows, by a simple continuity argument, if we can prove that (3.79) holds if B is replaced by $\tau(-R_0)$, the graph of $-R_0$. Note that $\tau(-R_0) \cap S_{(b-\epsilon, b)}$ coincides with $\tau(\hat{\phi}_I) \cap S_{(b-\epsilon, b)}$, where $\hat{\phi}_I(x) = \phi_I(-x)$. Pick $c_0 \in (c_I - \delta, c_I)$. Using Lemma 3.20 and the continuity of solutions with respect to parameters and initial data, the following properties are easily established. For each $\bar{c} \in (c_0, c_I)$ the set

$$P_{\bar{c}} := \bigcup_{c \in (c_0, \bar{c})} \tau(\psi^c) \cap \tau(\hat{\phi}_I)$$

is a curve in $\tau(\hat{\phi}_I)$ with the end points at $\tau(\psi^{c_0}) \cap \tau(\hat{\phi}_I)$, $\tau(\psi^{\bar{c}}) \cap \tau(\hat{\phi}_I)$. Moreover, $P_{\bar{c}}$ is increasing in $\bar{c}$ with respect to the set inclusion. Thus, as $\bar{c} \nearrow c_I$, the end point $\tau(\psi^{\bar{c}}) \cap \tau(\hat{\phi}_I)$ approaches a limit (ξ, η), and all we need to prove is that $(\xi, \eta) = (b, 0)$. We show this by contradiction. Assume that $(\xi, \eta) \neq (b, 0)$. Then (ξ, η) is on the trajectory $\tau(\hat{\phi}_I)$. As shown in Lemma 3.27, the trajectory of the solution of (3.2), (3.3) with $c = c_I$ intersects the v-axis at some point $(\xi_0, 0)$ with $\xi_0 < b$. But then, by continuity, for all $c \approx c_I$, $\tau(\psi^c)$ intersects the v-axis at a point near $(\xi_0, 0)$, in contradiction to Lemma 3.20(i). This contradiction completes the proof. $\square$

Proof of Proposition 3.25, Part 1. Here, we prove statement (i) of Proposition 3.25 assuming that $a > 0$. Since $f'(a) \leq 0$ (see (3.40)), Lemmas

3.20, 3.28 apply. Observe that (A3p) holds with $I_0 = I = (a, b)$, $\psi = \psi^c$, $c \in (c_I - \delta, c_I)$, if $\delta \in (0, c_I)$ and ψ^c are as in Lemma 3.20. The conclusion of statement (i) now follows directly from Lemmas 3.26, 3.28. $\square$

Lemma 3.29. *Let $I = (a, b) \in \mathcal{N}^+$. Suppose that $c > 0$ and ψ is a solution of (3.1) such that for some $x_0 < \hat{x} < \bar{x}$ the following relations are satisfied (cp. Figure 8):*

$$\psi(x_0) = 0, \quad \psi'(x) > 0 \ \ (x \in [x_0, \hat{x})), \quad \psi(\hat{x}) < b, \quad \psi'(\hat{x}) = 0, \tag{3.81}$$
$$\psi'(x) < 0 \ \ (x \in (\hat{x}, \bar{x})), \quad \psi(\bar{x}) \in (a, b), \quad \psi'(\bar{x}) = 0. \tag{3.82}$$

Then $a < \psi(x) < b$ for all $x \geq \bar{x}$ and, as $x \to \infty$, one has $(\hat{\psi}(x), \hat{\psi}_x(x)) \to (\zeta, 0)$ with $\zeta \in (a, b)$.

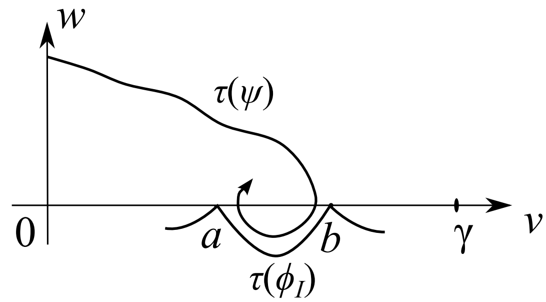

Figure 8: The trajectory of the solution as in Lemma 3.29.

Proof. Relations (3.82) (and the fact that (ψ, ψ_x) is not an equilibrium) imply that $(\psi(x), \psi_x(x))$ crosses the v-axis at the point $(\psi(\bar{x}), 0)$. Once $(\psi(x), \psi_x(x))$ is in the half-plane $\{(v, w) : v > 0\}$, ψ is increasing. Since $\tau(\psi)$ does not have self-intersections, relations (3.81) imply that $(\psi(x), \psi_x(x))$ must cross the v-axis at some point $(q, 0)$ between $(\psi(\bar{x}), 0)$ and $(\psi(\hat{x}), 0)$ and enter the lower half-plane $\{(v, w) : v < 0\}$. Once there, ψ is decreasing. Continuing by similar arguments, one shows easily that the function $x \to (\psi(x), \psi_x(x))$ is bounded on $(\bar{x}, \infty)$, with $\psi(\bar{x}) \leq \psi(x) \leq \psi(\hat{x})$, and that its limit equilibrium $(\zeta, 0)$ is between $(\psi(\bar{x}), 0)$ and $(\psi(\hat{x}), 0)$ (the limit equilibrium exists, as mentioned in Subsection 3.1, since $c > 0$). In particular, we have $\zeta \in (a, b)$. $\square$

Proof of Proposition 3.25, Part 2. Here we assume that $a = 0$. We prove statement (i) of Proposition 3.25 (assuming that $\theta = 0$ is stable from above, hence, in particular, $f'(0) \leq 0$), and also prove statement (iii) under the assumption that $f'(0) = 0$. The proofs share some arguments, so, for now, we just assume that $I = (0, b) \in \mathcal{N}^+$ and $f'(0) \leq 0$ with no further assumption on the stability of 0.

By Lemma 3.26, all points in $(\xi, \eta) \in S_{[0,\gamma]} \setminus B$, with B as in (3.78), are covered by trajectories of type (A2). Thus, we only need to consider points in B. Let $\epsilon > 0$ be as in Lemma 3.28. Fix any point

$$(\xi, \eta) \in B \cap S_{(b-\epsilon,b)} \setminus \tau(R_0). \tag{3.83}$$

By Lemma 3.28, $(\xi, \eta) \in \tau(\psi^{c_0})$ for some $c_0 \in (c_I - \delta, c_I)$. We claim that if $c \in (c_0, c_I)$ is sufficiently close to c_0 and ψ is the solution of (3.1), (3.3), then (A3p) holds if $\theta = 0$ is stable from above and (A3p0) holds if it is unstable from above (we take $I_0 = I$ in (A3p) and (A3p0)).

To prove the claim, we first note, appealing to Lemma 3.20 and the continuity of solutions with respect to c, that relations (3.81) hold for some points $x_0 < \hat{x}$ (depending on c). By Lemma 3.1, as long as $\psi'(x) < 0$, $(\psi(x), \psi'(x))$ stays between $\tau(\phi_I)$ and the v-axis. Therefore, one of the following possibilities occurs:

(a) $\psi_x < 0$ on $(\hat{x}, \infty)$ and, as $x \to \infty$, one has $\psi(x) \to \zeta$ for some $\zeta \in f^{-1}\{0\} \cap [0, b)$;

(b) there is $\bar{x} > \hat{x}$ such that relations (3.82) hold.

If alternative (b) occurs, then, using Lemma 3.29, we conclude that (A3p) holds. If (a) occurs, then (A3p) holds if we can verify that $\zeta > 0$. We now show that this is indeed the case if $\theta = 0$ is stable from above for the equation $\dot{\theta} = f(\theta)$. Indeed, under this stability assumption, statements (pi), (pii) of Lemma 3.2 imply that if (a) holds with $\zeta = 0$, then up to a translation, $\psi = \psi^c$, or, in other words, $\tau(\psi) = \tau(\psi^c)$. This is impossible, however, as the trajectories $\tau(\psi^c)$ and $\tau(\psi^{c_0})$ are disjoint (see Lemma 3.20(ii)) and $(\xi, \eta) \in \tau(\psi) \cap \tau(\psi^{c_0})$. Thus, under the stability assumption we are done and the proof of statement (i) of Proposition 3.25 is complete.

We now assume that $\theta = 0$ is unstable from above for the equation $\dot{\theta} = f(\theta)$ (hence, our assumption $f'(0) \geq 0$ now reduces to $f'(0) = 0$). In this case, we first show that alternative (a) holds if $c \in (c_0, c_I)$ is close enough to c_0. Indeed, the instability assumption implies that $f > 0$ in the interval $(0, d)$ for some $d > 0$. By continuity, if $c > c_0$ is close enough to c_0, then there is $\tilde{x}$ such that $\psi(\tilde{x}) \in (0, d)$, $\psi'(\tilde{x}) < 0$. Thus, for the alternative (b) to occur, $(\psi(x), \psi'(x))$ would have to cross the v-axis somewhere on the segment $(0, d) \times \{0\}$. However, due to $f > 0$, the direction of the vector field on this segment makes this impossible.

Thus, alternative (a) holds, as claimed. By the uniqueness statement of Lemma 3.2(pi), if ψ has the asymptotics (3.9) (with $\zeta = 0$), then necessarily $\tau(\psi) = \tau(\psi^c)$. However, as already noted above, this is impossible by Lemma 3.20(ii). Thus, by Lemma 3.2(pii), ψ must have the asymptotics (3.10) (which is the same as (3.16) when $f'(0) = 0$). Also, since $c_I > c$ and $f'(0) = 0$, $\lambda^+(c) = 0 > \lambda^-(c_I)$. We conclude that (A3p0) holds. This completes the proof of statement (iii) in the case $f'(0) = 0$. □

Proof of Proposition 3.25, Part 3. If remains to complete the proof of statement (iii) of Proposition 3.25 in the case $f'(0) > 0$.

Recall that, since the solution $\phi_I(x)$ converges to 0 and is decreasing, the eigenvalues $\lambda^\pm(c_I)$ are real, that is, $c_I \geq 2\sqrt{f'(0))}$. We will be working with $c > c_I$ this time (thus achieving (A3p) is no longer an option) and want to show that (A3p0) holds. Note that

$$\lambda^-(c) < \lambda^-(c_I) \tag{3.84}$$

for $c > c_I$ (see (3.69)) and that Lemma 3.4 with $\zeta = 0$ applies.

Fix a positive number d such that $f > 0$ in $(0, d)$.

Let ν and μ be as in Lemmas 3.24 and 3.27, respectively. We will use the following claim. There is $\epsilon > 0$ with $\epsilon < \min\{\nu, \mu\}$ such that if (ξ, η) is as in (3.83), then the solution of (3.1), (3.3) with $c = c_I$ satisfies $\psi(x) \searrow 0$ as $x \to \infty$.

Suppose for a while the claim is true. We prove that statement (iii) holds for such ϵ. As in Part 2 of the proof, we only need to consider points (ξ, η) satisfying (3.83). Fix any such point. Using Lemmas 3.24(i) and 3.27 for $c = c_I$, and then the continuity with respect to c, we obtain that for each $c > c_I$, $c \approx c_I$ the solution ψ of (3.1), (3.3) satisfies relations (3.81) for some $x_0 < \hat{x}$ (depending on c). Moreover, $(\psi(x), \psi'(x)) \in B$ for all $x \geq \hat{x}$ (see Lemma 3.27). Therefore, using the claim and the continuity with respect to c, we obtain that if $c > c_I$ is sufficiently close to c_I, then $(\psi(x), \psi'(x))$ enters the "wedge"

$$V := \{((r, s) \in \mathbb{R}^2 : 0 < r < d,\ R_0(r) < s < 0\}. \tag{3.85}$$

The invariance of B and the direction on the vector field on the segment $(0, d) \times \{0\}$ then implies that $(\psi(x), \psi'(x))$ stays in V for all large x and approaches $(0, 0)$ as $x \to \infty$.

Now, $\psi(x)$ cannot have the asymptotics (3.9) (with $\zeta = 0$). Otherwise, $(\psi(x), \psi_x(x))$ could not stay in V (above $\tau(\phi_I)$) for all large x because of the asymptotics of ϕ_I (see Lemma 3.15) and the relation (3.84) (cp. Remark 3.6(i)). Therefore, by Lemma 3.4, $\psi(x)$ has the asymptotics (3.16). This shows that relations (3.55) are all satisfied. Moreover, by (3.69), we have $\lambda^-(c_I) \le \lambda^+(c_I) < \lambda^+(c)$, hence all conditions in (A3p0) are satisfied.

It remains to prove the claim. Let V be as in (3.85). Set $\xi_0 := \min\{\nu, \mu\}/2$. If $\eta_0 \in (R_0(\xi_0), 0)$ is sufficiently close to $R_0(\xi_0)$ (so (ξ_0, η_0) is close to $\tau(\phi_I)$), then the solution $(\psi_0(x), \psi_0'(x))$ of (3.2) with $c = c_I$ and $(\psi_0(0), \psi_0'(0)) = (\xi_0, \eta_0)$ enters the wedge V. Then it stays in V for all large x and approaches $(0, 0)$ as $x \to \infty$. This follows, as above, from the invariance of B and the direction on the vector field on the segment $(0, d) \times \{0\}$. Since $(\xi_0, \eta_0) \in S_{(b-\nu, b)} \cap B$, it follows from Lemma 3.24 that the trajectory $\tau(\psi_0)$ intersects $\tau(\hat{\phi}_I)$. Consider now the region B_1 in B whose boundary consists of pieces of the trajectories $\tau(\hat{\phi}_I)$, $\tau(\psi_0)$, $\tau(\phi_I)$, and the point $(b, 0)$ (see Figure 9). Clearly, B_1 contains the set $B \cap S_{(b-\epsilon, b)} \setminus \tau(R_0)$ for some $\epsilon \in (0, \xi_0)$. For any $(\xi, \eta) \in B \cap S_{(b-\epsilon, b)} \setminus \tau(R_0)$, let ψ be the solution of (3.2), (3.3) with $c = c_I$. By Lemma 3.27, one has $\psi'(x) \le 0$ for some x. Since the trajectories $\tau(\psi_0)$, $\tau(\psi)$, and $\tau(\phi_I)$ of (3.2) with $c = c_I$ cannot not intersect, we have $\psi'(x) < 0$ for large x and $(\psi(x), \psi'(x)) \to (0, 0)$ as $x \to \infty$. The claim is proved. $\square$

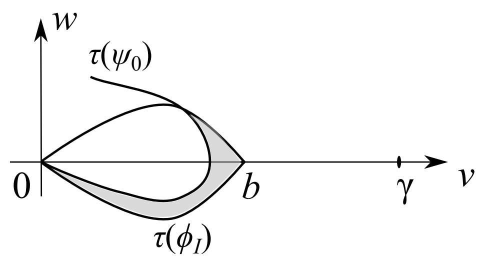

Figure 9: The shaded region depicts the region in the set B, as described in Part 3 of the proof of Proposition 3.25.

Remark 3.30. (i) Let us summarize how solutions ψ^c from Lemma 3.20 were used in the proof of Proposition 3.25(i) in order to find pairs (c, ψ) satisfying (A3p) with $I_0 = I$. For $c < c_I$, $c \approx c_I$, we took ψ either equal to ψ^c (Part 1 of the proof of Proposition 3.25 dealing with the case $a > 0$) or a small perturbation of ψ^c (Part 2 of the proof concerning the case $a = 0$ when it is stable from above). Since $\psi^c(\infty) = a$ and $\psi^c(0) \nearrow b$ as $c \nearrow c_I$, we can choose $c < c_I$ and ψ satisfying (A3p) and the following relations with any desired proximities

$$\psi(0) \approx b, \quad \psi(x) \approx a \text{ for some } x > 0. \tag{3.86}$$

This observation will be useful below.

(ii) Similarly, in the proof of Proposition 3.25(iii), we have found pairs (c, ψ), with $c > c_I$ or $c < c_I$, such that $\psi(0) \in (b - \epsilon, b)$ and (A3p0) holds with $I_0 = I$ (see Part 2 of the proof for the case $f'(0) = 0$ and 0 unstable from above, and Part 3 for the case $f'(0) > 0$). Of course, taking $\epsilon > 0$ small, we achieve $\psi(0) \approx b$ with any desired proximity (and we have $\psi(\infty) = 0$ by (A3p0)).

The solutions in the following lemma (see also Figure 10) will be useful in estimating the speed of propagation of solutions of (1.1).

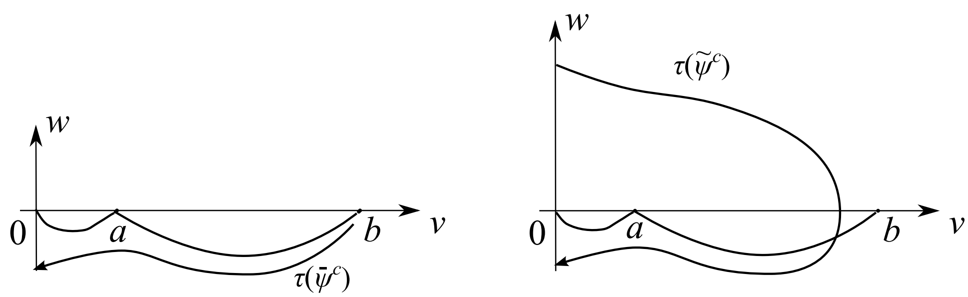

Figure 10: The trajectories of the solutions in Lemma 3.31

Lemma 3.31. *Let $I = (a, b) \in \mathcal{N}$. The following statements hold.*

(i) *For each $c < c_I$, there is a solution $\bar{\psi}^c$ of (3.1) such that for some $x_0 = x_0(c)$ one has*

$$\begin{aligned} &\bar{\psi}^c(x_0) = 0; \\ &\bar{\psi}^c_x < 0 \text{ on } (-\infty, x_0]; \\ &\lim_{x \to -\infty} \bar{\psi}^c(x) = b. \end{aligned} \tag{3.87}$$

(ii) *If $c_I > 0$, then for each $c \in (0, c_I)$ there is a solution $\tilde{\psi}^c$ of* (3.1) *such that for some $x_1 = x_1(c)$, $x_2 = x_2(c)$ with $x_1 < 0 < x_2$ one has*

$$\begin{aligned} &\tilde{\psi}^c(x_1) = \tilde{\psi}^c(x_2) = 0; \\ &\tilde{\psi}^c_x > 0 \text{ on } [x_1, 0); \quad \tilde{\psi}^c_x < 0 \text{ on } (0, x_2]; \\ &\tilde{\psi}^c(0) < b; \text{ and } \tilde{\psi}^c(0) \to b \text{ as } c \nearrow c_I. \end{aligned} \tag{3.88}$$

Proof. (i) We apply Lemma 3.8 with $c_0 = c_I$, $v = \phi_I$. Given $c < c_I$, we take $\bar{\psi}^c = \psi$, where ψ is as in Lemma 3.8(i). Using relation (3.26) and Lemma 3.17(ii), we find $x_0 = x_0(c)$ that $\bar{\psi}^c$ satisfies (3.87).

(ii) Assume that $c_I > 0$ and let ν be as in Lemma 3.24(i). Given $c \in (0, c_I)$, choose $\xi = \xi(c) \in (b - \nu, b)$ with $b - \xi < c_I - c$ (so that $\xi(c) \to b$ as $c \nearrow c_I$). Set $\eta := R_0(\xi) < 0$, so that the point (ξ, η) is on the trajectory $\tau(\phi_I)$, and let ψ be the solution of (3.1), (3.3). Using first Lemma 3.1 and then Lemma 3.17(ii), we find $\bar{x}_2$ such that $\psi'(x) < 0$ on $[0, \bar{x}_2]$ and $\psi(\bar{x}_2) = 0$. Next, for $s < 0$ sufficiently close to 0 we have $\psi' < 0$ on $[s, 0]$, and, by Lemma 3.1, $R_0(\psi(s)) < \psi'(s)$. Therefore, Lemma 3.24(i) applies to the initial condition $(\psi(s), \psi'(s))$. Consequently, replacing ψ by its translation so that its maximum point is placed at $x = 0$, we obtain a solution $\tilde{\psi}^c$ with all the desired properties. □

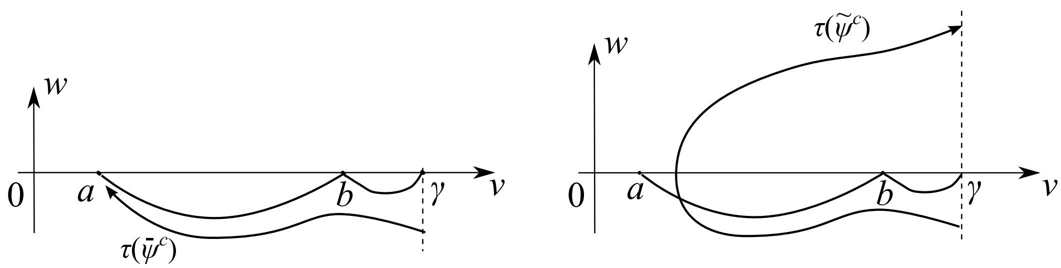

Figure 11: The trajectories of the solutions in Lemma 3.32

Here are analogous results, without proof, for the left-end point (see Figure 11).

Lemma 3.32. *Let $I = (a, b) \in \mathcal{N}$. The following statements hold.*

(i) *For each $c > c_I$, there is a solution $\bar{\psi}^c$ of* (3.1) *such that for some $x_0 = x_0(c)$ one has*

$$\begin{aligned} &\bar{\psi}^c(x_0) = \gamma; \\ &\bar{\psi}^c_x < 0 \text{ on } [x_0(c), \infty); \\ &\lim_{x\to\infty} \bar{\psi}^c(x) = a. \end{aligned} \tag{3.89}$$

(ii) *If $c_I < 0$, then for each $c \in (c_I, 0)$ there is a solution $\tilde{\psi}^c$ of* (3.1) *such that for some $x_1 = x_1(c)$, $x_2 = x_2(c)$ with $x_1 < 0 < x_2$ one has*

$$\begin{aligned} &\tilde{\psi}^c(x_1) = \tilde{\psi}^c(x_2) = \gamma; \\ &\tilde{\psi}^c_x < 0 \text{ on } [x_1, 0); \quad \tilde{\psi}^c_x > 0 \text{ on } (0, x_2]; \\ &\tilde{\psi}^c(0) > a; \text{ and } \tilde{\psi}^c(0) \to a \text{ as } c \searrow c_I. \end{aligned} \tag{3.90}$$

4 Proofs of Propositions 2.8, 2.12

Proof of Proposition 2.8. By statements (ii), (iii) of Proposition 3.11 and Lemma 3.7(iii), one has $I = (a, b) \in \mathcal{N}^0$ if and only if a, b are global maximizers of F in $[0, \gamma]$ and $F < F(a) = F(b)$ in (a, b). This clearly implies the first statement of Proposition 2.8.

Now, if $\mathcal{N}^+ \neq \emptyset$, then Proposition 3.11(ii) implies that 0 is not a maximizer of F in $[0, \gamma]$. Conversely, if 0 is not a global maximizer of F in $[0, \gamma]$, then, in the notation of Proposition 3.11, we have $0 < \gamma_* \leq \gamma^* \leq \gamma$. This and Proposition 3.11(iv) imply that the set $\mathcal{N}$ contains an interval $I = (0, b)$ for some $0 < b \leq \gamma_*$. By Proposition 3.11(iii), $c_I > 0$, that is $\mathcal{N}^+ \neq \emptyset$. The proof of the statement regarding $\mathcal{N}^-$ is analogous. □

Proof of Proposition 2.12. Statement (i) of Proposition 2.12 is a part of Proposition 3.11(v). We prove the remaining two statements.

Assume, as in statement (ii), that $F(u)$ has only finitely many maximizers in $[0, \gamma]$ and all of them are isolated zeros of f in $[0, \gamma]$. By Proposition 3.11(ii), γ_* is the smallest and γ^* the largest of these maximizers, and the set $R_0^{-1}\{0\} \cap [\gamma_*, \gamma^*]$ is finite. Further, since γ_*, γ^* are not accumulations points of $R_0^{-1}\{0\} \subset f^{-1}\{0\}$, statements (ii) and (iv) of Proposition 3.11 imply that the whole set $R_0^{-1}\{0\}$ is finite. If, moreover, ξ_{max} is the unique global maximizer of F in $[0, \gamma]$, then $\xi_{max} = \gamma_* = \gamma^*$. Therefore, according to Proposition 3.11(iii), we have $c_I \neq 0$ for each $I \in \mathcal{N}$. This completes the proof of statement (ii).

To prove statement (iii), assume that condition (DGM) from the introduction is satisfied. Then, in particular, γ is asymptotically stable from below as an equilibrium of the equation $\dot{\theta} = f(\theta)$. Indeed, by the comparison principle, its domain of attraction contains the interval $(\max \bar{u}_0, \gamma]$. Of course, γ is necessarily an isolated critical point of F in $[0, \gamma]$. Therefore, there is $a \in [0, \gamma)$ such that $I := (a, \gamma) \in \mathcal{N}$. We claim that $c_I > 0$. Indeed, assume to the contrary that $c_I \leq 0$. Consider the profile function ϕ_I. It has the limits $\phi_I(-\infty) = \gamma$, $\phi_I(\infty) = a$. Since $\bar{u}_0$ has compact support and $\bar{u}_0 < \gamma$, there is $x_0 > 0$ such that

$$\bar{u}_0(x) < \phi_I(x - x_0) \quad (x \in \mathbb{R}). \tag{4.1}$$

Take now the traveling front $U(x,t) = \phi_I(x - c_I t - x_0)$. By (4.1) and the comparison principle, for the solution of $\bar{u}$ of (1.1) with the initial condition $\bar{u}(\cdot, 0) = \bar{u}_0$ we have

$$\bar{u}(x,t) \leq U(x,t) \quad (x \in \mathbb{R},\, t > 0).$$

Using $c_I \leq 0$ and $\phi_I(\infty) = a$, we obtain from this that $\limsup_{t\to\infty} \bar{u}(x,t) \leq a$, in contradiction to (DGM).

Thus $c_I > 0$ as claimed. Proposition 3.11 now implies that $\gamma_* = \gamma^* = \gamma$. As remarked above, this means that γ is the unique global maximizer of F in $[0, \gamma]$. This completes the proof of statement (iii).

□

5 Preliminaries on the limit sets and zero number

In this section we recall several results concerning the Ω-limit sets and the zero number functional.

5.1 Properties of $\Omega(u)$

Consider the Cauchy problem

$$u_t = u_{xx} + c u_x + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{5.1}$$
$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{5.2}$$

where $f \in C^1(\mathbb{R})$, $c \in \mathbb{R}$, and $u_0 \in C(\mathbb{R})$. The Ω-limit set of a bounded solution u is defined as in (1.8) and denoted by $\Omega(u)$ or $\Omega(u_0)$. It will be useful to remember that if u is a bounded solution of (1.1), then the function $\tilde{u}(x,t) := u(x+ct,t)$ is a bounded solution of (5.1). Clearly, u and $\tilde{u}$ have the same initial value at $t = 0$ and $\Omega(u) = \Omega(\tilde{u})$. In other words, if u_0 is given, then $\Omega(u_0)$ is independent of the choice of c in the problem (5.1), (5.2).

Assume that the solution u of (5.1), (5.2) is bounded. Then, the usual parabolic regularity estimates imply that the derivatives u_t, u_x, u_{xx} are bounded on $\mathbb{R} \times [1,\infty)$ and they are globally α-Hölder on this set for each $\alpha \in (0,1)$. The following results are standard consequences of this regularity property: $\Omega(u_0)$ is a nonempty, compact, connected subset of $L^\infty_{loc}(\mathbb{R})$. Moreover, in (1.8) one can take the convergence in $C^1_{loc}(\mathbb{R})$, and $\Omega(u_0)$ is compact and connected in that space as well. The latter implies that the set

$$K_\Omega(u) := \{(\varphi(x), \varphi_x(x)) : \varphi \in \Omega(u_0),\ x \in \mathbb{R}\}$$

is connected in $\mathbb{R}^2$.

We now recall the invariance property of $\Omega(u_0)$. Let $\varphi \in \Omega(u)$, so that $u(x_n+\cdot, t_n) \to \varphi$ for some sequence $\{(x_n, t_n)\}$ with $t_n \to \infty$. Then, passing to a subsequence if necessary, one shows easily that the sequence $u(x_n+\cdot, t_n+\cdot)$ converges in $C^1_{loc}(\mathbb{R}^2)$ to a function U which is an entire solution of (5.1) (that is, a solution of (5.1) on $\mathbb{R}^2$). Obviously, $U(\cdot, 0) = \varphi$.

Finally, we note that $\Omega(u_0)$ is also translation-invariant: with each $\varphi \in \Omega(u_0)$, $\Omega(u_0)$ contains the whole translation group orbit of φ, $\{\varphi(\cdot + \xi) : \xi \in \mathbb{R}\}$. This follows directly from the definition of $\Omega(u_0)$. Combining the translation invariance with the compactness of $\Omega(u_0)$, we get that the set $K_\Omega(u)$ is compact in $\mathbb{R}^2$.

5.2 Zero number

Here we consider solutions of the linear equation

$$v_t = v_{xx} + cv_x + a(x,t)v, \quad x \in \mathbb{R},\ t \in (s,T), \tag{5.3}$$

where $-\infty < s < T \leq \infty$, a is a bounded measurable function on $\mathbb{R} \times [s,T)$, and c is a constant. In the next section we use the following fact, often without notice. If u, $\bar{u}$ are bounded solutions of the nonlinear equation (5.1) with a Lipschitz nonlinearity, then their difference $v = u - \bar{u}$ satisfies a linear equation (5.3).

We denote by $z(v(\cdot,t))$ the number, possibly infinite, of the zero points $x \in \mathbb{R}$ of the function $x \to v(x,t)$. The following intersection-comparison principle holds (see [1, 6]).

Lemma 5.1. *Let $v \in C(\mathbb{R} \times [s,T))$ be a nontrivial solution of* (5.3) *on $\mathbb{R} \times (s,T)$. Then the following statements hold true:*

(i) *For each $t \in (s,T)$, all zeros of $v(\cdot,t)$ are isolated.*

(ii) *$t \mapsto z(v(\cdot,t))$ is a monotone nonincreasing function on $[s,T)$ with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.*

(iii) *If for some $t_0 \in (s,T)$, the function $v(\cdot,t_0)$ has a multiple zero and $z(v(\cdot,t_0)) < \infty$, then for any $t_1, t_2 \in (s,T)$ with $t_1 < t_0 < t_2$ one has*

$$z(v(\cdot,t_1)) > z(v(\cdot,t_2)). \tag{5.4}$$

If (5.4) holds, we say that $z(v(\cdot,t))$ drops in the interval (t_1,t_2).

Remark 5.2. The previous lemma clearly implies that if $z(v(\cdot,s_0)) < \infty$ for some $s_0 \in (s,T)$, then $z(v(\cdot,t))$ can drop at most finitely many times in (s_0,T), and if it is constant on (s_0,T), then $v(\cdot,t)$ has only simple zeros for each $t \in (s_0,T)$.

Corollary 5.3. *Assume that v is a solution of* (5.3) *such that for some $s_0 \in (s,T)$ one has*

$$\liminf_{|x|\to\infty} |v(x,s_0)| > 0. \tag{5.5}$$

Then there is $t_0 \in (s,T)$ such that for $t \in [t_0,T)$ the function $v(\cdot,t)$ has only simple zeros, and their number is finite $z(v(\cdot,t))$ and independent of t.

Proof. Since the zeros of $v(\cdot,s_0)$ are isolated, (5.5) implies that there is only a finite number of them. The conclusion now follows directly from Lemma 5.1 and Remark 5.2. □

We shall also use the following property related to the monotonicity of the zero number:

Lemma 5.4. *Let v be a solution of* (5.3), *and $s < t_1 < t_2 < T$. Assume that $z_0 \in \mathbb{R}$ is a zero of $v(\cdot,t_2) = 0$. Then there is a continuous function σ on $[t_1,t_2]$ such that $u(\sigma(t),t) = 0$ for all $t \in [t_1,t_2]$ and $\sigma(t_2) = z_0$.*

For the proof see [10, Section 2].

The next lemma shows that the property for a solution to have multiple zeros is robust.

Lemma 5.5. *Assume that v is a nontrivial solution of* (5.3) *such that for some $s_0 \in (s, T)$ the function $v(\cdot, s_0)$ has a multiple zero at some x_0, that is, $v(x_0, s_0) = v_x(x_0, s_0) = 0$. Assume further that for some $\delta, \epsilon > 0$, v_n is a sequence in $C^1([x_0 - \delta, x_0 + \delta] \times [s_0 - \epsilon, s_0 + \epsilon])$ which converges in this space to the function v. Then for all sufficiently large n the function $v_n(\cdot, t)$ has a multiple zero in $(x_0 - \delta, x_0 + \delta)$ for some $t \in (s_0 - \epsilon, s_0 + \epsilon)$.*

This can be proved using a version of Lemma 5.1 on a small interval around x_0 and the implicit function theorem, see [7, Lemma 2.6] for details. Note that the v_n are not required to be solutions of any equation.

6 Proofs of the main theorems

Throughout this section we assume that the standing hypotheses (H) are satisfied. In addition, we make the following hypothesis:

$$f > 0 \text{ in } (-\infty, 0); \qquad f < 0 \text{ in } (\gamma, \infty). \tag{6.1}$$

Carefully note that this extra assumption is at no cost to generality. Indeed, in all our theorems, we assume that the initial datum u_0 of the solution considered satisfies, at the least, conditions (2.2), (2.3). This means that $\eta^- \le u_0 \le \eta^+$ for some $\eta^- \le 0$, $\eta^+ \ge \gamma$, with the property that if $\eta^- < 0$, then $f > 0$ in $[\eta^-, 0)$ (that is, η^- is in the domain of attraction of 0), and if $\eta^+ > \gamma$, then $f < 0$ in $(\gamma, \eta^+]$. By the comparison principle, $\eta^- \le u(x,t) \le \eta^+$ for all $x \in \mathbb{R}$ and $t \ge 0$. Hence, we can modify f outside the interval $[\eta^-, \eta^+]$ containing the range of the solution so as to achieve (6.1). In some cases—for example, if $f(u) \ge 0$ for $u < 0$, $u \approx 0$—after such a modification f may not be of class C^1 in a full neighborhood of 0, even if the original nonlinearity was, but this is of no concern (cp. hypothesis (H1)). We remark that all results from Section 3 that will be used in this section only concern the behavior of solutions of (3.1) while they stay in $[0, \gamma]$ (see relations (3.88)-(3.89) and statements (A1)-(A3p0) in Subsection 3.3). These, of course, are unaffected by any modification of f outside $[0, \gamma]$ made here or in Section 3.

The advantage we gain from assumption (6.1) is that we can now assume a certain behavior of solutions (3.2) once they leave the strip $S_{[0,\gamma]} = \{(v, w) :$

$v \in [0, \gamma]\}$. This will simplify the exposition slightly. Specifically, assumption (6.1) implies that for any $c \in \mathbb{R}$, the quadrants

$$Q_1 := \{(v, w) : v \geq \gamma, w \geq 0\} \setminus \{(\gamma, 0)\},$$
$$Q_3 := \{(v, w) : v \leq 0, w \leq 0\} \setminus \{(0, 0)\}$$

are positively invariant for system (3.2) in the sense that if a solution satisfies $(v(0), w(0)) \in Q_i$, for $i = 1$ or $i = 3$, then for all $x > 0$ one has $(v(x), w(x)) \in \operatorname{Int} Q_i$ (the interior of Q_i). Similarly, the quadrants

$$Q_2 := \{(v, w) : v \leq 0, w \geq 0\} \setminus \{(0, 0)\},$$
$$Q_4 := \{(v, w) : v \geq \gamma, w \leq 0\} \setminus \{(\gamma, 0)\}$$

are negatively invariant. From these properties, we get the following information on the solutions ψ as in (A2), (A3p), (A3n), or (A3p0) (see Section 3.3).

$$\text{if (A2) holds, then } |\psi'| > 0 \text{ on } \mathbb{R}; \tag{6.2}$$
$$\text{if (A3p) or (A3p0) holds, then } \psi' > 0 \text{ on } (-\infty, x_0]; \tag{6.3}$$
$$\text{if (A2n) holds, then } \psi' > 0 \text{ on } [x_0, \infty). \tag{6.4}$$

Also, if $\bar{\psi}^c$, $\tilde{\psi}^c$ are as in Lemmas 3.31, 3.32, then

$$\bar{\psi}^c_x(x) < 0 \quad (x \in \mathbb{R}), \qquad |\tilde{\psi}^c_x(x)| > 0 \quad (x \in \mathbb{R} \setminus \{0\}). \tag{6.5}$$

In the remainder of this section we assume that u is a solution of (1.1), (1.2) with the initial datum satisfying conditions (2.2), (2.3). Also, R_0 stands for the minimal $[0, \gamma]$-system of waves, $\tau(R_0)$ for its graph, and $\{c_I : I \in \mathcal{N}\}$, $\{\phi_I : I \in \mathcal{N}\}$ for the corresponding families of speeds and profile functions.

6.1 Some estimates: behavior at $x = \pm\infty$ and propagation

Lemma 6.1. *One has*

$$\lim_{t\to\infty} (\liminf_{x\to-\infty} u(x, t)) = \lim_{t\to\infty} (\sup_{x\in\mathbb{R}} u(x, t)) = \gamma, \quad \lim_{t\to\infty} (\limsup_{x\to-\infty} |u_x(x, t)|) = 0; \tag{6.6}$$

$$\lim_{t\to\infty} (\limsup_{x\to\infty} u(x, t)) = \lim_{t\to\infty} (\inf_{x\in\mathbb{R}} u(x, t)) = 0, \quad \lim_{t\to\infty} (\limsup_{x\to\infty} |u_x(x, t)|) = 0. \tag{6.7}$$

Proof. We prove (6.6) and omit the proof of (6.7), which is completely analogous. It is sufficient to prove the first two relations in (6.6), the second one then follows by standard parabolic regularity estimates for the function $\gamma - u$ (which solves a linear equation (5.3)).

One easily finds a nonincreasing continuous function $\bar{u}_0$ satisfying conditions (2.2), (2.3) such that $\bar{u}_0 \leq u_0$. The solution of (1.1) with the initial datum $\bar{u}_0$ is denoted by $\bar{u}$. By the comparison principle, $\bar{u} \leq u$ and $\bar{u}(x,t)$ is nondecreasing in x for each $t \geq 0$. Therefore, the limit $\rho(t) := \lim_{y \to -\infty} \bar{u}(y,t)$ exists for each $t \geq 0$. The function ρ is continuous on $[0,\infty)$ and it solves the equation $\dot{\rho} = f(\rho)$ on $(0,\infty)$ (see, for example, [33, Theorem 5.5.2]). Since $\bar{u}_0$ satisfies (2.2), we have $\rho(0) \in D_\gamma$. Therefore, $\rho(t) \to \gamma$, as $t \to \infty$. This yields a lower estimate. For an upper estimate, we let $\zeta_0 := \sup u_0 \in D_\gamma$ and take the solution $\dot{\theta} = f(\theta)$ with $\theta(0) = \zeta_0$. Then $\theta(t) \geq u(\cdot,t)$ for all $t > 0$, and $\theta(t) \to \gamma$, as $t \to \infty$. Combining these estimates we obtain (6.6). $\square$

Relations (6.6), (6.7), and the definition of $\Omega(u)$ immediately give the following.

Corollary 6.2. *The constant steady states* 0 *and* γ *are elements of* $\Omega(u)$*, and* $0 \leq \varphi \leq \gamma$ *for each* $\varphi \in \Omega(u)$*.*

The next lemma is an estimate of the speed of propagation of the solution to or above a constant b, and decay to or below another constant a.

Lemma 6.3. (i) *Assume that* $I = (a,b) \in \mathcal{N}^+$*. Then for every* $c < c_I$ *and* $x_0 \in \mathbb{R}$ *one has*

$$\liminf_{x \leq x_0,\, t \to \infty} u(x + ct, t) \geq b. \tag{6.8}$$

(ii) *Assume that* $I = (a,b) \in \mathcal{N}^-$*. Then for every* $c > c_I$ *and* $x_0 \in \mathbb{R}$ *one has*

$$\limsup_{x \geq x_0,\, t \to \infty} u(x + ct, t) \leq a. \tag{6.9}$$

Proof. We prove statement (i), the proof of (ii) is analogous. Since (6.8) is a lower estimate, it is sufficient to prove it for nonincreasing functions u_0. (In the general case, the result then follows from the comparison principle, upon taking a nonincreasing continuous function $\bar{u}_0 \leq u_0$ satisfying conditions (2.2), (2.3) and $\bar{u}_0 \leq u_0$). Thus, we may continue assuming that $u(\cdot,t)$ is nonincreasing for each $t \geq 0$.

Take a sequence $c_n \nearrow c_I$ and let $\psi_n := \tilde{\psi}^{c_n}$ be as in Lemma 3.31(ii). Given any $\epsilon > 0$, we can choose n such that $c_n > c$ and $\max \psi_n = \psi_n(0) \in (b - \epsilon, b)$. Since ψ_n is negative outside a compact interval (see Lemma 3.31 and (6.5)), using (6.6), (6.7) we find positive constants t_0 and y_0 such that

$$u(x + c_n t_0, t_0) > \psi_n(x - y_0) \quad (x \in \mathbb{R}).$$

Now, the functions $u(x + c_n t, t)$ and $\psi_n(x - y_0)$ satisfy the same equation, equation (5.1) with $c = c_n$. The comparison principle therefore gives

$$u(x + c_n t, t) > \psi_n(x - y_0) \quad (x \in \mathbb{R},\ t \geq t_0).$$

Using the monotonicity of $u(\cdot, t)$, we in particular obtain

$$u(x + ct, t) \geq u(y_0 + c_n t, t) > \psi_n(0) > b - \epsilon \quad (x \leq y_0 + (c_n - c)t,\ t \geq t_0).$$

Since $c_n > c$ and ϵ can be taken arbitrarily small, it is clear that (6.8) holds for any x_0. $\square$

In the case when 0 is unstable from above for the equation $\dot{\theta} = f(\theta)$, the following estimate will be useful.

Lemma 6.4. *If u_0 satisfies the additional hypotheses $u_0 \geq 0$ and $u_0 \equiv 0$ on some interval $[m, \infty)$, then for all (finite) $t_1 > t_0 > 0$ and $\mu > 0$, there is a constant $\kappa > 0$ such that*

$$u(x, t) \leq \kappa e^{-\mu x} \quad (x \in \mathbb{R},\ t \in [0, t_1]), \tag{6.10}$$

$$|u_x(x, t)| \leq \kappa e^{-\mu x} \quad (x \in \mathbb{R},\ t \in [t_0, t_1]). \tag{6.11}$$

Proof. It is sufficient to prove the estimate for u; (6.11) then follows from parabolic regularity estimates (possibly, after making κ larger).

The assumption $u_0 \geq 0$ and the comparison principle imply that $u \geq 0$. Since u is bounded and $f(0) = 0$, we have $|f(u(x, t))| \leq M u(x, t)$ for some constant $M \geq 0$. Therefore, by comparison,

$$u(x, t) \leq e^{Mt} v(x, t), \tag{6.12}$$

where v is the solution of $v_t = v_{xx}$ with $v(\cdot, 0) = u_0$. Since $u_0 \equiv 0$ on (m, ∞), for each $x \in \mathbb{R}$ and $t > 0$ one has

$$0 \leq v(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{m} e^{-\frac{|x-s|^2}{4t}} u_0(s)\, ds.$$

The substitution $r = (s-x)/\sqrt{4t}$ then yields

$$0 \leq v(x,t) \leq \frac{\|u_0\|_{L^\infty(\mathbb{R})}}{\sqrt{\pi}} \int_{-\infty}^{\frac{m-x}{\sqrt{4t}}} e^{-r^2}\, dr.$$

Using this, (6.12), and some elementary considerations, one shows that (6.10) holds if κ is sufficiently large. $\square$

6.2 A key lemma: no intersection of spatial trajectories

A key step in the proof of our theorems is the next lemma. It says that asymptotically, as $t \to \infty$, the spatial trajectory $\tau(u(\cdot,t))$ has to become disjoint from the trajectory $\tau(\psi)$ of any solution ψ of (3.1) satisfying one of the statements (A1)-(A3n) listed in Section 3.3.

Lemma 6.5. *Let ψ be a solution of* (3.1) *for some $c \in \mathbb{R}$. Assume that one of the statements* (A1)-(A3n) *holds. Then for each $\varphi \in \Omega(u)$ one has*

$$\tau(\varphi) \cap \tau(\psi) = \emptyset. \tag{6.13}$$

Proof. For the whole proof, we fix ψ and c satisfying one of the statements (A1)-(A3n) .

First of all we show that $\psi \notin \Omega(u)$. In view of Corollary 6.2, this is obvious if ψ is of type (A2)-(A3n), for in this case ψ assumes either negative values or values greater than γ. Assume now that ψ is a nonconstant periodic solution of (3.1) (in particular, $c = 0$). Then there is an equilibrium $(\alpha, 0)$ of (3.1) inside the closed curve $\tau(\psi)$. Clearly, the function $\psi - \alpha$ has infinitely many zeros, all of them simple, due to the uniqueness for the Cauchy problem for (3.1). Also, the assumption $0 < \psi < \gamma$ gives $0 < \alpha < \gamma$. If $\psi \in \Omega(u)$, then there are sequences $x_n \in \mathbb{R}$ and $t_n \to \infty$ such that $u(\cdot + x_n, t_n) - \alpha \to \psi - \alpha$ in $L^\infty_{loc}(\mathbb{R})$. This implies that $\lim z(u(\cdot, t_n) - \alpha) \to \infty$. However, Lemma 6.1 implies that $|(u(x,t) - \alpha)| > 0$ if $|x|$ and t are large enough. Therefore, by Corollary (5.3), $z(u(\cdot,t) - \alpha)$ is finite and independent of t if t is large enough. This contradiction shows that $\psi \in \Omega(u)$ cannot hold. Thus, $\psi \notin \Omega(u)$ is proved in all cases.

We now prove (6.13) in the cases (A1)-(A3p); the proof in the case (A3n) is omitted as it is analogous to the proof for (A3p).

We argue by contradiction. Assume that for some $\varphi \in \Omega(u)$ one has $\tau(\varphi) \cap \tau(\psi) \neq \emptyset$. This means that there is y_0 such that the function $\varphi - \psi(\cdot - y_0)$ has a multiple zero. Replacing φ by a translation (which is still an element of $\Omega(u)$) and ψ by a translation (which is also a solution of (3.1) satisfying the same condition from (A1)-(A3p) as ψ), we may assume without loss of generality that

$$x = 0 \text{ is a multiple zero of } \varphi(x) - \psi(x). \tag{6.14}$$

Let now $\tilde{u}(x,t) = u(x + ct, t)$, so that $\tilde{u}$ and ψ satisfy the same equation (5.1). Recalling that $\Omega(\tilde{u}) = \Omega(u)$ (see Sect. 5.1), we choose sequences $x_n \in \mathbb{R}$ and $t_n \to \infty$ such that $\tilde{u}(\cdot + x_n, \cdot + t_n) \to U$ in $C^1_{loc}(\mathbb{R}^2)$, where U is an entire solution of (5.1) with $U(\cdot, 0) = \varphi$. Since $\psi \not\equiv \varphi$ (because $\psi \notin \Omega(u)$), $V := U - \psi$ is a nontrivial entire solution of a linear equation (5.3). Therefore, (6.14) and Lemma 5.5 imply that for all sufficiently large n

$$\tilde{u}(\cdot + x_n, s_n + t_n) - \psi \text{ has a multiple zero } z_n \approx 0 \text{ for some } s_n \in (-1, 1). \tag{6.15}$$

We first find a contradiction if the sequence $\{x_n\}$ contains a bounded subsequence. Passing to a subsequence of (x_n, t_n), we may then assume that $x_n \to x_0$ for some $x_0 \in \mathbb{R}$. Then, by parabolic estimates, $\tilde{u}(\cdot + x_n, \cdot + t_n) - \tilde{u}(\cdot + x_0, \cdot + t_n) \to 0$ in $C^1_{loc}(\mathbb{R}^2)$. Hence, $\tilde{u}(\cdot + x_0, \cdot + t_n) - \psi$ has the same limit $V := U - \psi$ and, as in (6.15), we conclude that the function $\tilde{u}(\cdot + x_0, s_n + t_n) - \psi$ has a multiple zero near $x = 0$ for some $s_n \in (-1, 1)$. However, using Lemma 6.1, one shows easily that under any of the conditions (A1)-(A3n) (see also (6.2)-(6.4)), there is t_0 such that $|(\tilde{u}(x, t_0) - \psi(x))| > 0$ for all $x \approx \pm\infty$. Hence, by Corollary 5.3, for large t the function $\tilde{u}(\cdot + x_0, t) - \psi$ has only simple zeros and we have a contradiction.

Next, we seek a contradiction in the case $|x_n| \to \infty$.

Assume first that (A1) holds (and $c = 0$). Let $\rho > 0$ be the minimal period of ψ. Write $x_n = k_n\rho + \sigma_n$, where $k_n \in \mathbb{Z}$ and $\sigma_n \in [0, \rho)$. We may assume, passing to a subsequence if necessary, that $\sigma_n \to \sigma_0 \in [0, \rho]$. Then, by parabolic estimates, $\tilde{u}(\cdot + x_n, \cdot + t_n) - \tilde{u}(\cdot + k_n\rho + \sigma_0, \cdot + t_n) \to 0$ in $C^1_{loc}(\mathbb{R}^2)$. Hence, $\tilde{u}(\cdot + k_n\rho + \sigma_0, \cdot + t_n) - \psi$ has the same limit $V := U - \psi$ as $\tilde{u}(\cdot + k_n\rho + x_n, \cdot + t_n) - \psi$. Therefore, as in (6.15), we conclude that the function $\tilde{u}(\cdot + k_n\rho + \sigma_0, s_n + t_n) - \psi$ has a multiple zero near $x = 0$ for some $s_n \in (-1, 1)$. This means, since $\psi \equiv \psi(\cdot - k_n\rho)$, that $\tilde{u}(\cdot + \sigma_0, t) - \psi$ has a multiple zero for $t = t_n + s_n \to \infty$. This is impossible by Corollary 5.3 (as we already saw at the beginning of this proof). Thus, under condition (A1), we have derived a contradiction, as desired.

Now assume that (A2) holds. By (6.2), if we fix a small constant $\delta > 0$, then there is $\epsilon > 0$ such that the following relations hold:

$$\begin{aligned} &|\psi'(x)| > \epsilon \quad (x \in [x_1 - \delta, x_2 + \delta]), \\ &\psi(x) \in \mathbb{R} \setminus (-\epsilon, \gamma + \epsilon) \quad (x \in \mathbb{R} \setminus [x_1 - \delta, x_2 + \delta]). \end{aligned} \tag{6.16}$$

By Lemma 6.1, there are positive constants r, t_0 such that $|\tilde{u}(x, t_0) - \gamma| + |\tilde{u}_x(x, t_0)| < \epsilon$ for $x < -r$ and $|u(x, t_0)| + |u_x(x, t_0)| < \epsilon$ if $x > r$. It follows (using $|x_n| \to \infty$) that for all sufficiently large n the function $\tilde{u}(\cdot, t_0) - \psi(\cdot - x_n)$ has a unique zero. Clearly, by (6.6), (6.7), $z(\tilde{u}(\cdot, t) - \psi(\cdot - x_n)) \geq 1$ for all t, hence the equality must hold here by the monotonicity of the zero number (see Lemma 5.1). The unique zero of $\tilde{u}(\cdot, t) - \psi(\cdot - x_n)$ has to be simple for all $t > t_0$ (see Remark 5.2). Since this holds for all sufficiently large n, we can choose n so that also $t_n + s_n > t_0$. We thus have a contradiction to (6.15).

Finally, we assume that (A3p) holds (and continue to assume that $|x_n| \to \infty$). The possibility $x_n \to \infty$ can be treated as in the previous case. One finds t_0 such that $z(\tilde{u}(\cdot, t_0) - \psi(\cdot - x_n)) = 1$ for all sufficiently large n, and then arguments similar to the ones above yield a contradiction. We proceed assuming that x_n (replaced by a subsequence) converges to $-\infty$. By (A3p), we have $c < c_{I_0}$ for some $I_0 = (a, b) \in \mathcal{N}^+$ and there is $\epsilon > 0$ such that $b - \epsilon > \psi$ everywhere. By (6.8), there is t_0 such that

$$\tilde{u}(x, t) > b - \epsilon \quad (x \leq 0,\, t \geq t_0).$$

This implies that if t is large enough, then all zeros of $\tilde{u}(\cdot, t) - \psi(\cdot - x_n)$ are located in $(0, \infty)$. In particular, if $z_n \approx 0$ is a multiple zero as in (6.15), then $z_n + x_n \geq 0$, which is absurd when $x_n \to -\infty$. With this last contradiction, the proof is complete. □

A similar result concerning condition (A3p0) is proved in the following lemma.

Lemma 6.6. *Assume that $I_0 = (0, b) \in \mathcal{N}^+$ for some $b \in (0, \gamma]$ and $\theta = 0$ is unstable from above for the equation $\dot{\theta} = f(\theta)$. Assume further that $u_0 \geq 0$ and $u_0 \equiv 0$ on an interval (m, ∞). Let ψ be a solution of* (3.1) *for some c such that statement* (A3p0) *holds. Then for each $\varphi \in \Omega(u)$ one has*

$$\tau(\varphi) \cap \tau(\psi) = \emptyset. \tag{6.17}$$

Proof. As before, we let $\tilde{u}(\cdot,t) = u(\cdot + ct, t)$.

According to (A3p0), ψ satisfies (3.16) with $\lambda^+(c) > \lambda^-(c_{I_0})$, and $\zeta = 0$. This implies (see Remark 3.6(iii)) that for some $\delta > 0$ and $x_1 \in \mathbb{R}$ one has

$$\psi(x),\ -\psi_x(x) > e^{(\lambda^-(c_{I_0})+\delta)x} \quad (x \geq x_1). \tag{6.18}$$

On the other hand, given any $t > 0$, Lemma 6.4 yields a constant $\kappa = \kappa(t)$ such that

$$u(x,t),\ |u_x(x,t)| \leq \kappa e^{\lambda^-(c_{I_0})x} \quad (x \in \mathbb{R}). \tag{6.19}$$

Further, with x_0 as in (A3p0) (see also (6.3)), we have

$$(x - x_0)\psi(x) > 0, \tag{6.20}$$

and there are constants $\delta, \epsilon > 0$ such that

$$\psi'(x) > \epsilon \quad (x \in [x_0 - \delta, x_0 + \delta]). \tag{6.21}$$

We note, first of all, that the above inequalities imply that for each fixed $t_0 > 0$ one has $|(\tilde{u}(x,t_0) - \psi(x))| > 0$ for all $x \approx \pm\infty$. Indeed, for $x \approx \infty$ this follows from (6.18), (6.19); and for $x \approx -\infty$ it follows from (6.20) and the fact that $\tilde{u} > 0$ (which is a consequence of the assumption $u_0 \geq 0$ and the comparison principle). Therefore, by Corollary 5.3, $z(\tilde{u}(\cdot,t) - \psi)$ is finite for each $t > 0$.

We next show that there is $t_0 > 0$ such that for all $y \in \mathbb{R}$ with sufficiently large $|y|$ and for all $t \geq t_0$ one has

$$z(\tilde{u}(\cdot,t) - \psi(\cdot - y)) = 1. \tag{6.22}$$

Once this is done, essentially the same arguments as given in the proof of Lemma 6.5 for the case (A2) can be used to prove (6.17); these details are left to the reader.

We first show that

$$z(\tilde{u}(\cdot,t) - \psi(\cdot - y)) \geq 1 \quad (t > 0,\ y \in \mathbb{R}). \tag{6.23}$$

Indeed, since $\tilde{u} \geq 0$, (6.20) implies that $\tilde{u}(x,t) > \psi(x - y)$ if x is negative and sufficiently large (depending on y). On the other hand, relations (6.19), (6.18) give the opposite inequality if x is positive and sufficiently large. This proves (6.23).

For the rest of the proof, we fix t_0 such that

$$\liminf_{x \to -\infty} \tilde{u}(x, t_0) > \sup_{x \in \mathbb{R}} \psi(x)$$

(t_0 exists by Lemma 6.1 and the relations in (A3p0), (6.3)). Relations (6.19), (6.18) then clearly imply that (6.22) holds for $t = t_0$ if y is negative and sufficiently large. Consequently, by (6.23) and the monotonicity of the zero number, (6.22) continues to hold for all $t \geq t_0$. Similarly, using (6.19), (6.18), and (6.21), one shows easily that if $y > 0$ is sufficiently large, then (6.22) for holds for $t = t_0$, hence for all $t \geq t_0$. The proof is now complete. □

Remark 6.7. The above proof shows how the assumption that $u_0 \equiv 0$ on $[m, \infty)$ is used; see (6.18), (6.19). In fact, this assumption is used here and in other arguments below only via estimates on u, as given in Lemma 6.4. Thus, the assumption can be replaced by any other assumption which guarantees a sufficiently fast exponential decay of $u(x, t)$ as $x \to \infty$.

6.3 The spatial trajectories of the functions in $\Omega(u)$

Set

$$K_\Omega(u) := \bigcup_{\varphi \in \Omega(u)} \tau(\varphi) = \{(\varphi(x), \varphi_x(x)) : \varphi \in \Omega(u),\ x \in \mathbb{R}\}. \tag{6.24}$$

This is a compact, connected subset of $\mathbb{R}^2$ (cp. Sect. 5.1). The definition of $\Omega(u)$ implies via a simple compactness argument that $K_\Omega(u)$ can be equivalently defined by

$$\begin{aligned} K_\Omega(u) = \{(\xi, \eta) \in \mathbb{R}^2 : \ & (u(x_n, t_n), u_x(x_n, t_n)) \to (\xi, \eta) \\ & \text{for some sequences } t_n \to \infty \text{ and } x_n \in \mathbb{R}\}. \end{aligned} \tag{6.25}$$

Our ultimate goal is to prove that $K_\Omega(u) = \tau(R_0)$. The following lemma is the first step in that direction. Recall that γ_* and γ^* were introduced in (3.31), (3.32), and S_J in (3.77).

Lemma 6.8. *The following statements hold.*

(i) *For each $\xi \in R_0^{-1}\{0\}$ one has*

$$K_\Omega(u) \cap \{(\xi, \eta) : \eta \in \mathbb{R}\} = \{(\xi, 0)\}. \tag{6.26}$$

(ii) $\tau(R_0) \cap S_{[\gamma_*,\gamma^*]} \subset K_\Omega(u) \cap S_{[\gamma_*,\gamma^*]} \subset \tau(R_0) \cup \tau(-R_0)$.

(iii) *Assume that $I = (a, b) \in \mathcal{N}^+$. In case $a = 0$, assume also that $\theta = 0$ is stable from above for the equation $\dot{\theta} = f(\theta)$. Then there is $\epsilon > 0$ such that*

$$K_\Omega(u) \cap S_{(b-\epsilon,b)} = \tau(\phi_I) \cap S_{(b-\epsilon,b)}. \tag{6.27}$$

(iv) *Assume that $I = (a, b) \in \mathcal{N}^-$. In case $b = \gamma$, assume also that $\theta = \gamma$ is stable from below for the equation $\dot{\theta} = f(\theta)$. Then there is $\epsilon > 0$ such that*

$$K_\Omega(u) \cap S_{(a,a+\epsilon)} = \tau(\phi_I) \cap S_{(a,a+\epsilon)}. \tag{6.28}$$

(v) *Assume that $I = (0, b) \in \mathcal{N}^+$ for some $b \in (0, \gamma]$ and that $\theta = 0$ is unstable from above for the equation $\dot{\theta} = f(\theta)$. Assume also that u_0 satisfies the additional assumptions $u_0 \geq 0$, $u_0 \equiv 0$ on some interval (m, ∞). Then there is $\epsilon > 0$ such that*

$$K_\Omega(u) \cap S_{(b-\epsilon,b)} = \tau(\phi_I) \cap S_{(b-\epsilon,b)}. \tag{6.29}$$

Proof. Lemma 6.1 guarantees that if $\xi \in (0, \gamma)$, then for each sufficiently large t, there is $x(t)$ such that $u(x(t), t) = \xi$, $u_x(x(t), t) \leq 0$. This implies the following property of $K_\Omega(u)$:

$$\text{For each } \xi \in (0, \gamma) \text{ there is } \eta \leq 0 \text{ such that } (\xi, \eta) \in K_\Omega(u). \tag{6.30}$$

To prove statement (i), take an arbitrary $\xi \in R_0^{-1}\{0\}$. If $\xi = 0$ or γ, Corollary 6.2 gives $(0, 0)$, $(\gamma, 0) \in K_\Omega(u)$. There is no $(0, \eta) \in K_\Omega(u)$ with $\eta \neq 0$, for that would imply the existence of a function $\varphi \in \Omega(u)$ with $\varphi(0) = 0$, $\varphi'(0) = \eta \neq 0$. The range of such a function cannot be contained in $[0, \gamma]$, hence, $\varphi \notin \Omega(u)$ by virtue of Corollary 6.2. Thus, (6.26) is proved for $\xi = 0$ and similarly one proves it for $\xi = \gamma$. Let now $\xi \in (0, \gamma)$. Since $R_0(\xi) = 0$, Lemma 6.5, Corollary 3.18, and Remark 3.19 imply that $(\xi, \eta) \notin K_\Omega(u)$ for any $\eta \neq 0$. This and (6.30) imply (6.26). Statement (i) is proved.

Statements (iii) -(iv) are proved in a similar fashion. Lemma 6.5 and Proposition 3.25 imply that, for some $\epsilon > 0$, (6.27)-(6.28) hold with the equality signs replaced by the inclusions $\subset$. Using this and (6.30), we see that these inclusions have to actually be equalities. The same arguments can be repeated for the proof of statement (v), only this time one also uses

Lemma 6.6, in addition to Lemma 6.5 and Proposition 3.25. Statements (iii)-(v) are thus proved.

Lemma 6.5, in conjunction with Corollary 3.18 and Remark 3.19, further implies that

$$K_\Omega(u) \cap S_{[\gamma_*,\gamma^*]} \subset B := \{(v,w) : v \in [\gamma_*,\gamma^*],\ R_0(v) \le w \le -R_0(v)\}.$$

In view of this and (6.26), to prove the second inclusion in statement (ii), we just need to show that $K_\Omega(u)$ cannot contain interior points of B. Suppose it does; let (ξ,η) be such a point. Then $\xi \in I$ for some $I = (a,b) \in \mathcal{N}^0$ and (ξ,η) is inside the heteroclinic loop of (3.1) (with $c = 0$) formed by $\tau(\phi_I)$, $\tau(-\phi_I)$, and the equilibria $(a,0)$, $(b,0)$. By Lemma 3.10, there is a periodic orbit $\tau(\psi)$ of (3.1) inside this loop, with (ξ,η) inside the closed orbit $\tau(\psi)$. Since $(\xi,\eta) \in K_\Omega(u)$ and $K_\Omega(u)$ also contains the points $(0,0)$, $(\gamma,0)$, which are outside $\tau(\psi)$, the connectedness of $K_\Omega(u)$ implies that $K_\Omega(u)$ contains a point on $\tau(\psi)$. This means that there is $\phi \in \Omega(u)$ such that $\tau(\phi)\cap\tau(\psi) \neq \emptyset$, in contradiction to Lemma 6.5. This contradiction proves the second inclusion in (ii). Combining this with (6.30), we next obtain the first inclusion in (ii). The proof is now complete. $\square$

6.4 $\Omega(u)$ contains the minimal propagating terrace

With Lemma 6.8 at hand, we are in position we prove the following inclusions. Recall that the set $\tilde{\mathcal{N}} \subset \mathcal{N}$ was introduced in (2.25). It differs from $\mathcal{N}$ only if 0 is unstable from above for the equation $\dot\theta = f(\theta)$ or γ is unstable from below for this equation. In particular, if $I = (0,b)$ is as in as in statement (v) of Lemma 6.8, then $I \in \mathcal{N} \setminus \tilde{\mathcal{N}}$.

Lemma 6.9. *For each $I \in \tilde{\mathcal{N}}$ one has $\phi_I \in \Omega(u)$. Moreover, if the assumptions of statement (v) of Lemma 6.8 are satisfied and $I = (0,b)$ is as in that statement, then also $\phi_I \in \Omega(u)$.*

In the proof of this lemma the following unique-continuation result will be useful.

Lemma 6.10. *Let $I \in \mathcal{N}$, $c := c_I$, and let U be a solution of* (5.1) *on some time interval (s,T). Assume that there exist $t_0 \in (s,T)$ and an open set $G \subset \mathbb{R}^2$ such that*

$$\emptyset \neq \tau(U(\cdot,t_0)) \cap G \subset \tau(\phi_I). \tag{6.31}$$

Then there is $\theta \in \mathbb{R}$ such that $U \equiv \phi_I(\cdot+\theta)$.

Proof. Set $\varphi := U(\cdot, t_0)$. Relations (6.31) imply that there exists $x_0 \in \mathbb{R}$ with the following property. For each $x \approx x_0$ there is $\vartheta(x)$ such that

$$\varphi(x) = \phi_I(\vartheta(x)), \quad \varphi'(x) = \phi_I'(\vartheta(x)). \tag{6.32}$$

Since $\phi_I' < 0$, the value $\vartheta(x)$ is defined uniquely, and the implicit function theorem implies $\vartheta \in C^1$. Differentiating the first identity in (6.32) and comparing to the second one, we obtain that $\vartheta' \equiv 1$. Thus, in a neighborhood of x_0 we have $\varphi \equiv \phi_I(\cdot + \theta)$ for some $\theta \in \mathbb{R}$. Consider now the function $V := U - \phi_I(\cdot + \theta)$. It is a solution of a linear equation (5.3) on the time interval (s, T), and $V(\cdot, t_0)$ vanishes on a neighborhood of x_0. By Lemma 5.1, this is possible only if $V \equiv 0$, that is, $U \equiv \phi_I(\cdot + \theta)$. □

Remark 6.11. Clearly, if $c = c_I = 0$, then Lemma 6.10 remains valid, with the same proof, if $\phi_I(x)$ is replaced by $\hat{\phi}_I(x) = \phi_I(-x)$.

Proof of Lemma 6.9. Assume that $I = (a, b) \in \tilde{\mathcal{N}}$ or $I = (0, b) \in \mathcal{N} \setminus \tilde{\mathcal{N}}$ is as in the statement (v) of Lemma 6.8. Using Lemma 6.8, we find a point $(\xi_0, \eta_0) \in \tau(\phi_I)$ and a neighborhood G of (ξ_0, η_0) such that

$$G \cap K_\Omega(u) = G \cap \tau(\phi_I). \tag{6.33}$$

In particular, $(\xi_0, \eta_0) \in K_\Omega(u)$, which implies that there is $\varphi \in \Omega(u)$ and $x_0 \in \mathbb{R}$ such that $(\varphi(x_0), \varphi'(x_0)) = (\xi_0, \eta_0)$. Then $\tau(\varphi) \in K_\Omega(u)$ and (6.33) gives

$$(\xi_0, \eta_0) \in G \cap \tau(\varphi) \subset \tau(\phi_I). \tag{6.34}$$

Take now the entire solution U of (3.1) with $c = c_I$ such that that $U(\cdot, 0) = \varphi$ (cp. Sect. 5.1). Using (6.34) and Lemma 6.10, we conclude that $\varphi \equiv \phi_I(\cdot - \theta)$ for some θ. This and the translation invariance of $\Omega(u)$ give $\phi_I \in \Omega(u)$. □

Lemma 6.12. *If $\xi \in f^{-1}\{0\}$ and $(\xi, 0) \in K_\Omega(u)$, then (the constant steady state) ξ is an element of $\Omega(u)$. In particular, by Lemma 6.8(i), $R_0^{-1}\{0\} \subset \Omega(u)$.*

Proof. If $\xi = 0$ or $\xi = \gamma$, then $\xi \in \Omega(u)$ by Corollary 6.2 and there is nothing to prove. Next we consider the case $\xi \in (0, \gamma)$. From $(\xi, 0) \in K_\Omega(u)$ we infer that there is $\varphi \in \Omega(u)$ such that $(\varphi(0), \varphi'(0)) = (\xi, 0)$. We claim that $\varphi \equiv \xi$, which gives the desired conclusion $\xi \in \Omega(u)$. Assume $\varphi \not\equiv \xi$. There are sequences x_n and $t_n \to \infty$, such that $u(\cdot + x_n, \cdot + t_n) \to U$ in $C^1_{loc}(\mathbb{R}^2)$, where U is an entire solution of (5.1) with $c = 0$ and with $U(\cdot, 0) = \varphi$. Clearly,

$U-\xi$ is a nontrivial entire solution of a linear equation (5.3) with a multiple zero $x=0$ at $t=0$. By Lemma 5.5, for all sufficiently large n, the function $u(\cdot+x_n, s_n+t_n)-\xi$ has a multiple zero near $x=0$ for some $s_n \in (-1,1)$. Thus $u(\cdot, s_n+t_n)-\xi$ has a multiple zero (near $-x_n$) for all large n. However, Lemma 6.1 and Corollary 5.3 clearly imply that $u(\cdot,t)-\xi$ has only simple zeros for all large t. This contradiction proves that $\xi \in \Omega(u)$, as claimed. $\square$

We remark that if $\xi \in R_0^{-1}\{0\}$ is the limit $\phi_I(\infty)$ or $\phi_I(-\infty)$ of some ϕ_I, $I \in \mathcal{N}$, then the conclusion that $\xi \in \Omega(u)$ also follows from Lemma 6.9 by means of the translation invariance and closedness of $\Omega(u)$ in $L^\infty_{loc}(\mathbb{R})$. In general, however, this argument does not cover all elements of $R_0^{-1}\{0\}$ ($R_0^{-1}\{0\}$ may contain an interval).

Although not needed for our proofs, it may be worthwhile to remark that at this point it is easy to prove that that $\tau(R_0) = K_\Omega(u)$ holds under the extra assumption that u_0 is nonincreasing. Indeed, Lemmas 6.9, 6.12 give $\tau(R_0) \subset K_\Omega(u)$. If $K_\Omega(u) \setminus \tau(R_0) \neq \emptyset$, then using Lemma 6.9 one shows that for some large values of t, the function $u(\cdot,t)$ would have to be increasing on some intervals. This this is impossible if u_0 is nonincreasing.

6.5 Ruling out other points from $K_\Omega(u)$

Another step toward the proofs of the main results is the following strengthening of statements (iii)-(v) of Lemma 6.8.

Lemma 6.13. *Assume that the hypotheses of one of the statements (iii)-(v) of Lemma 6.8 are satisfied and let $I=(a,b)$ be as in that statement. Then*

$$K_\Omega(u) \cap S_{(a,b)} = \tau(\phi_I). \tag{6.35}$$

We first prove the following general result, which will be needed at several places below.

Lemma 6.14. *Let $J \subset (0,\gamma)$ be an open interval. Assume that*

$$K_\Omega(u) \cap S_J = \tau(\phi) \cap S_J, \tag{6.36}$$

where ϕ is a solution of (3.1) for some $c \in \mathbb{R}$ such that $\phi' < 0$ and J is included in the range of ϕ. Then for each $\theta \in J$ there exist $s_0 > 0$ and a C^1 function $\zeta(t)$ on (s_0,∞) such that

$$\big(\theta - u(x+ct+\zeta(t),t)\big)x > 0 \quad (x \in \mathbb{R} \setminus \{0\}), \tag{6.37}$$

$\zeta'(t) \to 0$ *as* $t \to \infty$, *and*

$$\lim_{t\to\infty} u(\cdot + ct + \zeta(t), t) = \phi(\cdot + x_0) \ \text{ in } C^1_{loc}(\mathbb{R}), \tag{6.38}$$

where x_0 *is the unique point with* $\phi(x_0) = \theta$.

Remark 6.15. Note that this lemma applies in particular if the hypotheses of one of the statements (iii)-(v) of Lemma 6.8 are satisfied, $\phi = \phi_I$, $c = c_I$, and $J := (b - \epsilon, b)$ (in the case of statements (iii)-(iv)) or $J := (a, a + \epsilon)$ (the case of statements (v)). This remark will be used in the proof of Lemma 6.13.

Proof of Lemma 6.14. Fix an arbitrary $\theta \in J$ and take a neighborhood J_0 of θ such that $\bar{J}_0 \subset J$. Relations (6.36) and $\phi' < 0$ in particular imply that if t is sufficiently large, say $t > s_0$, then $u_x(x,t) < 0$ whenever $u(x,t) \in J_0$. This property and Lemma 6.1 imply that, possibly after making s_0 larger, for each $t > s_0$ the equation $u(x + ct, t) = \theta$ has a unique solution, which we denote by $\zeta(t)$. The uniqueness of $\zeta(t)$ and Lemma 6.1 clearly imply (6.37). By the implicit function theorem, ζ is a C^1-function.

Further, with x_0 defined as in the lemma, one has $u(ct + \zeta(t), t) = \theta = \phi(x_0)$. Now, any sequence $t_n \to \infty$ has a subsequence such that $u(\cdot + ct_n + \zeta(t_n), t_n)$ converges in $C^1_{loc}(\mathbb{R})$ to an element $\varphi \in \Omega(u)$ with $\varphi(0) = \theta$. We claim that $\varphi = \phi(\cdot + x_0)$. To prove this, let $\eta = \varphi'(0)$. We have $\tau(\varphi) \subset K_\Omega(u)$, in particular $(\theta, \eta) \in K_\Omega(u)$. Since $\theta \in J$, relation (6.36) implies that for some neighborhood G of (θ, η) one has

$$(\theta, \eta) \in G \cap \tau(\varphi) \subset \tau(\phi). \tag{6.39}$$

Take now the entire solution U of (5.1) such that that $U(\cdot, 0) = \varphi$ (cp. Sect. 5.1). Using (6.39) and Lemma 6.10, we conclude that $\varphi \equiv \phi(\cdot + y)$ for some y, and since $\varphi(0) = \theta$, we have $y = x_0$. This proves the claim. As the claim is valid with the same limit for any sequence $\{t_n\}$, we have proved (6.38).

It remains to show that $\zeta'(t) \to 0$ as $t \to \infty$. To simplify the notation, set $\tilde{u}(x,t) := u(x + ct, t)$. Recall from Section 5.1 that any sequence $t_n \to \infty$ can be replaced by a subsequence such that $\tilde{u}(\cdot + \zeta(t_n), \cdot + t_n)$ converges in $C^1_{loc}(\mathbb{R}^2)$ to an entire solution U of equation (5.1). By (6.38), we have $U(\cdot, 0) = \phi(\cdot + x_0)$. Since ϕ is a steady state of (5.1), we have $U \equiv \phi$, by uniqueness and backward uniqueness for (5.1). Thus, the convergence in

$C^1_{loc}(\mathbb{R}^2)$ yields

$$(\tilde u(\cdot+\zeta(t_n),\cdot+t_n),\tilde u_x(\cdot+\zeta(t_n),\cdot+t_n),\tilde u_t(\cdot+\zeta(t_n),\cdot+t_n)) \\ \to (\phi(\cdot+x_0),\phi'(\cdot+x_0),0).$$

Since this is true for any sequence $t_n\to\infty$, the convergence takes place with t_n replaced by t, with $t\to\infty$. In particular, at $x=0$ we have

$$\big(\tilde u(\zeta(t),t),\tilde u_x(\zeta(t),t),\tilde u_t(\zeta(t),t)\big)\to(\theta,\phi_x(x_0),0), \tag{6.40}$$

as $t\to\infty$. By the definition of ζ, $\tilde u(\zeta(t),t)=\theta$. Differentiating this relation, we obtain $\tilde u_x(\zeta(t),t)\zeta'(t)+u_t(\zeta(t),t)=0$. Since $\phi'(x_0)\neq 0$, from (6.40) we conclude that $\zeta'(t)\to 0$ as $t\to\infty$.

The proof of Lemma 6.14 is now complete. □

We will prove the conclusion of Lemma 6.13, assuming that either the hypotheses of statement (iii) or the hypothesis of statement (v) of Lemma 6.8 are satisfied. Analogous arguments can be used if the hypothesis of statement (iv) of Lemma 6.8 are satisfied and this part of the proof will be omitted. We need one more technical result for the proof.

Lemma 6.16. *Assume that the hypotheses of statement (iii) or statement (v) of Lemma 6.8 are satisfied and let $I=(a,b)$ be as in that statement. Then for any $a_1\in(a,b)$, there exist $c>0$, $c\neq c_I$, and a solution ψ of (3.1) such that*

$$\psi<b,\quad \limsup_{x\to-\infty}\psi(x)<0,\quad \psi(0)>a_1,\quad \inf_{x>0}\psi(x)<a_1, \tag{6.41}$$

and for some $t_0>0$ and $y_0\in\mathbb{R}$ one of the following statements (d1), (d2) is valid:

(d1) *$c<c_I$ and $z(u(\cdot+ct_0,t_0)-\psi(\cdot-y))=1$ $(y\geq y_0)$.*

(d2) *$c>c_I$, $\psi(\infty)=0$, and $z(u(\cdot+ct_0,t_0)-\psi(\cdot-y))=1$ $(y\leq -y_0)$.*

(More specifically, (d1) holds if the hypotheses of Lemma 6.8(iii) are satisfied and either (d1) or (d2) holds if the hypotheses of Lemma 6.8(iv) are satisfied.)

Proof. Assume first that the hypotheses of statement (iii) of Lemma 6.8 are satisfied. As in Remark 3.30(i), we find $c \in (0, c_I)$ and a solution ψ of (3.1) satisfying (A3p) with $I_0 = I$, and such that

$$\psi(0) \approx b, \quad \psi(x) \approx a \ \text{ for some } x > 0,$$

with any desired proximities. In particular, we may choose ψ such that $\psi(0) > a_1$, $\inf_{x>0} \psi(x) < a_1$. By (A3p) (see also (6.3)), the first two relations in (6.41) are satisfied as well. Finally, property (A3p) and Lemma 6.1 imply (as already shown in the proof of Lemma 6.5), that (d1) holds for some t_0, y_0.

Next assume that the hypotheses of statement (v) of Lemma 6.8 are satisfied. As in Remark 3.30(ii), we find a solution ψ of (3.1) for some $c \neq c_I$, such that $\psi(0) > a_1$ and (A3p0) holds with $I_0 = I$ (in particular $c > 0$). By (A3p0) (and (6.3)), all relations in (6.41) are satisfied and, moreover, $\psi(\infty) = 0$. As shown in the proof of Lemma 6.6, see (6.22), there are t_0, y_0 such that

$$z(\tilde{u}(\cdot, t_0) - \psi(\cdot - y)) = 1 \quad (|y| \geq y_0).$$

Thus one of the statements (d1), (d2) holds (depending on whether $c < c_I$ or $c > c_I$). $\square$

Proof of Lemma 6.13. Assume that either the hypotheses of statement (iii) or the hypotheses of statement (v) of Lemma 6.8 are satisfied (the proof for the case of statement (iv) is omitted). Let $I = (a, b)$ and $\epsilon > 0$ be as in the corresponding statement of Lemma 6.8.

The proof is by contradiction. Assume that (6.35) is not true: there are $\xi \in (a, b)$ and $\eta \in \mathbb{R}$ such that $(\xi, \eta) \in K_\Omega(u) \setminus \tau(\phi_I)$. Then there are sequences $t_n \to \infty$ and z_n such that

$$(u(z_n, t_n), u_x(z_n, t_n)) \to (\xi, \eta). \tag{6.42}$$

In view of the statements (iii), (v) of Lemma 6.8, we necessarily have $\xi \in (a, b - \epsilon]$. As noted in Remark 6.15, Lemma 6.14 applies if we take $J = (b - \epsilon, b)$, $\phi = \phi_I$, $c = c_I$, and $\theta \in (b - \epsilon, b)$. Fix any such θ and let s_0 and $\zeta(t)$ be as in the conclusion of Lemma 6.14. Also let x_0 be the unique value such that $\phi_I(x_0) = \theta$. From the convergence properties (6.38), (6.42), it follows that, given any $k > 0$, for all sufficiently large n one has $(u(z_n, t_n), u_x(z_n, t_n)) \approx (\xi, \eta)$ and

$$\big(u(x + c_I t_n + \zeta(t_n), t_n), u_x(x + c_I t_n + \zeta(t_n), t_n)\big) \approx (\phi_I(x + x_0), \phi_I'(x + x_0)) \quad (x \in [-k, k]). \tag{6.43}$$

Since $(\xi, \eta) \notin \tau(\phi_I)$ (and $\xi \in (a, b)$), (ξ, η) has a positive distance from $\tau(\phi_I)$. Therefore, (6.43) together with the relations (6.37) (with $c = c_I$) and $\theta > b - \epsilon \geq \xi$ imply that for all large n we have $z_n > c_I t_n + \zeta(t_n) + k$; that is, as x increases, $(u(x + c_I t_n + \zeta(t_n), t_n), u_x(x + c_I t_n + \zeta(t_n), t_n))$ stays close to the trajectory $\tau(\phi_I)$ for $x \in [-k, k]$ *before* it gets close to (ξ, η) at $x = z_n$. Setting $a_1 := (a + \xi)/2$, we claim that for all sufficiently large n there are x_n, $\bar{x}_n$, such that the following relations are valid (cp. Figure 12):

$$c_I t_n + \zeta(t_n) < x_n < \bar{x}_n < z_n \tag{6.44}$$

and

$$\begin{aligned} u(x_n, t_n) = u(\bar{x}_n, t_n) = a_1, \\ u(x, t_n) < a_1 \quad (x \in (x_n, \bar{x}_n)). \end{aligned} \tag{6.45}$$

Indeed, for x_n we take the minimal zero of $u(\cdot, t_n) - a_1$ (for large n); it is found near the point $z_0 + c_I t_n + \zeta(t_n)$, where z_0 is defined by $\phi_I(z_0 + x_0) = a_1$. Note that $z_0 > 0$ because $a_1 < \xi < \theta = \phi_I(x_0)$ and ϕ_I is decreasing. Also, by (6.43), $u_x(x_n, t_n) < 0$ for large n. For $\bar{x}_n > x_n$ we then take the next zero of $u(\cdot, t_n) - a_1$ in $[x_n, \infty)$, which exists since $u(x, t_n)$ is close to $\xi > a_1$ for $x \approx z_n$. This choice of x_n, $\bar{x}_n$ yields the following additional properties:

$$\text{the sequence } \{x_n - c_I t_n - \zeta(t_n)\} \text{ is bounded,} \tag{6.46}$$

$$\bar{x}_n - x_n \to \infty \tag{6.47}$$

(the latter follows from (6.43)).

We next employ a solution ψ as in Lemma 6.16. Let $\tilde{u}(x, t) := u(x + ct, t)$, so $\tilde{u}$ satisfies equation (5.1) of which ψ is a steady state and, consequently, for any y, the map $t \mapsto z(\tilde{u}(\cdot, t) - \psi(\cdot - y))$ is nonincreasing (see Section 5.2). Lemma 6.16 yields $t_0 > 0$ and $y_0 \in \mathbb{R}$ such that one of the statements (d1), (d2) holds.

Assume first that (d1) holds. Then, by the nonincrease of the zero number,

$$z(\tilde{u}(\cdot, t) - \psi(\cdot - y)) \leq 1 \quad (t \geq t_0,\ y \geq y_0). \tag{6.48}$$

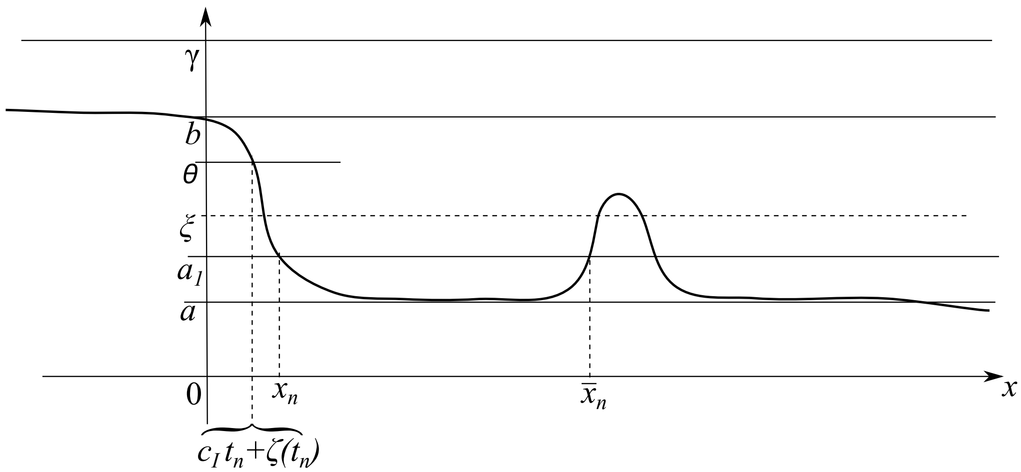

Figure 12: The supposed structure of the graph of $u(\cdot, t_n)$.

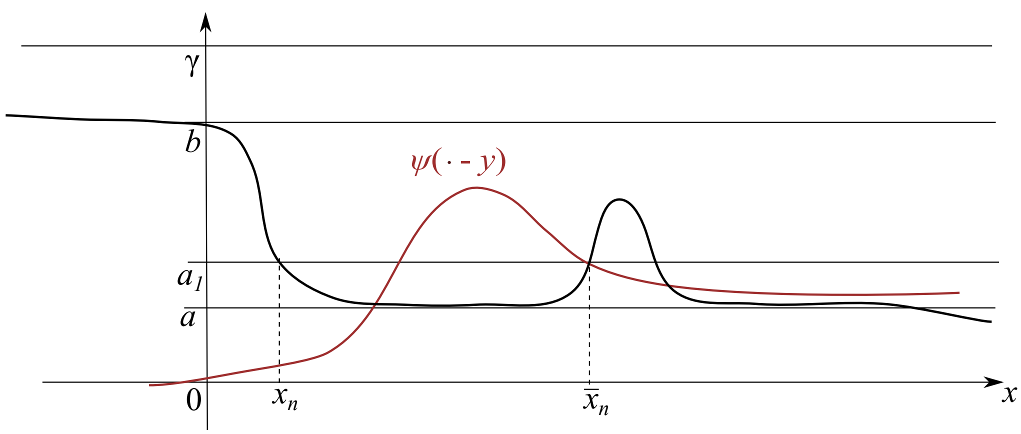

Figure 13: Intersections of $\tilde{u}(\cdot, t_n)$ and $\psi(\cdot - y)$

Obviously, there is $y_1 > 0$ such that $\psi(y_1) = a_1$ and $\psi > a_1$ on $[0, y_1)$. We choose a large enough n so that $t_n > t_0$ and

$$y := \bar{x}_n - ct_n - y_1 > y_0. \tag{6.49}$$

Such a choice is possible since

$$\bar{x}_n - ct_n - y_1 > (c_I - c)t_n + \zeta(t_n) - y_1$$

(see (6.44)), $c_I > c$ (see (d1), $t_n \to \infty$, and $\zeta(t)/t \to 0$ as $t \to \infty$ (cp. Lemma 6.14). Thus, $z(\tilde{u}(\cdot, t_n) - \psi(\cdot - y)) \le 1$. On the other hand, at $x = y_1 + y = \bar{x}_n - ct_n$, we find

$$\tilde{u}(y_1 + y, t_n) = u(\bar{x}_n, t_n) = a_1 = \psi(y_1). \tag{6.50}$$

Hence, $\tilde{u}(\cdot, t_n) - \psi(\cdot - y)$ has a zero at $y_1 + y$. Moreover, by the last relation in (6.45) and the fact that $\psi > a_1$ on $[0, y_1)$, we have $\tilde{u}(x, t_n) - \psi(x - y) < 0$ for $x < y_1 + y$ sufficiently close to $y_1 + y$ (cp. Figure 13). Finally, using (6.41) and the fact that $\tilde{u}(x, t_n) > 0$ for $x \approx -\infty$, we obtain that $\tilde{u}(x, t_n) - \psi(x - y) > 0$ for all sufficiently large negative x. Therefore, there is another zero of $\tilde{u}(\cdot, t_n) - \psi(\cdot - y)$ in $(-\infty, y_1 + y)$, and we have a contradiction to (6.48). This contradiction completes the proof if (d1) holds.

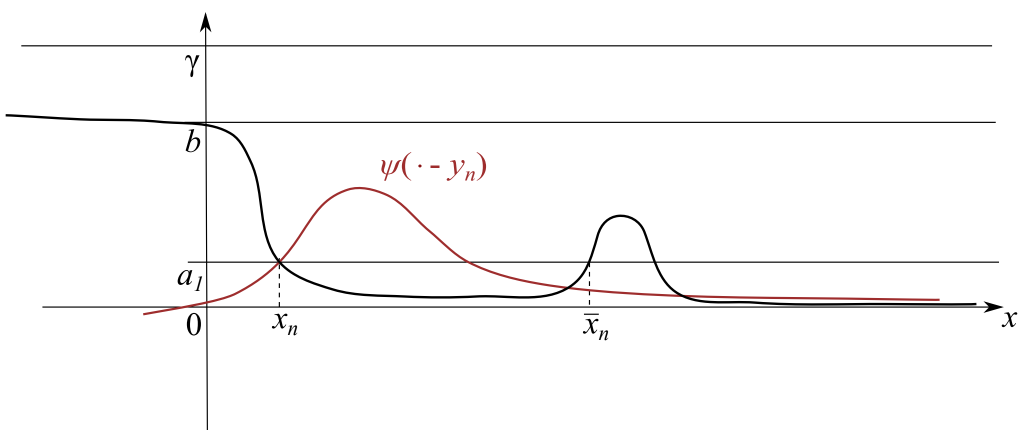

Figure 14: Intersections of $\tilde{u}(\cdot, t_n)$ and $\psi(\cdot - y_n)$ in the case that (d2) holds, $a = 0$, and the hypotheses of Lemma 6.8(iv) are satisfied.

If (d2) holds (and the statement (v) of Lemma 6.8 are satisfied), similar arguments apply, but there are some differences. We give full details. By (d2), we have $z(\tilde{u}(\cdot, t) - \psi(\cdot - y)) \le 1$ for all $t \ge t_0$ and $y \le -y_0$. Obviously, there is $y_1 < 0$ such that $\psi(y_1) = a_1$ and $\psi > a_1$ on $(y_1, 0]$. Observe that if

n_0 is large enough, then for all $n \geq n_0$ one has $t_n > t_0$ and

$$y_n := x_n - ct_n - y_1 < -y_0. \tag{6.51}$$

Indeed, this follows from (6.46) and the facts that $c_I < c$ (see (d2)), $t_n \to \infty$, and $\zeta(t)/t \to 0$ as $t \to \infty$ (Lemma 6.14). Thus, for any $n \geq n_0$ we have $z(\tilde{u}(\cdot, t_n) - \psi(\cdot - y_n)) \leq 1$. On the other hand, at $x = y_1 + y_n = x_n - ct_n$, we have

$$\tilde{u}(y_1 + y_n, t_n) = u(x_n, t_n) = a_1 = \psi(y_1). \tag{6.52}$$

Moreover, for $x > y_1 + y_n$ sufficiently close to $y_1 + y_n$, we have $\tilde{u}(x, t_n) - \psi(x - y_n) < 0$ by the last relation in (6.45) and the fact that $\psi > a_1$ on $[y_1, 0)$ (cp. Figure 14).

Finally, at $x = y_1 + y_n + \bar{x}_n - x_n = \bar{x}_n - ct_n$, we have

$$\tilde{u}(y_1 + y_n + \bar{x}_n - x_n, t_n) = u(\bar{x}_n, t_n) = a_1,$$

whereas, due to (6.47) and $\psi(\infty) = 0$ (see (d2)),

$$\psi(x - y_n) = \psi(y_1 + \bar{x}_n - x_n) < a_1$$

if $n \geq n_0$ is taken sufficiently large. For such n, the function $\tilde{u}(\cdot, t_n) - \psi(\cdot - y_n)$ has a second zero in $(y_1 + y_n, \infty)$ and we have a contradiction in this case as well. $\square$

6.6 Completion of the proofs of Theorems 2.5, 2.13, and 2.15

Recall that $\gamma_0 \geq 0$, $\gamma_1 \leq \gamma$, and $\tilde{\mathcal{N}}$ were introduced in Section 2.3 and γ_*, γ^* in Section 3.2. Relation $\gamma_0 > 0$ means that is unstable from above for the equation $\dot{\theta} = f(\theta)$ and $\gamma < \gamma_1$ means that γ is unstable from below for this ODE.

Proof of Theorems 2.5 and 2.13. For the proof of Theorem 2.13, we need to show that

$$R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\ \xi \in \mathbb{R}\} \subset \Omega(u) \tag{6.53}$$

and

$$\Omega(u) \subset R_0^{-1}\{0\} \cup \{\phi_I(\cdot - \xi) : I \in \tilde{\mathcal{N}},\ \xi \in \mathbb{R}\} \\ \cup \{\hat{\phi}_I(\cdot - \xi) : I \in \mathcal{N}^0,\ \xi \in \mathbb{R}\} \cup \Omega_0 \cup \Omega_1, \tag{6.54}$$

where $\hat{\phi}_I(x) := \phi_I(-x)$, Ω_0 is a set of functions with range in $(0, \gamma_0)$, and Ω_1 is a set of functions with range in (γ_1, γ).

If $\gamma_0 = 0$ and $\gamma_1 = \gamma$ (that is, the stability assumption (S) of Theorem 2.5 is satisfied), then $\Omega_0 = \emptyset = \Omega_1$, $\mathcal{N} = \tilde{\mathcal{N}}$, and (6.53), (6.54) give the conclusion of Theorem 2.5. Thus by proving (6.53), (6.54), we will have proved both theorems.

Inclusion (6.53) is a consequence of Lemmas 6.12, 6.9, and the translation invariance of $\Omega(u)$.

In the proof of (6.54), we use the following inclusion obtained directly from Lemmas 6.8(ii): and 6.13

$$K_\Omega(u) \cap S_{[\gamma_0,\gamma_1]} \subset \tau(R_0) \cup (\tau(-R_0) \cap S_{[\gamma_*,\gamma^*]}). \tag{6.55}$$

Take any $\varphi \in \Omega(u)$, we show that it belongs to the right-hand side of (6.54).

We have $\tau(\varphi) \subset K_\Omega(u)$, by the definition of $K_\Omega(u)$. If $\varphi \equiv \xi$ for a constant ξ, then $\tau(\varphi) = \{(\xi, 0)\}$. By (6.55), this is possible only if $\xi \in R_0^{-1}\{0\}$.

In the rest of the proof we assume that φ is nonconstant. Let U be the entire solution of (5.1) with $U(\cdot, 0) = \varphi$.

If the range of φ contains a point $\varphi(x_0)$ in (γ_0, γ_1), then (6.55), implies that there exist $I \in \tilde{\mathcal{N}}$ and a neighborhood G of the point $(\varphi(x_0), \varphi'(x_0))$ such that

$$G \cap \tau(\varphi) \subset \tau(\psi),$$

where $\psi = \phi_I$, or $c_I = 0$ and $\psi = \hat{\phi}_I$. Applying Lemma 6.10 (see also Remark 6.11), we obtain that $\varphi \equiv \psi(\cdot - \xi)$ for some $\xi \in \mathbb{R}$, showing that φ belongs to the right-hand side of (6.54) and also that $\gamma_0 < \varphi < \gamma_1$.

Consider now the function $\tilde{\varphi} := U(\cdot, -1)$. Note that that $\tilde{\varphi} \in \Omega(u)$ by the invariance of $\Omega(u)$, and $\tilde{\varphi}$ is not identical to a constant (otherwise φ would be). If the range of $\tilde{\varphi}$ contains a point in (γ_0, γ_1), then by the previous conclusion $\gamma_0 < \tilde{\varphi} < \gamma_1$. But then, by the comparison principle $\gamma_0 < \varphi = U(\cdot, 0) < \gamma_1$, and the previous conclusion applies to φ itself. Thus, φ belongs to the right-hand side of (6.54).

If, on the other hand, the range of $\tilde{\varphi}$ contains no points in (γ_0, γ_1), then either $0 \le \tilde{\varphi} \le \gamma_0$ or $\gamma \ge \tilde{\varphi} \ge \gamma_1$. In this case, by the strong comparison principle (and the fact that $\tilde{\varphi}$ is not identical to any constant), we have $0 < \varphi < \gamma_0$ or $\gamma > \varphi > \gamma_1$. This completes the proof of inclusion (6.54). $\square$

Proof of Theorem 2.15. Under the hypotheses of Theorem 2.15, the interval $I := I_* = (0, \gamma_0)$ satisfies the hypotheses of statement (v) of Lemma 6.8.

Therefore, by Lemma 6.9 we have $\phi_{I_*} \in \Omega(u)$, and by Lemma 6.13

$$K_\Omega(u) \cap S_{(0,\gamma_0)} = \tau(\phi_{I_*}). \tag{6.56}$$

Using (6.56) and Lemma 6.10, similarly as in the previous proof, one shows easily that each function $\varphi \in \Omega(u)$ with range in I_* coincides with a shift of ϕ_{I_*}. Since we already know that $\phi_{I_*} \in \Omega(u)$, using the translation invariance of $\Omega(u)$ we conclude that the set Ω_0 in (6.54) is given by

$$\Omega_0 = \{\phi_{I_*}(\cdot - \xi) : \xi \in \mathbb{R}\}.$$

The theorem is proved. □

The following corollary will be used below to justify applications of Lemma 6.14.

Corollary 6.17. *Let $I = (a, b) = \mathcal{N}$. One has*

$$K_\Omega(u) \cap S_I = \tau(\phi_I) \tag{6.57}$$

in any of the following cases (ci)–(civ):

(ci) $I \in \tilde{\mathcal{N}} \setminus \mathcal{N}^0$*;*

(cii) $\gamma_0 > 0$*,* $I = (0, \gamma_0)$*, and* u_0 *satisfies (besides the standing assumptions* (2.2), (2.3)*) the following relations:* $u_0 \geq 0$ *and* $u_0 \equiv 0$ *on an interval* $[m, \infty)$*;*

(ciii) $\gamma_1 < \gamma$*,* $I = (\gamma_1, \gamma)$*, and* u_0 *satisfies the following relations:* $u_0 \leq \gamma$ *and* $u_0 \equiv \gamma$ *on an interval* $(-\infty, n]$*;*

(civ) $I \in \mathcal{N}^0$ *and* $\hat{\phi}_I \notin \Omega(u)$*.*

Proof. If (ci) or (civ) holds, (6.57) follows directly from Theorem 2.13. If (cii) holds, (6.57) was verified in the proof of Theorem 2.15 (see (6.56)), and (ciii) is analogous to (cii). □

6.7 Completion of the proofs of Theorems 2.7, 2.9, and 2.17

To prove the theorems, we just need to rule out the functions $\hat{\phi}_I$, $I \in \mathcal{N}^0$, from $\Omega(u)$:

Lemma 6.18. *Assume that the hypotheses of any one of Theorems 2.7, 2.9, 2.17 are satisfied. If $\mathcal{N}^0 \neq \emptyset$, then one has*

$$\hat{\phi}_I \notin \Omega(u) \quad (I \in \mathcal{N}^0). \tag{6.58}$$

Once this lemma is proved, Theorems 2.7, 2.9 follow from Theorem 2.5, and Theorem 2.17 follows from Theorem 2.13.

It remains to prove Lemma 6.18. Under the hypotheses of Theorem 2.7, this can been done using reflection arguments, which are simpler than the arguments given below. However, the arguments we use take care of all three theorems.

First, we derive a common consequence of the hypotheses of the theorems.

Lemma 6.19. *Assume that the hypotheses of any one of Theorems 2.7, 2.9, 2.17 are satisfied. If $I = (a, b) \in \mathcal{N}^0$, then there exist $t_0 \geq 0$ and $\alpha_0 \in f^{-1}\{0\} \cap (a, b)$ such that for each $t \geq t_0$ one has*

$$\limsup_{x\to\infty} u(x,t) < \min_{y_1(t)\leq x\leq y_0(t)} u(x,t) \leq \max_{y_1(t)\leq x\leq y_0(t)} u(x,t) < \liminf_{x\to-\infty} u(x,t), \tag{6.59}$$

where

$$y_0(t) := \max\{x \in \mathbb{R} : u(x,t) = \alpha_0\}, \quad y_1(t) := \min\{x \in \mathbb{R} : u(x,t) = \alpha_0\}. \tag{6.60}$$

Remark 6.20. (i) Recall that neither of the possibilities $\gamma_0 > 0$ (meaning that 0 is unstable from above for the equation $\dot{\theta} = f(\theta)$), $\gamma_1 < \gamma$ (meaning that γ is unstable from below) is allowed in Theorems 2.7, 2.9. Hence if any of them occurs, then according to the assumption of Lemma 6.19 the hypotheses of Theorem 2.17 are in effect.

(ii) Also recall that if $\gamma_0 > 0$, then $J = (0, \gamma_0) \in \mathcal{N}$ and $c_J > 0$ (see the remarks preceding Theorem 2.4). Hence $J \in \mathcal{N}^+$. Likewise, if $\gamma_0 < \gamma$, then then $J = (\gamma_1, \gamma) \in \mathcal{N}^-$.

(iii) If $\mathcal{N}^0 \neq \emptyset$, then the hypotheses of any one of Theorems 2.7, 2.9, 2.17 imply that one of the following combinations of assumptions holds:

(e1) $\mathcal{N}^+ \neq \emptyset$ (which includes the case $\gamma_0 > 0$) and $\mathcal{N}^- \neq \emptyset$ (which includes the case $\gamma_1 < \gamma$);

(e2) $\mathcal{N}^+ \neq \emptyset$ and (Z1) holds for each $I \in \mathcal{N}^0$.

(e3) $\mathcal{N}^- \neq \emptyset$ and (Z0) holds for each $I \in \mathcal{N}^0$.

(e4) $\mathcal{N}^+ = \mathcal{N}^- = \emptyset$ and for each $I = (a, b) \in \mathcal{N}$ conditions (2.21), (2.22) hold for some $\alpha = \beta \in (a, b) \cap f^{-1}\{0\}$.

Indeed, the hypotheses of Theorem 2.7 (and $\mathcal{N}^0 \neq \emptyset$) yield (e1); any of the assumptions (a1), (a2), (a3) of Theorem 2.9 gives (e2), (e3), or (e4), respectively; and the hypotheses of Theorem 2.17 imply that one of the conditions (e1), (e2), (e3) holds.

If $I \in \mathcal{N}^0$ and $\alpha, \beta \in I \cap f^{-1}\{0\}$, consider the following relations:

$$\limsup_{x\to\infty} u(x,t) \leq \min_{x<y_0(t)} u(x,t), \quad \text{where } y_0(t) = \max\{x \in \mathbb{R} : u(x,t) = \alpha\}, \tag{6.61}$$

$$\liminf_{x\to-\infty} u(x,t) \geq \max_{x>y_1(t)} u(x,t) \quad \text{where } y_1(t) = \min\{x \in \mathbb{R} : u(x,t) = \beta\}. \tag{6.62}$$

Notice that for $t = 0$, these are the same relations as (2.21), (2.22). We will establish the following "invariance properties:"

Lemma 6.21. *Suppose that $I \in \mathcal{N}^0$ and $\alpha, \beta \in I \cap f^{-1}\{0\}$. Let $t_0 \in [0, \infty)$. If* (6.61) *holds for $t = t_0$, then it holds with the strict inequality for each $t > t_0$. If* (6.62) *holds for $t = t_0$, then it holds with the strict inequality for each $t > t_0$.*

Before proving this statement, we show how it implies Lemma 6.19.

Proof of Lemma 6.19. If $\mathcal{N}^0 = \emptyset$, there is nothing to prove. We proceed assuming that $\mathcal{N}^0 \neq \emptyset$ and fix an arbitrary $I = (a, b) \in \mathcal{N}^0$.

Lemma 6.21 in particular implies that if the relation (2.21) in (Z0) holds, then (6.61) holds (with the same α) for all $t \geq 0$; and if the relation (2.22) in (Z1) holds, then (6.61) holds for all $t \geq 0$.

Next we show that (6.61) holds for each $\alpha \in f^{-1}\{0\} \cap (a,b)$ and $t > 0$ sufficiently large if $\mathcal{N}^+ \neq \emptyset$. (This applies in particular if $\gamma_0 > 0$, see Remark 6.20(ii).) By Proposition 3.11(iii),(iv), the set $\mathcal{N}^+ \neq \emptyset$ contains an interval $J := (0, b_0)$ for some $b_0 > 0$. Of course, if $\gamma_0 > 0$, then $J := (0, \gamma_0)$. Since $I = (a,b) \in \mathcal{N}^0$, relations (2.13), (2.12) give $b_0 \leq a$. We now apply Lemma 6.14 to the interval J, which is justified by Corollary 6.17(ci),(cii) ((ci) applies if $\gamma_0 = 0$, for in this case $J \in \tilde{\mathcal{N}} \cap \mathcal{N}^+$; and (cii) applies if $\gamma_0 > 0$ due to the hypotheses of Theorem 2.17, cp. Remark 6.20(i)). Fixing any $\theta \in (0, b_0)$, Lemma 6.14 implies (see (6.37)) that there is $s_0 \geq 0$ such that for all $t > s_0$ the function $u(\cdot, t) - \theta$ has exactly one zero. Making s_0 larger, if necessary, for all $t > s_0$ we also have $\theta > \limsup_{x\to\infty} u(x,t)$ (cp. Lemma 6.1). It is then clear that for all $t > s_0$ and $\alpha \geq \theta$ the relation (6.61) is satisfied. This in particular applies to any $\alpha \in (a,b) \cap f^{-1}\{0\}$, since $\theta < b_0 \leq a$, and we have the desired conclusion.

Similarly one shows that if $\gamma_1 = \gamma$ and $\mathcal{N}^- \neq \emptyset$, or $\gamma_1 < \gamma$, then (6.62) holds for each $\beta \in f^{-1}\{0\} \cap (a,b)$ if t is sufficiently large.

Using the above conclusions in conjunction with Lemma 6.21 (and remembering that $(a,b) \cap f^{-1}\{0\} \neq \emptyset$ because ϕ_I is a standing wave with range (a,b)), one shows easily that any of the conditions (e1)–(e4) stated in Remark 6.20(iii) implies that both relations (6.61), (6.62) hold with some $\alpha, \beta \in I \cap f^{-1}\{0\}$ with $\alpha \leq \beta$, if t is large enough. As already noted in Section 2.2, condition (6.61) remains valid if α is replaced by any larger element of $I \cap f^{-1}\{0\}$. Thus, we can take $\alpha = \beta := \alpha_0$ and then, the strict inequalities in (6.61), (6.62), as provided by Lemma 6.21, yield (6.59), (6.60). $\square$

Proof of Lemma 6.21. We only prove the invariance property for (6.62); the proof for (6.61) is analogous. To start with, we recall that for all $t \geq 0$ one has

$$\liminf_{x\to-\infty} u(x,t) \in D_\gamma. \tag{6.63}$$

This follows from the standing hypothesis (2.2) (see the proof of Lemma 6.1). In particular, since $f(\beta) = 0$ (β is as in (6.62)),

$$\liminf_{x\to-\infty} u(x,t) > \beta \quad (t \geq 0). \tag{6.64}$$

Assuming that (6.62) holds for $t = t_0$, set

$$\eta_0 := \liminf_{x\to-\infty} u(x,t_0). \tag{6.65}$$

By (6.62) and (6.64),

$$\eta_0 \geq \max_{x \geq y_1(t_0)} u(x, t_0), \quad \eta_0 > \beta. \tag{6.66}$$

Let $\eta(t)$ be the solution of $\dot{\eta} = f(\eta)$ with $\eta(t_0) = \eta_0$. Then, obviously, $\eta(t) > \beta$ for all $t \geq t_0$. Moreover,

$$\eta(t) \leq \liminf_{x \to -\infty} u(x, t) \quad (t \geq t_0). \tag{6.67}$$

To show this, take a continuous nonincreasing bounded function $\bar{u}_0$ such that $\bar{u}_0 \leq u_0$ and $\bar{u}_0(-\infty) = \eta_0$. Let $\bar{u}$ be the solution of (1.1) with the initial condition $\bar{u}(\cdot, t_0) = \bar{u}_0$. As in the proof of Lemma 6.1, one has $\bar{u}(-\infty, t) = \eta(t)$ and $\bar{u}(\cdot, t) \leq u(\cdot, t)$ for all $t > t_0$. This implies (6.67).

To complete the proof of Lemma 6.21, it is now sufficient to prove that for any $\bar{t} > t_0$ one has

$$\eta(\bar{t}) > \max_{x \geq y_1(\bar{t})} u(x, \bar{t}). \tag{6.68}$$

We distinguish two cases: $t_0 > 0$ and $t_0 = 0$. Assume first that $t_0 > 0$. Applying Lemma 5.4 to $v := u - \beta$, we find a continuous function σ on $[t_0, \bar{t}]$ such that $\sigma(\bar{t}) = y_1(\bar{t})$ and $u(\sigma(t), t) = \beta$ for all $t \in [t_0, \bar{t}]$. In particular, $\sigma(t_0) \geq y_1(t_0)$ and therefore, by (6.66),

$$\eta(t_0) = \eta_0 \geq u(x, t_0) \quad (x \geq \sigma(t_0)). \tag{6.69}$$

Further, for all $t \in [t_0, \bar{t}]$ we have $\eta(t) > \beta = u(\sigma(t), t)$. This, (6.69), and the comparison principle imply that $\eta(t) > u(x, t)$ for all (x, t) in the set $\{(x, t) : x \geq \sigma(t),\ t \in [t_0, \bar{t}]\}$. In particular, from $\sigma(\bar{t}) = y_1(\bar{t})$ we obtain that (6.68) holds.

Now we treat the case $t_0 = 0$. It is sufficient to prove that (6.68) holds for all sufficiently small $\bar{t} > 0$ (and then use the previous conclusion for a small positive t_0). Since $\eta(0) = \eta_0 > \beta = u(y_1(0), 0)$ and u is continuous on compact subsets of $\mathbb{R} \times [0, \infty)$, there exist $\epsilon > 0$, $z_1 < y_1(0)$, and $t_1 > 0$ such that $u(x, t) < \eta_0 - \epsilon$ for all $(x, t) \in [z_1, y_1(0)] \times [0, t_1]$. Making $t_1 > 0$ smaller if necessary, we also have $\eta(t) > \eta_0 - \epsilon$ for $t \in [0, t_1]$. Using these relations and (6.62) with $t = 0$, we obtain, via the comparison principle, that

$$\eta(t) > u(x, t) \quad ((x, t) \in [z_1, \infty] \times (0, t_1]). \tag{6.70}$$

Now, for all sufficiently small $t \geq 0$ one has $y_1(t) \geq z_1$. Indeed, if not, then there is a sequence $t_n \searrow 0$ such that $y_1(t_n) \leq z_1$. Since $u(y_1(t), t) = \beta$

for all t, it is not difficult to show that the sequence $\{y_1(t_n)\}$ is bounded (for example, this can be shown using (6.64) and a comparison with a spatially decreasing solution $\bar{u}$ as above). Thus, passing to a subsequence we may assume that $y_1(t_n) \to z_0$ for some $z_0 \leq z_1 < y_1(0)$. The continuity of u on compact subsets of $\mathbb{R} \times [0, \infty)$ then implies that $u_0(z_0) = \beta$, contradicting the minimality of $y_1(0)$ (see (6.62)). Thus, indeed, we have $y_1(t) \geq z_1$ for all $t \in [0, t_1]$, possibly after making t_1 smaller. This and (6.70) imply that (6.68) holds for all sufficiently small $\bar{t} > 0$. This completes the proof. □

Proof of Lemma 6.18. If $\mathcal{N}^0 = \emptyset$, there is nothing to be proved, thus we assume $\mathcal{N}^0 \neq \emptyset$. Fix any $I \in \mathcal{N}^0$ and let t_0 and α_0 be as in Lemma 6.19.

We prove Lemma 6.18 by contradiction. Suppose that $\hat{\phi}_I \in \Omega(u)$. This means that there are are $x_n \in \mathbb{R}$, $t_n > 0$, $n = 1, 2, \ldots$ such that $t_n \to \infty$ and $u(\cdot + x_n, t_n) \to \hat{\phi}_I$ in $C^1_{loc}(\mathbb{R})$. Remember that $\hat{\phi}_I' > 0$ and the range of $\hat{\phi}_I$ is the interval I containing α_0. Therefore, denoting by z the unique zero of $\hat{\phi}_I - \alpha_0$, for all sufficiently large n there is an interval $J_n := [z_n - d_n, z_n + d_n]$, with $z_n \approx z$ and $d_n \to \infty$, such that $u(\cdot + x_n, t_n) - \alpha_0$ is increasing on J_n and vanishes at z_n. Using this and Lemma 6.1, we infer that for all large enough n, the function $u(\cdot, t_n) - \alpha_0$ has at least three zeros whose mutual distances go to infinity as $n \to \infty$. The monotonicity of the zero number (see Lemma 5.1) implies that $z(u(\cdot, t) - \alpha_0) \geq 3$ for all $t > 0$. By Lemma 6.1 and Corollary 5.3, we can make $t_0 > 0$ larger so that for each $t \geq t_0$, the zeros of $u(\cdot, t) - \alpha_0$ are all simple, and their number, say k, is finite and independent of t. We denote by $\vartheta_1(t) < \cdots < \vartheta_k(t)$ the zeros of $u(\cdot, t) - \alpha_0$ for $t \geq t_0$. Since they are simple, the functions $\vartheta_1, \ldots \vartheta_k$ are C^1 on $[t_0, \infty)$. Moreover, as noted above, $k \geq 3$ and

$$\vartheta_k(t_n) - \vartheta_1(t_n) \to \infty. \tag{6.71}$$

Obviously, $\vartheta_1(t) = y_1(t)$, $\vartheta_k(t) = y_0(t)$, where $y_1(t)$, $y_0(t)$ are as in (6.60).

Taking t_0 as in Lemma 6.19 and using relations (6.59), one easily constructs a smooth decreasing function $\bar{u}_0$ such that

$$\begin{aligned} \liminf_{x \to -\infty} u(x, t_0) > \lim_{x \to -\infty} \bar{u}_0(x) &> \max_{\vartheta_1(t_0) \leq x \leq \vartheta_k(t_0)} u(x, t_0), \\ \limsup_{x \to \infty} u(x, t_0) < \lim_{x \to \infty} \bar{u}_0(x) &< \min_{\vartheta_1(t_0) \leq x \leq \vartheta_k(t_0)} u(x, t_0). \end{aligned} \tag{6.72}$$

Clearly, if η is large enough, then

$$u(x, t_0) > \bar{u}_0(x + \eta) \quad (x \leq \vartheta_k(t_0)), \tag{6.73}$$

$$u(x, t_0) < \bar{u}_0(x - \eta) \quad (x \geq \vartheta_1(t_0)). \tag{6.74}$$

Let $\bar{u}$ be the solution of (1.1) on (t_0, ∞) with the initial condition $\bar{u}(\cdot, t_0) = \bar{u}_0$. Then $\bar{u}(x,t)$ is continuous on $\mathbb{R} \times [t_0, \infty)$ and decreasing in x. By (6.72), the relations $\bar{u}(\infty, t) < \alpha_0 < \bar{u}(-\infty, t)$ hold for $t = t_0$, hence they continue to hold for all for all $t \geq t_0$ (see the proof of Lemma 6.1). Therefore, for each $t \geq t_0$ the function $\bar{u}(x,t) - \alpha_0$ has a unique zero $\xi(t)$ and $t \mapsto \xi(t)$ is continuous on $[t_0, \infty)$.

Consider now the relations

$$\xi(t) - \eta < \vartheta_1(t), \quad \vartheta_k(t) < \xi(t) + \eta. \tag{6.75}$$

They are both satisfied for $t = t_0$ (use the monotonicity of $u(\cdot, t)$ and the relations (6.73), (6.74), with $x = \vartheta_1(t_0)$, $x = \vartheta_k(t_0)$, respectively). By continuity, they are also satisfied it $t > t_0$ is sufficiently close to t_0. On the other hand, by (6.71), the relations (6.75) cannot be satisfied for all $t > t_0$. Thus, there is $t_1 > t_0$ such that relations (6.75) hold for all $t \in [t_0, t_1)$ and either $\xi(t_1) - \eta = \vartheta_1(t_1)$ or $\vartheta_k(t_1) = \xi(t_1) + \eta$. Assume that the former holds (the latter can be dealt with in an analogous way). Then

$$\bar{u}(\vartheta_1(t_1) + \eta, t_1) = \bar{u}(\xi(t_1), t_1) = \alpha_0 = u(\vartheta_1(t_1), t_1). \tag{6.76}$$

Since $\xi(t) - \eta$ is the unique zero of the decreasing function $\bar{u}(\cdot + \eta, t) - \alpha_0$ and $\vartheta_k(t) > \vartheta_1(t)$, the first relation in (6.75) yields

$$\bar{u}(\vartheta_k(t) + \eta, t) < \alpha_0 = u(\vartheta_k(t), t) \quad (t_0 \leq t \leq t_1). \tag{6.77}$$

Using this, (6.73), and the strong comparison principle, we obtain

$$\bar{u}(x + \eta, t) < u(x,t) \quad (x < \vartheta_k(t),\, t_0 \leq t \leq t_1), \tag{6.78}$$

contradicting (6.76). This contradiction shows that $\hat{\phi}_I \in \Omega(u)$ is impossible. □

6.8 Completion of the proofs of Theorems 2.11 and 2.19

Recall that for each $I \in \mathcal{N}$, ϕ_I was chosen so that $\phi_I(0) = (a+b)/2$.

The essential part of the proofs of Theorems 2.11, 2.19, the existence of functions ζ_I, is provided by Lemma 6.14. More precisely, we have the following result, which follows directly from Lemma 6.14 and Corollary 6.17.

Corollary 6.22. *Let $I = (a,b) \in \mathcal{N}$. Assume that one of the conditions (ci)–(civ) stated in Corollary 6.17 holds. Then there is a C^1 function ζ_I defined on some interval (s_I, ∞) such that the following statements hold:*

(j) $\lim_{t\to\infty} \zeta_I'(t) = 0 \quad (I \in \mathcal{N})$;

(jj) $\big((a+b)/2 - u(x+ct+\zeta(t),t)\big)x > 0 \quad (x \in \mathbb{R}\setminus\{0\},\ t > s_I)$;

(jjj) $\lim_{t\to\infty} u(\cdot + c_I t + \zeta_I(t), t) - \phi_I = 0$, *locally uniformly on* $\mathbb{R}$.

Recalling also that $\mathcal{N} = \tilde{\mathcal{N}}$ under hypothesis (S) assumed in Theorem 2.11, Corollary 6.22 proves the validity of statements (i) and (ii) of Theorem 2.11, except for assertions (d) and (e); as well as the validity of statements (iii), (iv), and (v) of Theorem 2.19, except for assertions (d) and (e).

Assertion (d) concerns elements $I_1, I_2 \in \mathcal{N}$, with $I_1 < I_2$, $c_{I_1} = c_{I_2}$, and with additional restrictions, as imposed in the above statements: $c_{I_2} \neq 0$ in Theorem 2.11(i) and Theorem 2.19(iii), and $I_1, I_2 \in \tilde{\mathcal{N}}$ in Theorem 2.19(iii),(iv) (no restrictions in Theorem 2.11(ii) and Theorem 2.19(v)). Note that under these restrictions or the additional assumptions made in the above statements ((2.16) in Theorem 2.11(ii) and Theorem 2.19(iii),(iv); (U) in Theorem 2.19(v)), one of the conditions (ci)–(civ) of Corollary 6.17 is satisfied when we take $I = I_1$ or $I = I_2$. Hence, Corollary 6.22 applies to $I = I_1$ and $I = I_2$. We use this to prove that $\zeta_{I_1}(t) - \zeta_{I_2}(t) \to \infty$ as $t \to \infty$, as stated in assertions (d) of Theorems 2.11, 2.19.

First we note that for any sequence $t_n \to \infty$, the sequence $\{\zeta_{I_1}(t_n) - \zeta_{I_2}(t_n)\}$ cannot be bounded. Indeed, by Corollary 6.22, the boundedness and the relation $c_{I_1} = c_{I_2}$ would give

$$\begin{aligned} &u(c_{I_1}t_n + \zeta_{I_1}(t_n), t_n) - \phi_{I_2}(\zeta_{I_1}(t_n) - \zeta_{I_2}(t_n)) \\ &\quad = u(\zeta_{I_1}(t_n) - \zeta_{I_2}(t_n) + c_{I_2}t_n + \zeta_{I_2}(t_n), t_n) - \phi_{I_2}(\zeta_{I_1}(t_n) - \zeta_{I_2}(t_n)) \to 0. \end{aligned}$$

At the same time,

$$u(c_{I_1}t_n + \zeta_{I_1}(t_n), t_n) \to \phi_{I_1}(0),$$

which is absurd, as the ranges I_1, I_2 of ϕ_{I_1}, ϕ_{I_1} are open and do not overlap.

The relations $I_1 < I_2$ and $c_{I_1} = c_{I_2}$ further imply that $\zeta_{I_1}(t) - \zeta_{I_2}(t)$ cannot be negative for any sufficiently large t. Indeed, that would imply that $u(\cdot, t)$ has to increase from the value $\theta_1 := u(c_{I_1}t + \zeta_{I_1}(t), t)$ at the center of I_1 to the greater value $u(c_{I_2}t + \zeta_{I_2}(t), t)$ at the center of I_2 . If that occurred for a sequence of times $t = t_n \to \infty$, we would obtain an element of $(\theta_2, \eta) \in K_\Omega(u) \cap S_{I_2}$ with $\eta \geq 0$. However, we already know, by Corollary 6.22, that there is no such element. Thus, we have proved that $\zeta_{I_1}(t) - \zeta_{I_2}(t) \to \infty$, as desired.

We proceed by noting that statements (i) and (ii) of Theorem 2.19 follow directly from Lemma 6.3 (and the relations $c_{I_*} > 0$, $c_{I^*} < 0$ mentioned at the beginning of Section 2.3).

To complete the proofs of statement (ii) of Theorem 2.11 and statement (v) of Theorem 2.19, we need to prove assertion (e). Note that, under the assumptions of these statements, Corollary 6.22 now applies to all $I \in \mathcal{N}$. Let $\{(x_n, t_n)\}$ be any sequence in $\mathbb{R}^2$ such that $t_n \to \infty$ and for each $I \in \mathcal{N}$ one has

$$\lim_{n\to\infty} |c_I t_n + \zeta_I(t_n) - x_n| = \infty. \tag{6.79}$$

Then, passing to a subsequence, we may assume that $u(\cdot + x_n, t_n) \to \varphi$ in $L^\infty_{loc}(\mathbb{R})$, for some $\varphi \in \Omega(u)$. By Theorems 2.7, 2.9, 2.15, either $\varphi \equiv \xi$ for some $\xi \in R_0^{-1}\{0\}$, or $\varphi = \phi_I(\cdot + \eta)$ for some $I \in \mathcal{N}$ and $\eta \in \mathbb{R}$. The former is the desired conclusion; we need to rule out the latter. Assume that it holds with $I = (a, b) \in \mathcal{N}$. Since we also have $u(\cdot + ct_n + \zeta_I(t_n), t_n) \to \phi_I$ in $L^\infty_{loc}(\mathbb{R})$ (see Corollary 6.22(jjj)), assumption (6.79) clearly implies that the function $u(\cdot, t_n) - (a + b)/2$ has at least two zeros, in contradiction to Corollary 6.22(jj).

It remains to prove statement (iii) of Theorem 2.11 and statement (vi) of Theorem 2.19. These proofs are nearly identical, so for definiteness we just give the proof for statement (iii) of Theorem 2.11. Thus, we assume that (2.16) holds and the set $R_0^{-1}\{0\}$ is finite: $R_0^{-1}\{0\} = \{a_1, \dots, a_{k+1}\}$, with $0 = a_1 < a_2 < \dots < a_{k+1} = \gamma$. Then, $\mathcal{N} = \{I_1, \dots, I_k\}$ with $I_j = (a_j, a_{j+1})$, $j = 1, \dots, k$, and all the functions $\zeta_j(t) := \zeta_{I_j}(t)$, $j = 1, \dots, k$, are defined for sufficiently large t. Moreover, the relations $c_{I_j} \geq c_{I_{j+1}}$ (cp. (2.12)) and the properties of the functions ζ_j proved above imply that, as $t \to \infty$, one has

$$c_{I_j} t + \zeta_j(t) - (c_{I_{j+1}} t + \zeta_{j+1}(t)) \to \infty \quad (j = 1, \dots, k). \tag{6.80}$$

To prove that (2.33) holds we go by contradiction. Assume it does not, that is, there exist sequences $x_n \in \mathbb{R}$ and $t_n \to \infty$ such that

$$\Big|u(x_n, t_n) - \Big(\sum_{j=1,\dots,k} \phi_{I_j}(x_n - c_{I_j} t_n - \zeta_{I_j}(t_n)) - \sum_{1\leq j\leq k-1} a_{j+1}\Big)\Big| \geq \epsilon, \tag{6.81}$$

for some $\epsilon > 0$. Passing to a subsequence of $\{(x_n, t_n)\}$, we may assume that there is i such that for each n the point x_n is contained in the i-th interval in the following sequence of intervals

$$\begin{aligned}
&(-\infty,\, c_{I_k}t_n+\zeta_k(t_n)],\\
&[c_{I_{j+1}}t_n+\zeta_{j+1}(t_n),\, c_{I_j}t_n+\zeta_j(t_n)], \quad j=1,\dots,k-1,\\
&[c_{I_1}t_n+\zeta_1(t_n),\, \infty).
\end{aligned} \tag{6.82}$$

Passing to a further subsequence, we obtain that either there is exactly one j such $c_{I_j}t_n+\zeta_j(t_n)-x_n$ converges to a finite value, or for all j one has $|c_{I_j}t_n+\zeta_j(t_n)-x_n|\to\infty$. Finally, passing to a yet further subsequence, we may assume that $u(\cdot+x_n,t_n)\to\varphi$ for some $\varphi\in\Omega(u)$.

Consider the case when for each n the point x_n belongs to the first of the intervals (6.82), and let

$$\rho := \lim_{n\to\infty}\big(x_n-(c_{I_k}t_n+\zeta_k(t_n))\big)\in[-\infty,0].$$

Then for $j<k$ one has

$$\phi_{I_j}(x_n-c_{I_j}t_n-\zeta_j(t_n))\to\phi_{I_j}(-\infty)=a_{j+1}. \tag{6.83}$$

If now $\rho>-\infty$, then, by Corollary 6.22(jjj), $u(x_n,t_n)-\phi_{I_k}(x_n-c_{I_k}t_n-\zeta_k(t_n))\to 0$. Thus, the limit of the left-hand side of (6.81) is 0, and we have a contradiction. If $\rho=-\infty$, then (6.83) also holds for $j=k$ (and $a_{k+1}=\gamma$). Also, by statement (e) already proved above, for the limit function φ we have $\varphi\equiv\xi\in R_0^{-1}\{0\}$. If $\xi=\gamma$, then, again, the limit of the left-hand side of (6.81) is 0, and we have a contradiction. We show that the case $\xi<\gamma$ leads to a contradiction as well. Suppose it holds. Then for large n one has $u(x_n,t_n)\approx\xi$, whereas for large negative x one has $u(x,t_n)\approx\gamma$ (see Lemma 6.1). Thus, for each large enough n, there is $\tilde{x}_n<x_n$ such that $u(\tilde{x}_n,t_n)=\gamma-\delta$ where $\delta>0$ is chosen so small that $\gamma-\delta>a_j$, $j=1,\dots,k$. In particular, $\gamma-\delta\notin R_0^{-1}\{0\}$. However, statement (e) of Theorems 2.11, 2.19, applies to $\tilde{x}_n$ as well and we get, possibly after passing to another subsequence, that $u(x+\tilde{x}_n,t_n)\to\tilde{\xi}\in R_0^{-1}\{0\}$ for each $x\in\mathbb{R}$. Taking $x=0$, we obtain $\gamma-\delta=\xi\in R_0^{-1}\{0\}$ and we have a contraction.

The proof in the case when for each n the point x_n belongs to the i-the interval in (6.82), for $i=2,\dots,k+1$, is similar and we omit its details.

6.9 Proof of Theorem 2.22

Throughout this subsection, we assume that the hypotheses of the theorem are satisfied and $I:=(a,b)$ is as in the hypotheses.

Similarly as in [31], we shall use properties of solutions of an asymptotically autonomous equation

$$v_t = v_{xx} + c_I v_x + f(v) + h(x,t), \quad x \in \mathbb{R},\ t > 0. \tag{6.84}$$

Here h is a uniformly continuous function on $\mathbb{R} \times [0,\infty)$ such that for some positive constants κ and σ one has

$$\|h(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \kappa e^{-\sigma t} \quad (t \ge 0). \tag{6.85}$$

Since $f'(a) < 0$, $f'(b) < 0$, classical results (see [11, 33]) show that the family $\phi_I(x - c_I t - \eta)$, $\eta \in \mathbb{R}$, of fronts of the bistable nonlinearity of $f\Big|_{[a,b]}$ is asymptotically stable with asymptotic phase. As observed in [31], a form of this conclusion remains valid when an exponentially decaying inhomogeneity is added in the equation. Namely, we have the following lemma.

Lemma 6.23. *Under the above assumptions on h, assume that v is a solution of* (6.84) *such that*

$$\inf_{\eta \in \mathbb{R}} \|v(\cdot,t) - \phi_I(\cdot - \eta)\|_{L^\infty(\mathbb{R})} \to 0, \quad \text{as } t \to \infty. \tag{6.86}$$

Then there are $\eta \in \mathbb{R}$ and $\vartheta > 0$ such that

$$\lim_{t\to\infty} e^{\vartheta t} \|v(\cdot,t) - \phi_I(\cdot - \eta)\|_{L^\infty(\mathbb{R})} = 0. \tag{6.87}$$

This lemma can be proved, for example, by estimates very similar to those given in Exercise 6 in [18, Section 5.1]. Also, the lemma is a consequence of [31, Theorem 3.1]. Although, we do not assume here that $a \le v \le b$, which is an assumption in [31, Theorem 3.1], one can use the assumptions $f'(a) < 0$, $f'(b) < 0$ and a simple modification of v to reduce the proof of Lemma 6.23 to the case $a \le v \le b$. We omit the details.

We shall also use the following lemma.

Lemma 6.24. *Let u be a solution of* (1.1), (1.2), *where $u_0 \in C(\mathbb{R})$ satisfies* (2.2), (2.3). *There is a C^1 function ζ_I defined on some interval on (s_I, ∞) such that the following statements hold:*

(a) $\lim_{t\to\infty} \zeta_I'(t) = 0$;

(b) $\big((a+b)/2 - u(x + c_I t + \zeta_I(t), t)\big)x > 0 \quad (x \in \mathbb{R} \setminus \{0\},\ t > s_I)$;

(c) $\lim_{t\to\infty} u(\cdot + c_I t + \zeta_I(t), t) - \phi_I = 0$, *locally uniformly on* $\mathbb{R}$*;*

Moreover, for each $\theta \in (a,b)$ *there is* s_θ *such that*

(d) $z(u(\cdot,t) - \theta) = 1 \quad (t > s_\theta)$.

If $c_I \neq 0$, statements (a)-(c) are obtained from Theorem 2.19(iii), and statement (d) from Lemma 6.14. We will just need to prove that under the present assumptions the statements hold if $c_I = 0$. We will do it at the end of this subsection. First we prove that if the conclusion of the lemma holds for a solution u, then the conclusion of Theorem 2.22 holds for this solution.

Completion of the proof of Theorem 2.22. Let u be as in Lemma 6.24. Set

$$\tilde{u}(x,t) = u(x + c_I t, t),$$

so that $\tilde{u}$ is a solution of

$$\tilde{u}_t = \tilde{u}_{xx} + c_I \tilde{u}_x + f(\tilde{u}), \quad x \in \mathbb{R},\ t > 0. \tag{6.88}$$

In order to prove Theorem 2.22, we find a function v satisfying the hypotheses of Lemma 6.23 and such that for some $t_0 > 0$ one has

$$v(x,t) = \tilde{u}(x,t) \quad ((c_I^- - c_I)t < x < (c_I^+ - c_I)t,\ t \geq t_0), \tag{6.89}$$

where $c_I^- < c_I < c_I^+$ are as in (2.41). This and Lemma 6.23 clearly yield conclusion (2.43) of Theorem 2.22. The remaining conclusions, (2.42), (2.44) will be verified in the process of establishing (6.89).

First, we show that there are positive constants s, ϵ, k, and ς such that

$$u(x,t) \leq b + ke^{-\varsigma t} \quad (x > (c_I^- - \epsilon)t,\ t \geq s), \tag{6.90}$$
$$u(x,t) \geq b - ke^{-\varsigma t} \quad (x < (c_I^- + \epsilon)t,\ t \geq s), \tag{6.91}$$

and

$$u(x,t) \geq a - ke^{-\varsigma t} \quad (x < (c_I^+ + \epsilon)t,\ t \geq s), \tag{6.92}$$
$$u(x,t) \leq a + ke^{-\varsigma t} \quad (x > (c_I^+ - \epsilon)t,\ t \geq s). \tag{6.93}$$

We prove estimates (6.90), (6.91) by comparison arguments. Analogous arguments show that (6.92), (6.93) hold and this part will be omitted. Of course, making adjustment to the constants, if necessary, we may take the same ς, ϵ, k, s in all four estimates.

Assume first that $b = \gamma$. Then, the assumptions (2.2) and $f'(b) < 0$, imply, via comparison with a solution of the equation $\dot{\theta} = f(\theta)$, that for some constants k, $\varsigma > 0$ one has $\tilde{u}(x,t) \le b + ke^{-\varsigma t}$ for all $x \in \mathbb{R}$ and $t > 0$. Thus (6.90) holds with arbitrary $s > 0$ and $\epsilon > 0$ and there is nothing to be proved in (6.91) (recall that $c_I^- = -\infty$ if $b = \gamma$).

Assume that $b < \gamma$, so that $\bar{I}$ is defined (cp. (2.40)) and the quantities $c_{\bar{I}} < c_I$ and $c_I^- = (c_{\bar{I}} + c_I)/2$ are finite. We have $\bar{I} = (\bar{a}, \bar{b})$ with $\bar{a} = b$ and some $\bar{b} \in (b, \gamma]$. Take any $c \in (c_{\bar{I}}, c_I^-)$. We apply Lemma 3.32(i) to the interval $\bar{I} = (\bar{a}, \bar{b})$. This yields a solution $\bar{\psi}^c$ of (3.1) such that

$$\bar{\psi}^c_x(x) < 0 \ \ (x \in \mathbb{R}), \quad \bar{\psi}^c(x) > \gamma \ \ (x < x_0), \quad \lim_{x\to\infty} \bar{\psi}^c(x) = \bar{a} = b \tag{6.94}$$

(see (3.89) and (6.5)). Since $f'(b) < 0$, the convergence in (6.94) is exponential.

Using (6.94) and Lemma 6.1, one shows easily that if s is sufficiently large, then there is $y > 0$ such that the following estimate holds for $t = s$:

$$u(x,t) < \psi^c(x - ct - y) \quad (x \in \mathbb{R}).$$

By the comparison principle, this estimate continues to hold for all $t \ge s$. In particular, since ψ^c is decreasing, if $0 < \epsilon < c_I^- - c$, we have

$$u(x,t) < \psi^c((c_I^- - \epsilon)t - ct - y) \quad (x \ge (c_I^- - \epsilon)t).$$

The exponential convergence in (6.94) and the fact that $c_I^- - \epsilon - c > 0$ imply that (6.90) holds if s, k are sufficiently large and $\varsigma > 0$, ϵ are sufficiently small.

We now prove the lower estimate (6.91). Fix $c \in (c_I^-, c_I)$. We use the fact that for suitable positive constants δ and β, the function $\phi_I(x - ct) - \delta e^{-\beta t}$ is a subsolution of equation (1.1) (see Lemma 6.25 below). Of course, the space and time translations of this function are subsolutions as well.

By Lemma 6.1, if t_0 is large enough, then

$$\liminf_{x\to-\infty} u(x,t_0) > \gamma - \delta \ge b - \delta.$$

We fix such t_0 satisfying also $t_0 > s_I$, with s_I as in Lemma 6.24. Since $\phi_I < b$, for all sufficiently large $\xi > 0$, we have

$$u(x,t_0) > \phi_I(y) - \delta \quad (x \le -\xi, \ y \in \mathbb{R}). \tag{6.95}$$

We choose ξ so large that (6.95) holds and in addition

$$-\xi + c(t - t_0) < c_I t + \zeta_I(t) \quad (t \geq t_0). \tag{6.96}$$

Such a choice is possible as $c < c_I$ and $\zeta_I(t)/t \to 0$ as $t \to \infty$ (see Lemma 6.24(a)). By Lemma 6.24(b) and the normalization $\phi_I(0) = (a+b)/2$ (cp. (2.10)),

$$u(x,t) > \frac{(a+b)}{2} = \phi_I(0) \quad (x < c_I t + \zeta_I(t),\ t \geq t_0). \tag{6.97}$$

By (6.96), this holds in particular, if $x = -\xi + c(t - t_0)$ and $t \geq t_0$. Thus the subsolution

$$U(x,t) := \phi_I(x - c(t - t_0) + \xi) - \delta e^{-\beta(t-t_0)}$$

satisfies $u(x,t) > U(x,t)$ if $t \geq t_0$ and $x = -\xi + c(t - t_0)$, or if $t = t_0$ and $x \leq -\xi$ (cp. (6.95)). Therefore, by the comparison principle,

$$u(x,t) \geq U(x,t) \quad (x \leq -\xi + c(t - t_0),\ t \geq t_0). \tag{6.98}$$

If now $\epsilon > 0$ is sufficiently small (so that $c_I^- + \epsilon < c$), then for all large enough t we have $(c_I^- + \epsilon)t < -\xi + c(t - t_0)$. Therefore, (6.98) and the fact that ϕ_I is decreasing yield the following relation

$$u(x,t) \geq \phi_I((c_I^- + \epsilon)t - c(t - t_0) + \xi) - \delta e^{-\beta(t-t_0)} \quad (x < (c_I^- + \epsilon)t). \tag{6.99}$$

Since $\phi_I(x) \to b$ as $x \to -\infty$ with an exponential rate (as $f'(b) < 0$) and $c_I^- + \epsilon - c < 0$, we obtain from (6.99) that (6.91) holds for suitable constants.

Thus relations (6.90)-(6.93) are valid.

Obviously, making s larger if necessary, we have, for all $t > s$,

$$\begin{aligned}(c_I^- t - 1, c_I^- t] &\subset ((c_I^- - \epsilon)t, (c_I^- + \epsilon)t),\\ [c_I^+ t, c_I^+ t + 1) &\subset ((c_I^+ - \epsilon)t, (c_I^+ + \epsilon)t\end{aligned}$$

(if $c_I^- = -\infty$, we define $(c_I^- t - 1, c_I^- t] = \emptyset$, similarly for $c_I^+ = \infty$). Now, the function $u - b$ solves a linear parabolic equation. Therefore, using (6.90)-(6.93), and parabolic estimates one obtains, possibly after making adjustments to the constants k, ς, and s, that

$$|u_x(x,t)| \leq k e^{-\varsigma t} \quad (x \in (c_I^- t - 1, c_I^- t] \cup [c_I^+ t, c_I^+ t + 1),\ t \geq s). \tag{6.100}$$

Take now a smooth function ρ on $\mathbb{R}$ such that $0 \leq \rho \leq 1$, $\rho \equiv 0$ on $(-\infty, 0)$, and $\rho \equiv 1$ on $(1, \infty)$. Define a function w on $\mathbb{R} \times (s, \infty)$ by

$$w(x,t) = \begin{cases} \big(1 - \rho(x - (c_I^- t - 1))\big)b + \rho(x - (c_I^- t - 1))u(x,t) & (x \leq c_I t), \\ \big(1 - \rho(x - c_I^+ t)\big)u(x,t) + \rho(x - c_I^+ t)a & (x \geq c_I t). \end{cases}$$

It is understood here that $\rho(-\infty) = 0$ so that in the case $c_I^- = -\infty$ we have $w(x,t) = u(x,t)$ on $(-\infty, c_I t)$. An analogous remark applies to the case $c_I^+ = \infty$. Notice that

$$v(x,t) := w(x + c_I t, t) = \begin{cases} b & (x < (c_I^- - c_I)t - 1), \\ \tilde{u}(x,t) & ((c_I^- - c_I)t < x < (c_I^+ - c_I)t), \\ a & (x > (c_I^+ - c_I)t + 1). \end{cases} \quad (6.101)$$

Clearly, v satisfies equation (6.84) with

$$h(x,t) = v_t(x,t) - v_{xx}(x,t) - c_I v_x(x,t) - f(v_t(x,t)).$$

Since $\tilde{u}$ is a bounded solution of (6.88), parabolic estimates imply that h is uniformly continuous. Clearly, h vanishes in the regions indicated in (6.101). In the remaining part of $\mathbb{R} \times (s, \infty)$, straightforward estimates using (6.90)-(6.93) and (6.100) (and the fact that $\tilde{u}$ is a solution of (6.88)) show that h satisfies (6.85) for some constants κ and σ.

Finally, using Lemma 6.24(c),(d), estimates (6.90)-(6.93), and the relations $\phi_I(-\infty) = b$, $\phi_I(\infty) = a$, one shows easily that

$$\lim_{t \to \infty} \|v(\cdot, t) - \phi_I(\cdot - \zeta_I(t))\|_{L^\infty(\mathbb{R})} = 0. \quad (6.102)$$

Thus v satisfies all hypotheses of Lemma 6.23 and (6.89) holds by (6.101). As noted at the beginning of the proof, this implies conclusion (2.43) of Theorem 2.22. Conclusions (2.42), (2.44) follow from (6.91), (6.93). The proof of Theorem 2.22 is now complete. □

Proof of Lemma 6.24. As noted above, if $c_I \neq 0$ the conclusions follow from Theorem 2.19(iii) and Lemma 6.14. If $c_I = 0$, the statements also follow from Lemma 6.14, provided we can show that $K_\Omega(u) \cap S_{(a.b)} = \tau(\phi_I)$. This is satisfied, by Corollary 6.17 if

$$\hat{\phi}_I \notin \Omega(u). \quad (6.103)$$

This obviously holds if u_0 is monotone nonincreasing, for then all elements of $\Omega(u)$ are such. Thus, Lemma 6.24 and Theorem 2.22 are proved under the extra condition that u_0 is monotone.

We now remove the monotonicity restriction on u_0. Given u_0, we choose two monotone initial data $\bar{u}_0, \underline{u}_0 \in C(\mathbb{R})$ satisfying conditions (2.2), (2.3), and such that $\underline{u}_0 \leq u_0 \leq \bar{u}_0$. Then, by the comparison principle, for the corresponding solutions of (1.1) we have $\underline{u} \leq u \leq \bar{u}$. As proved above, Theorem 2.22 applies to the functions $\underline{u}$, $\bar{u}$. Thus each of them satisfies relations (2.42)-(2.44). This and the relations $\underline{u} \leq u \leq \bar{u}$ imply that if t is sufficiently large, then there is no room for $u(\cdot, t)$ to be close to the decreasing function $\hat{\phi}_I$ on a large spatial interval. Hence, (6.103) holds. This completes the proof in the nonmonotone case. □

The following result was used in the proof of Theorem 2.22.

Lemma 6.25. *Let* $I = (a, b) \in \mathcal{N}$, $f'(a) < 0$, $f'(b) < 0$, *and* $0 < \beta < \min\{-f'(a), -f'(b)\}$. *If* $c < c_I$, *then there is* $\delta_0 > 0$ *such that for each* $\delta \in (0, \delta_0)$ *the function*

$$U(x, t) = \phi_I(x - ct) - \delta e^{-\beta t}, \quad (x, t) \in \mathbb{R} \times [0, \infty),$$

is a subsolution of (1.1). *Similarly, if* $c > c_I$, *then there is* $\delta_0 > 0$ *such that for each* $\delta \in (0, \delta_0)$ *the function*

$$U(x, t) = \phi_I(x - ct) + \delta e^{-\beta t}, \quad (x, t) \in \mathbb{R} \times [0, \infty),$$

is a supersolution of (1.1).

Proof. We prove the result for $c < c_I$, the proof for $c > c_I$ is analogous. We essentially repeat an argument from [11], which was used there to show that for suitable positive constants β, δ, and σ, the function $\phi_I(x - c_I t - \sigma\delta e^{-\beta t}) - \delta e^{-\beta t}$ is a subsolution.

To simplify the notation, set $\phi := \phi_I$. Remember that $\phi' < 0$ and $\phi'' + c_I\phi' + f(\phi) \equiv 0$. We have (omitting the argument $x - ct$ of ϕ)

$$\begin{aligned} U_t - U_{xx} - f(U) &= (c_I - c)\phi' + \delta\beta e^{-\beta t} + f(\phi) - f(\phi - \delta e^{-\beta t}) \\ &= (c_I - c)\phi' + \delta e^{-\beta t}(f'(q(x, t)) + \beta), \end{aligned} \tag{6.104}$$

where $\phi(x - ct) - \delta e^{-\beta t} \leq q(x, t) \leq \phi(x - ct)$.

Since $\beta < \min\{-f'(a), -f'(b)\}$, there is δ_1 such that $f'(v) + \beta < 0$ if $|v - a| < \delta_1$ or $|v - b| < \delta_1$. If $0 < \delta < \delta_1/2$, and x, t are such that $|\phi(x - ct) - a| < \delta_1/2$ or $|\phi(x - ct) - b| < \delta_1/2$, then the expression on the last line of (6.104) is negative. For the remaining values of x and t, the function $(c_I - c)\phi'(x - ct)$ is bounded above by a negative constant. Therefore if $\delta > 0$ is sufficiently small, $U_t - U_{xx} - f(U) \leq 0$ on $\mathbb{R} \times [0, \infty)$, that is, U is a subsolution. $\square$

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