Convergence to a steady state for asymptotically autonomous semilinear heat equations on $\mathbb{R}^N$

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Abstract

We consider parabolic equations of the form

$$u_t = \Delta u + f(u) + h(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty),$$

where f is a C^1 function with $f(0) = 0$, $f'(0) < 0$, and h is a suitable function on $\mathbb{R}^N \times [0,\infty)$ which decays to zero as $t \to \infty$ (hence the equation is asymptotically autonomous). We show that, as $t \to \infty$, each bounded localized solution $u \geq 0$ approaches a set of steady states of the limit autonomous equation $u_t = \Delta u + f(u)$. Moreover, if the decay of h is exponential, then u converges to a single steady state. We also prove a convergence result for abstract asymptotically autonomous parabolic equations.

Key words: parabolic equation; asymptotically autonomous; convergence; quasiconvergence.

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1 Introduction

We consider parabolic equations of the following form

$$u_t = \Delta u + f(u) + h(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,\infty)\,. \tag{1.1}$$

Here, f is a C^1 function on $\mathbb{R}$ with $f(0) = 0$, $f'(0) < 0$, and h is a suitable function on $\mathbb{R}^N \times [0,\infty)$ which decays to zero as $t \to \infty$. The decay means that the equation is asymptotically autonomous (both in space and time). Our goal is to examine how the presence of the nonautonomous term h affects convergence properties of nonnegative localized solutions of (1.1).

To motivate the problem, let us first assume $h \equiv 0$:

$$u_t = \Delta u + f(u), \quad (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{1.2}$$

Let u be a nonnegative global bounded solution of (1.2) which satisfies

$$\lim_{|x|\to\infty} \sup_{t\in(0,\infty)} u(x,t) = 0. \tag{1.3}$$

We emphasize that the spatial decay of u is required to be uniform with respect to time, in that sense u is a localized solution. Global solutions satisfying this requirement will typically converge to zero as $t \to \infty$; such are all nonnegative solutions strictly below a spatially decaying steady state or supersolution. Global solutions satisfying (1.3) which do not converge to zero are usually found as threshold solutions on the boundary of the domain of attraction of the asymptotically stable trivial solution, see for example [11, 12, 15, 16, 34, 37].

Under the above assumptions, it is known that the solution u converges, as $t \to \infty$, to a steady state φ of (1.2). The converge takes place in the supremum norm and the limit steady state φ, if nontrivial, is a ground state of the elliptic equation

$$\Delta\varphi + f(\varphi) = 0, \quad x \in \mathbb{R}^N\,. \tag{1.4}$$

We use the term *ground state* to refer to any positive solution φ of (1.4) such that $\varphi(x) \to 0$ as $|x| \to \infty$. Recall that any ground state is a radially symmetric and radially decreasing function with respect to some center in $\mathbb{R}^N$ (see [19, 28, 29]).

The convergence result for (1.2) quoted above was proved in [4] under the stronger assumption that the spatial decay in (1.3) is exponential (we remark that if u has the initial state $u_0 := u(\cdot, 0)$ with compact support, then the exponential decay is no extra restriction). There are earlier results in [11, 15] where specific nonlinearities were considered. In one dimension, that is for $N = 1$, convergence results are available for an even larger class of nonlinearities, see [12, 37] (earlier convergence results for specific one-dimensional problems can be found in [13, 14, 16, 17]). See also [25] and [16] for convergence results dealing with radial problems in higher space dimension and time-periodic problems on $\mathbb{R}$, respectively.

Let us now consider the nonautonomous problem: $h \not\equiv 0$. Even though the effect of the nonautonomous perturbation diminishes as $t \to \infty$, its presence renders key arguments of [4] unusable and different techniques have to be sought. In [10], Chill and Jendoubi succeeded in adapting energy arguments based on the concentrated compactness and Lojasiewicz inequality to asymptotically autonomous problems. For their arguments to apply to (1.1), rather restrictive hypotheses have to be made; in particular, it is assumed in [10] that $h(t, \cdot)$ has its support contained in a compact set independent of t. Also, as usual with techniques involving the Lojasiewicz inequality (see [7, 8, 35], for example), the nonlinearity f has to be of a very specific form or analytic; in [10] the nonlinearity f is chosen such that (1.4) has a unique radial ground state. Under these assumptions, the convergence of localized solutions to a ground state is proved in [10]. We remark, that techniques based on the Lojasiewicz inequality have also been used in [9, 23] in proofs of convergence results for asymptotically autonomous equations on bounded domains.

In this paper, we prove a convergence result for (1.1) using a completely different approach. It has three main ingredients:

I) Adapting some arguments from [6], we show that $\omega(u)$, the ω-limit set of the solution u, consists of steady states of (1.2). This amounts to showing that chain recurrent points of (1.2) are steady states. The key tool here is the energy functional of (1.2) which is defined on $\omega(u)$, although it may not be finite along the solution u itself.

II) By an asymptotic symmetrization result of [18], all functions in $\omega(u)$ are radially symmetric about the same center. This allows us to show, similarly as in [4], that if $\omega(u)$ is not a single steady state, then some of its elements are contained on a normally hyperbolic manifold of steady states of (1.2).

III) We rule out the latter possibility by applying a convergence result for autonomous equations [3, 20]. This is facilitated by a trick which shows that the solution u can be viewed as a solution of an auxiliary autonomous problem to which a convergence theorem of [3] applies.

With these techniques, we can treat general C^1 nonlinearities f (with $f(0) = 0 > f'(0)$) and we do not need to make any assumptions on the support of $h(\cdot, t)$. In fact, $h(\cdot, t)$ does not even have to decay at spatial infinity. On the other hand, for the last two steps in the above outline, we need the decay of h in t to be exponential. Note, however, that for more specific problems (see Section 2.1 for an example), we can prove convergence results under a weaker decay assumption.

The paper is organized as follows. In the next section we state our main results. They include a convergence theorem under the assumption of exponential decay and a quasiconvergence theorem in which h is assumed to decay with no particular rate. As an application of the main results, we show the convergence of threshold solutions for a more specific class of parabolic problems. In the same section, we also prove a convergence result for abstract asymptotically autonomous parabolic equations. The proofs of our main theorems are finalized in Sections 6, 7; Sections 3-5 contain preliminary steps toward the proofs. Proofs of some technical lemmas are given in appendices.

2 Main results

Throughout the paper, the standing hypothesis on f is that it is a C^1 function on $\mathbb{R}$ satisfying $f(0) = 0$, and $f'(0) < 0$.

2.1 Convergence and quasiconvergence for (1.1)

We always assume that h is a function defined on $\mathbb{R}^N \times (0, \infty)$ such that $t \mapsto h(\cdot, t)$ belongs to $L^\infty((0, \infty), X)$. Here we choose $X := L^\infty(\mathbb{R}^N)$ (hence $h \in L^\infty(\mathbb{R}^N \times (0, \infty))$. We remark that other spaces, for example, $X = L^p(\mathbb{R}^N)$, with $p \geq N + 1$, could be chosen with slightly different assumptions on u.

The minimal additional assumption on h, which is sufficient for our quasiconvergence theorem, Theorem 2.1, is the following one:

$$\lim_{t \to \infty} \|h(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = 0 \,. \tag{2.1}$$

For the convergence theorem, we need a stronger hypotheses on h involving Hölder continuity (in space or time) and exponential decay. Specifically, we assume that for some constants $\alpha \in (0, 1]$, $\mu > 0$, and $C^* > 0$, the function $\tilde{h}(x,t) := e^{\mu t} h(x,t)$ satisfies

$$\text{either} \quad \|\tilde{h}\|_{C^\alpha((0,\infty),X)} \le C^*, \tag{2.2a}$$

$$\text{or} \quad \|\tilde{h}\|_{L^\infty((0,\infty),C^\alpha(\mathbb{R}^N))} \le C^*. \tag{2.2b}$$

Here $C^\alpha((0,\infty), L^\infty(\mathbb{R}^N))$ and $C^\alpha(\mathbb{R}^N)$ stand for the spaces of bounded, α-Hölder functions from $(0,\infty)$ to $L^\infty(\mathbb{R}^N)$, and from $\mathbb{R}^N$ to $\mathbb{R}$, respectively. They are equipped with the usual norms.

By a global solution of (1.1) we mean a function $u \in W^{2,1}_{N+1,\mathrm{loc}}(\mathbb{R}^N \times (0,\infty))$ such that the equation is satisfied almost everywhere. In particular, a global solution is a continuous function on $\mathbb{R}^N \times (0,\infty)$ [27, Lemma II.3.3].

Our convergence result for (1.1) is as follows.

Theorem 2.1. *Assume that there are constants $\alpha \in (0,1)$, $\mu > 0$, and $C^* > 0$ such that* (2.2) *holds. Let u be a global, bounded, nonnegative solution of* (1.1) *which satisfies* (1.3)*. Then, as $t \to \infty$, either $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ or else there exists a ground state φ of* (1.4) *such that $\|u(\cdot,t) - \varphi\|_{L^\infty(\mathbb{R}^N)} \to 0$.*

This result extends the convergence theorem of [4] in two aspects: the nonautonomous term h is allowed and the spatial decay of u does not have to be exponential. Compared to [10], we allow much more general nonlinearities f (in particular, we do not require the uniqueness for the radial ground states of (1.4)), the solution u is not required to be in the energy space, and we do not assume $h(\cdot,t)$ to have compact support, not even to decay at $|x| = \infty$. On the other hand, the exponential decay of h in time is more restrictive than the assumptions in [10].

While the exponential decay of the function h is hardly an optimal condition in the theorem above, it cannot, in general, be replaced with a mere decay of h. Indeed, it is not difficult to show (see [18, Example 2.3] for details) that if f is such that (1.4) has a ground state φ, then there exist a continuous function h on $\mathbb{R}^N \times (0,\infty)$ and a bounded nonconvergent function $\xi : (0,\infty) \to \mathbb{R}^N$ such that $t\|h(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}$ is bounded on $(0,\infty)$ and $u(x,t) = \varphi(x - \xi(t))$ is a nonconvergent solution of (1.1). In this construction, h is exponentially decaying in x. However, the temporal decay of h is too slow for $\|h(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}$ to be even integrable on $(1,\infty)$. The integrability of this function may be sufficient for the convergence, but we cannot prove this using our method.

We next formulate a quasiconvergence theorem under the weaker hypothesis (2.1) (and without the Hölder continuity assumption). We need some preparations. Assume u is a nonnegative bounded solution of (1.1) satisfying (1.3). We define the omega limit set of u by

$$\omega(u) := \{z : u(\cdot, t_k) \to z \text{ for some } t_k \to \infty\}, \tag{2.3}$$

where the convergence is in the supremum norm. To justify this definition, we show that the set $\gamma(u) := \{u(\cdot, t) : t \geq 1\}$ is relatively compact in $C_0(\mathbb{R}^N)$, the space of continuous functions $\mathbb{R}^N$ decaying to 0 at infinity, equipped with the supremum norm.

Let

$$M := \|u\|_{L^\infty(\mathbb{R}^N \times (0,\infty))} \tag{2.4}$$

and let β_0 be a Lipschitz constant of f in $[0, M]$. Since $|f(u)|$ is bounded by a constant determined by M and β_0, standard regularity estimates give

$$\|u\|_{W^{2,1}_{N+1}(B(x,1)\times(t-1,t+1))} \leq C(M, \|h\|_{L^{N+1}(B(x,2)\times(t-2,t+2))}, \beta_0) \qquad ((x,t) \in \mathbb{R}^N \times (2,\infty)), \tag{2.5}$$

where the right hand side is a constant determined by the indicated quantities and we use $B(x, R)$ to denote the open ball in $\mathbb{R}^N$ centered at x and having radius R. Since

$$\|h\|_{L^{N+1}(B(x,2)\times(t-2,t+2))} \leq C(N)\kappa, \quad \text{with } \kappa = \sup_{t>0} \|h\|_{L^\infty(\mathbb{R}^N\times(0,\infty))},$$

one has

$$\|u\|_{W^{2,1}_{N+1}(B(x,1)\times(t-1,t+1))} \leq C(N, M, \beta_0, \kappa)\,. \tag{2.6}$$

Using the imbedding $W^{2,1}_{N+1}(B(x, 1) \times (t-1, t+1)) \hookrightarrow C^{\sigma,\frac{\sigma}{2}}(B(x, 1) \times (t-1, t+1))$, for any $0 < \sigma \leq 1 - \frac{1}{N+1}$ (see [27, Lemma II.3.3]), we find a universal bound on u in these Hölder spaces. The Arzelà-Ascoli theorem, in conjunction with (1.3), now readily implies the relative compactness of $\gamma(u) := \{u(\cdot, t) : t \geq 1\}$ in $C_0(\mathbb{R}^N)$.

It follows by standard arguments that $\omega(u)$ is nonempty, compact, and connected in $C_0(\mathbb{R}^N)$, and it attracts the solution u in the following sense:

$$\lim_{t\to\infty} \operatorname{dist}_{C_0(\mathbb{R}^N)}(u(\cdot, t), \omega(u)) = 0\,. \tag{2.7}$$

Theorem 2.2. *Assume that* $h \in L^\infty((0,\infty), X)$ *satisfies* (2.1) *and* u *is a global, bounded, nonnegative solution of* (1.1) *which satisfies* (1.3)*. Then either* $\omega(u) = 0$*, that is,* $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$*, or else* $\omega(u)$ *consists of ground states of* (1.4)*.*

We remark that our theorems are general enough to apply to a (seemingly) larger class of equations of the form

$$u_t = \Delta u + f(u) + g(x,t,u), \quad (x,t) \in \mathbb{R}^N \times (0,\infty)\,, \tag{2.8}$$

where g is a continuous function on $\mathbb{R}^N \times (0,\infty) \times [0,\infty)$ such that for any finite $m > 0$ one has

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N,\, u\in[0,m]} g(t,x,u) = 0. \tag{2.9}$$

Indeed, if u is a nonnegative bounded solution of (2.8), then u is a solution of (1.1) with $h(x,t) = g(x,t,u(x,t))$. This function satisfies (2.1) and, under suitable additional conditions on g, it also satisfies (2.2).

We illustrate this application of our results to (2.9) in the following example, which also elucidates how solutions converging to a ground state of the limit equation are found on the "threshold" between decay to 0 and blow-up in finite time.

Consider the following problem

$$u_t = \Delta u + \lambda(t)(u^p - mu), \quad (x,t) \in \mathbb{R}^N \times (0,\infty)\,, \tag{2.10}$$

where m is a positive constant, λ is a continuous positive function on $[0,\infty)$, and $1 < p < p_S$, p_S being the Sobolev critical exponent: $p_S := (N+2)/(N-2)$ if $N > 2$, $p_S = \infty$ if $N \in \{1,2\}$. Assume that

$$\lambda(t) \to \lambda_0 \tag{2.11}$$

for some $\lambda_0 \in (0,\infty)$. For a fixed nonnegative function $\psi \in C(\mathbb{R}^N) \setminus \{0\}$ with compact support, let u^μ stand for the maximally defined solution of (2.10) satisfying the initial condition $u^\mu(\cdot,0) = \mu\psi$. For technical reasons (see [34] for the background), we also assume that $p < p_{BV}$, where $p_{BV} := N(N+2)/(N-1)^2$ if $N > 1$, $p_{BV} = \infty$ if $N = 1$ (this extra restriction can be omitted if ψ is radially symmetric about some center).

Proposition 2.3. *Under the above assumptions and notation, there exists* $\mu^* \in (0,\infty)$ *such that the following statements hold.*

(i) For each $\mu \in (0, \mu^*)$ one has

$$\lim_{t\to\infty} \|u^\mu(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = 0. \tag{2.12}$$

(ii) For each $\mu \in (\mu^*, \infty)$ the solution u^μ blows up in finite time.

(iii) The solution $u^* := u^{\mu^*}$ is global and there is a ground state φ of (1.4) such that $\|u^*(\cdot, t) - \varphi\|_{L^\infty(\mathbb{R}^N)} \to 0$.

Equation (2.10) is a special case of problems considered in [34], see Example 2.7 in that paper. Note that by (2.11), λ is uniformly continuous on $[0, \infty)$, as required in [34, Example 2.7].

The existence of $\mu^* \in (0, \infty)$ such that (i), (ii) hold is established in [34, Theorem 2.2]. By [34, Theorem 2.3], the solution u^* is global and bounded, and there is $\xi \in \mathbb{R}^N$ such that all elements of $\omega(u^*)$ are positive functions which are radially symmetric about ξ. We now combine this result with Theorem 2.2 of the present paper. To apply that theorem, observe that u^* satisfies equation (1.1) with $f(u) = \lambda_0(u^p - mu)$ and

$$h(x, t) = (\lambda(t) - \lambda_0)((u^*(x, t))^p - mu^*(x, t)).$$

Since u^* is bounded, h clearly satisfies (2.1). Using Theorem 2.2, we obtain that $\omega(u^*)$ consists of grounds states of the equation

$$\Delta u + \lambda_0(u^p - mu) = 0, \quad x \in \mathbb{R}^N, \tag{2.13}$$

all having the same center of symmetry ξ. It is well known (see [5, 26]), that (2.13) has a unique radially symmetric ground state (hence also a unique ground state radially symmetric around ξ). This implies statement (iii).

Note that, although statement (iii) is a convergence result, we did not need the decay of $\lambda(t) - \lambda_0$ to be exponential. The quasiconvergence theorem, Theorem 2.2, was strong enough for the result, thanks to the special structure of the problem. In more general problems, in particular those which are not spatially homogeneous, Theorem 2.2 would typically not give a convergence result. Instead, Theorem 2.1 would have to be used, requiring the exponential decay of the inhomogeneities.

2.2 An abstract convergence result

As mentioned above, in the proof of Theorem 2.1 we use a trick to transform (1.1) to an auxiliary autonomous system to which existing convergence results can be applied. In this section we show how this transformation can be done for abstract parabolic equations of the form

$$u_t = Au + f(u) + h(t), \quad t \geq 0\,. \tag{2.14}$$

We assume the following hypotheses.

(A) X is a Banach space and A is a linear sectorial operator on X [31] with domain $D(A) \subset X$ (as in [31], we do not require $D(A)$ to be dense in X).

(F) $f \in C^1(X_\beta, X)$, where $\beta \in [0,1)$ and X_β is a fractional power space corresponding to A or the space $X_\beta = D_A(\beta,p)$, for some $1 \leq p \leq \infty$. See [31, Section 2.2] for the definition of these spaces; we remark that $D_A(\beta,p)$ is an interpolation space between X and $D(A)$, when $D(A)$ is equipped with the graph norm. For $\beta = 0$, one defines $X_\beta = X$.

(H) There exist constants $\alpha \in (\beta,1)$, $\mu > 0$, and $C > 0$ such that the function $\tilde{h}(t) := e^{\mu t}h(t)$ satisfies the following condition:

$$\text{either} \quad \|\tilde{h}\|_{C^\alpha((0,\infty),X)} \leq C \quad \text{or} \quad \|\tilde{h}\|_{L^\infty((0,\infty),D_A(\alpha,p))} \leq C\,. \tag{2.15}$$

Note that β was defined in (F).

For $u_0 \in X_\beta$ and a finite T, a mild solution of (2.14) on $(0,T)$ with the initial condition $u(0) = u_0$ is a function $u \in L^\infty((0,T), X_\beta)$ which satisfies the integral equation

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}(f(u(s)) + h(s))\,ds\,,$$

where e^{At} is the analytic semigroup generated by the sectorial operator A. In general, one may not have $u(t) \to u_0$ as $t \to 0+$ in X_β, not even in X, if $D(A)$ is not dense, but this is of no concern here. We remark that one has $u \in C([0,T), X)$ if $u_0 \in \mathrm{cl}_X\, D(A)$ and $u \in C([0,T), X_\beta)$ if $u_0 \in \mathrm{cl}_{X_\beta}\, D(A)$. We refer the reader to [31, Section 7] for these and most of the forthcoming results regarding abstract semilinear equations. A mild solution is uniquely determined, up to extensions, by its initial condition. Given any $\bar{u} \in X_\beta$, there exist $T > 0$ and a neighborhood U of $\bar{u}$ in X_β such that for each $u_0 \in U$ the mild solution $u(t, u_0)$ with the initial condition $u(0) = u_0$ is defined on $(0, T)$ and, for each fixed $t \in (0, T)$, the map $u_0 \mapsto u(t, u_0)$ is C^1. This is shown by the usual uniform contraction mapping argument.

Thanks to (H), any mild solution is a classical solution on $(0, T)$, that is, a function $u \in C^1((0, T), X) \cap C((0, T), D(A))$ satisfying the equation (cp. [31, Theorem 4.3.1, Theorem 4.3.8]).

For a solution u on $(0, \infty)$, we define its ω-limit set by

$$\omega(u) := \{z : u(\cdot, t_k) \to z \text{ for some } t_k \to \infty\}, \tag{2.16}$$

where the convergence is in X_β. Similarly as with (2.3), if $\{u(t) : t \geq 1\}$ is relatively compact in X_β, then $\omega(u)$ is a nonempty, compact, and connected set in X_β, and it attracts $u(t)$ in X_β.

Let us now consider the limit equation

$$u_t = Au + f(u), \quad t \geq 0\,. \tag{2.17}$$

Denote by E the set of all equilibria of (2.17). We say that an equilibrium $\phi \in E$ satisfies the *normal hyperbolicity condition* if there exist an integer $k \geq 0$ and a k-dimensional submanifold of X_β such that the following two conditions hold.

(i) $\phi \in M \subset E$.

(ii) The linearized operator $A + f'(\phi)$ has 0 as an eigenvalue of (algebraic) multiplicity k and there is $\delta > 0$ such that the spectrum of $A + f'(\phi)$ contains no nonzero element λ with $|\operatorname{Re} \lambda| < \delta$. If $k = 0$, we in addition require that $0 \notin \sigma(A + f'(\phi))$ (that is, $A + f'(\phi)$ is an isomorphism of $D(A)$ onto X).

We can now formulate our convergence result for (2.14).

Theorem 2.4. *Assume* (A)–(H) *and let u be a solution of* (2.14) *such that $\{u(t) : t \geq 1\}$ is relatively compact in X_β. Assume further that $\omega(u) \subset E$ and there is $\phi \in \omega(u)$ satisfying the normal hyperbolicity condition. Then $\omega(u) = \{\phi\}$.*

This result extends convergence theorems of [3, 20] which deal with autonomous problems ([1] contains an ODE predecessor of these results).

Since the proof of Theorem 2.4 is independent from the rest of the paper and uses different notation, we give it here. First we recall a convergence result for autonomous equations (cp. [3, 20]).

Lemma 2.5. *The statement of Theorem 2.4 holds if* $h \equiv 0$.

Proof. Fix $\delta > 0$ and let Π be the time-δ map of (2.17): $\Pi u_0 = \bar{u}(\delta, u_0)$, where $\bar{u}(t, u_0)$ is the mild solution of (2.17) with the initial condition $\bar{u}(0) = u_0$. In view of the compactness of the set

$$K := \mathrm{cl}_{X_\beta}\{u(t) : t \geq 1\} = \{u(t) : t \geq 1\} \cup \omega(u),$$

we can certainly choose $\delta > 0$ such that Π is defined on an X_β-neighborhood U of K. Then $\Pi : U \to X_\beta$ is a C^1-map. We apply to this map and to its orbit $\Pi^n(u(1)) = u(1 + n\delta)$, $n = 1, 2 \ldots,$ the convergence result of [3]. As in [4, Section 2.2], using the spectral mapping theorem [31, Section 2.3.2], one shows easily that a normal hyperbolicity condition for the fixed point ϕ of Π assumed in [3, Theorem B] follows from the normal hyperbolicity assumption of Theorem 2.4. The conclusion of [3, Theorem B] is then that $u(1 + n\delta) \to \phi$. Consequently $\omega(u) = \{\phi\}$, as is easily verified using the assumption $\omega(u) \subset E$ and the continuity of Π. □

Proof of Theorem 2.4. We may assume that $\sigma(A) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < -(\mu + 1)\}$, otherwise, replace A with $A - k$ and $f(u)$ with $f(u) + ku$ for sufficiently large $k \in \mathbb{R}$. Then

$$\|e^{tA}\|_{L(X)} \leq Me^{-(\mu+1)t} \tag{2.18}$$

for some $M > 0$ (cf. [31, Proposition 2.1.1 (iii)]). Let

$$y(t) = \int_0^t e^{A(t-s)}h(s)\, ds\,,$$

that is, y is the mild (hence classical) solution of

$$\begin{aligned} y_t &= Ay + h(t), \qquad t \in (0, \infty) \\ y(0) &= 0\,. \end{aligned}$$

As shown in [31, Proposition 4.4.10], (H), $\alpha \geq \beta$, and the assumption on $\sigma(A)$ imply

$$\begin{aligned} \|y(t)\|_{X_\beta} &\leq C_0 e^{-\mu t} \quad (t>1)\,, \\ \|y'(t)\|_{X_\beta} &\leq C_0 e^{-\mu t} \quad (t>1)\,, \end{aligned} \tag{2.19}$$

for some constant C_0.

Set $v := u - y$. Then

$$v_t = u_t - y_t = A(u-y) + f(u) + h - h = Av + f(v+y), \quad t \in (0,\infty)\,.$$

Fix $0 < \nu < \mu$ and define

$$z(\eta) := \begin{cases} y\left(-\frac{\ln \eta}{\nu}\right) & \eta > 0\,, \\ 0 & \eta \leq 0\,. \end{cases}$$

We show that $z \in C^1(\mathbb{R}, X_\beta)$. Since $y \in C^1((0,\infty), X_\beta)$, $z \in C^1(\mathbb{R}\setminus\{0\}, X_\beta)$. Further, using (2.19) and the substitution $t = -\frac{\ln \eta}{\nu}$, we obtain

$$\lim_{\eta \to 0^+} \frac{z(\eta)}{\eta} = \lim_{t\to\infty} y(t) e^{\nu t} = 0\,,$$

with the limit in X_β. Of course, $\lim_{\eta\to 0^-} z(\eta)/\eta = 0$, hence $z'(0)$ exists and $z'(0) = 0$. By (2.19),

$$\lim_{\eta\to 0^+} z'(\eta) = \lim_{\eta \to 0^+} -y'\left(-\frac{\ln\eta}{\nu}\right)\frac{1}{\eta\nu} = \lim_{t\to\infty} -y'(t)\,\frac{e^{\nu t}}{\nu} = 0\,,$$

where the limits are again in X_β. This shows that $z \in C^1(\mathbb{R}, X_\beta)$.

The autonomous system

$$\begin{aligned} v_t &= Av + f(v + z(\eta)), && t > 0\,, \\ \eta_t &= -\nu\eta, && t > 0\,, \end{aligned} \tag{2.20}$$

has the solution $t \mapsto (v(t), \eta(t)) = (u(t) - y(t), e^{-\nu t})$ with compact trajectory $\{(v(t), \eta(t)) : t \geq 1\} \subset X_\beta \times \mathbb{R}$ and with the omega limit set $\omega(v,\eta) = \omega(u) \times \{0\}$. Clearly, the set of equilibria of (2.20) is $E \times \{0\}$. We intend to apply Lemma 2.5 to this autonomous system and the solution (v, η). Observe, that (2.20) fits the setup of Lemma 2.5 with the sectorial operator $\tilde{A} : (w,\xi) \mapsto (Aw, -\nu\xi)$ and the C^1 nonlinearity $F : X_\beta \times \mathbb{R} \to X \times \mathbb{R}$ given by $F(w,\xi) := (f(w + z(\xi)), 0)$ ($X_\beta \times \mathbb{R}$ is a fractional power space or an interpolation space corresponding to $\tilde{A}$, just as X_β is for A). We verify that $(\phi, 0) \in \omega(v, \eta)$ satisfies the normal hyperbolicity condition. The linearization of the right hand side of (2.20) at $(\phi, 0)$ is

$$L := \begin{bmatrix} A & 0 \\ 0 & -\nu \end{bmatrix} + F'(\phi, 0) = \begin{bmatrix} A + f'(\phi) & 0 \\ 0 & -\nu \end{bmatrix},$$

where we used $z'(0) = 0$. The spectrum of the operator L is given by $\sigma(L) = \sigma(A + f'(\phi)) \cup \{-\nu\}$. Moreover, the algebraic (geometric) multiplicity of the eigenvalue 0 for $A + f'(\phi)$ is the same as the algebraic (geometric) multiplicity of the eigenvalue 0 for L.

It follows that if $M \subset E$ is the manifold from the normal hyperbolicity condition for ϕ (with respect to (2.17)), then $(\phi, 0)$ satisfies the normal hyperbolicity condition with the manifold $M \times \{0\}$.

Hence Lemma 2.5 applies and we obtain $\omega(v, \eta) = \{(\phi, 0)\}$. Consequently, $\omega(u) = \{\phi\}$. □

Remark 2.6. As in [3, 20], the assumption that $\phi \in \omega(u)$ satisfies the normal hyperbolicity condition (assumptions (i) and (ii)) can be replaced with the assumption (ii) alone, provided one assumes $k \leq 1$ in (ii). Indeed, in that case one can easily show, using a one-dimensional center manifold of ϕ, that if ϕ lies on a continuum of equilibria of (2.17), then some $\tilde{\phi} \approx \phi$ on this continuum satisfies the normal hyperbolicity condition with $k = 1$ ($\tilde{\phi}$ needs to be taken in the relative interior of the continuum), see [20] or [3] for details. This implies $\omega(u) = \{\phi\}$, for otherwise $\omega(u)$, being connected, contains a continuum of equilibria with ϕ on it. Taking an element $\tilde{\phi}$ as above, we obtain from Theorem 2.4 that $\omega(u) = \{\tilde{\phi}\}$, which is a contradiction.

3 An estimate for linear equations

Lemma 3.1. *Fix $R > 0$ and set $\Omega := \mathbb{R}^N \setminus B(0, R)$. Assume that $v \in W^{2,1}_{N+1,loc}(\Omega \times \mathbb{R})$ is a bounded solution of the equation*

$$v_t = \Delta v + b_i(x, t) v_{x_i} + c(x, t) v, \quad (x, t) \in \Omega \times \mathbb{R},$$

where b_i, $i = 1, \cdots, N$, and c are bounded measurable functions defined on $\Omega \times \mathbb{R}$. If $c(x, t) \leq -\varepsilon < 0$ ($(x, t) \in \Omega \times \mathbb{R}$), then there are constants $\nu > 0$ and C_ν depending on ε, $\|v\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})}$, $\|b_i\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})}$, $\|c\|_{L^\infty(\mathbb{R}^N \times \mathbb{R})}$ and R such that $|v(x, t)| \leq C_\nu e^{-\nu |x|}$.

Lemma 3.1 is a special case of [33, Lemma 2.4] in which Ω is a general domain, with possibly unbounded boundary. The statement of [33, Lemma 2.4] contains the extra assumption that $v \leq 0$ on $\partial\Omega \times \mathbb{R}$. However, it is obvious from the proof, that this assumption can be omitted if $\partial\Omega$ is bounded, as is the case in Lemma 3.1.

4 $\omega(u)$ as a chain recurrent set

Throughout this section we assume the hypotheses of Theorem 2.2 (the exponential decay and Hölder continuity assumptions on h are not needed here). We use the notation of Section 2, in particular see (2.3) for the definition of the ω-limit set of the solution u. Our main goal in this section is to expose $\omega(u)$ as a chain recurrent set of the limit autonomous equation (1.2).

Since the statements of our main theorems deal with a fixed bounded solution u, by modifying f outside the range of u we may assume, without loss of generality, that both f and f' are bounded. We let

$$\beta_0 := \sup_{u \geq 0} |f'(u)|. \tag{4.1}$$

Then f is a (globally) Lipschitz function with Lipschitz constant β_0. This implies (see, for example, [22] or [31]) that for each $U_0 \in C_0(\mathbb{R}^N)$ the Cauchy problem

$$U_t = \Delta U + f(U), \qquad x \in \mathbb{R}^N,\ t \in (0,\infty), \tag{4.2}$$

$$U(x,0) = U_0(x), \qquad x \in \mathbb{R}^N. \tag{4.3}$$

has a unique global solution with $U(\cdot,t) \in C_0(\mathbb{R}^N)$ for each $t > 0$. We denote by S the solution semiflow of this problem. Specifically, setting $Y := C_0(\mathbb{R}^N)$, $S : Y \times [0,\infty) \to Y$ is defined by $S(U_0,t) = U(\cdot,t)$, where U is the solution of (4.2), (4.3). Then S is a continuous map [22, 31]. Below we often use the notation $S(t)U_0 := S(U_0,t)$.

We say that a subset $K \subset Y$ is *positively invariant* under S, if $U_0 \in K$ implies $S(t)U_0 \in K$ for each $t \geq 0$. We say that K is *invariant* under S, if for each $U_0 \in K$ there is an entire solution $\tilde{U}$ of $U_t = \Delta U + f(U)$ with $\tilde{U}(\cdot,t) \in Y$ for all $t \in \mathbb{R}$ and $\tilde{U}(\cdot,0) = U_0$. Here an *entire solution* refers to a solution defined on $\mathbb{R}^N \times \mathbb{R}$. Note that a function $\tilde{U} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfying $\tilde{U}(\cdot,t) \in Y$ for all $t \in \mathbb{R}$ is an entire solution if and only if $S(t)\tilde{U}(\cdot,s) =$

$\tilde{U}(\cdot, t+s)$ for all $s \in \mathbb{R}$ and $t \geq 0$. An invariant set K is *chain transitive under* $S\big|_K$ if for any $\phi, \psi \in K$ and any $\varepsilon > 0$, $T > 0$ there exist an integer $k \geq 1$, real numbers $t_1, \cdots, t_k \geq T$, and elements $\phi_0, \phi_1, \cdots, \phi_k \in K$ such that $\phi_0 = \phi$, $\phi_k = \psi$ and

$$\|S(t_{i+1})\phi_i - \phi_{i+1}\|_Y < \varepsilon \quad (0 \leq i < k)\,. \tag{4.4}$$

This in particular implies that each $\phi \in K$ is *chain recurrent under* $S\big|_K$, meaning that the previous assertion is valid with $\psi = \phi$.

Lemma 4.1. *Under the assumptions of Theorem 2.2, $\omega(u)$ is invariant under S, in particular all elements of $\omega(u)$ are C^2 functions. Moreover, there exist constants $\nu > 0$ and $C > 0$ such that*

$$z(x) \leq Ce^{-\nu|x|} \quad (x \in \mathbb{R}^N, z \in \omega(u))\,, \tag{4.5}$$

$$|\nabla z(x)| \leq Ce^{-\nu|x|} \quad (x \in \mathbb{R}^N, z \in \omega(u))\,, \tag{4.6}$$

$$|D^2 z(x)| \leq C \qquad (x \in \mathbb{R}^N, z \in \omega(u))\,. \tag{4.7}$$

Proof. First note that (1.3) implies the following universal decay of the elements of $\omega(u)$:

$$\lim_{|x|\to\infty} \sup_{z\in\omega(u)} z(x) = 0\,. \tag{4.8}$$

To prove the invariance of $\omega(U)$, we follow a standard scheme. Fix $z \in \omega(u)$ and choose a sequence $(t_i)_{i\in\mathbb{N}}$ such that $t_i \to \infty$ and $u(\cdot, t_i) \to z$ in Y. Set $\tilde{u}_i(x,t) := u(x, t+t_i)$ for $(x,t) \in \mathbb{R}^N \times (-t_i, \infty)$. Passing to a subsequence if necessary, we may assume that $t_i > i$. By similar estimates as in (2.5), for any R, the functions $\tilde{u}_i$ with $i > R$ form a bounded sequence in $W^{2,1}_{N+1}(B(0,R)\times(-R,R))$, and consequently in $C^{\sigma,\sigma/2}(B(0,R)\times(-R,R))$ for $\sigma := 1 - 1/(N+1)$. Using Arzelà-Ascoli theorem and a diagonalization procedure one finds a subsequence of u_i (still denoted by u_i,) which converges in $C_{\mathrm{loc}}(\mathbb{R}^N \times \mathbb{R})$ to a continuous function U. Of course, U inherits the following property from u: $0 \leq U \leq M$, where M is an upper bound on u. Moreover, for any fixed $t \in \mathbb{R}$, $u(\cdot, t+t_i) \to U(\cdot, t)$ with convergence in $C_{\mathrm{loc}}(\mathbb{R}^N)$. Since the sequence $u(\cdot, t+t_i)$, $i > \max\{-t, 0\}$ is relatively compact in Y (as shown in Section 2), the converge $u(\cdot, t+t_i) \to U(\cdot, t)$ takes place in Y, and hence $U(\cdot, t) \in \omega(u)$.

Clearly, $U(\cdot, 0) = \lim u(\cdot, t_i) = z$. Further, since $\tilde{u}_i$ satisfies

$$(\tilde{u}_i)_t = \Delta\tilde{u}_i + f(\tilde{u}_i) + h(x, t+t_i), \quad (x,t) \in \mathbb{R}^n \times (-t_i, \infty)\,,$$

taking $i \to \infty$ and using (2.1), we obtain that U is a bounded, weak solution of (4.2). Consequently, since $f \in C^1$ parabolic estimates imply that U is a classical solution. This proves the invariance of $\omega(u)$.

We next prove an exponential decay estimate for the solution U (and in particular for $z = U(\cdot, 0)$). Since $f(0) = 0$, Hadamard's formula shows that U satisfies

$$U_t = \Delta U + c(x,t)U, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where

$$c(x,t) := \int_0^1 f'(sU(x,t))\, ds \quad ((x,t) \in \mathbb{R}^N \times \mathbb{R}).$$

Since $0 \le U \le M$, c is bounded by a constant determined by M (hence independent of z). Moreover, since $U(\cdot, t) \in \omega(u)$ for each t, using (4.8) and $f'(0) < 0$ we find positive constants R and $\varepsilon_0 > 0$, independent of z, such that $c(x,t) \le -\varepsilon_0 < 0$ for each $|x| > R$ and $t \in \mathbb{R}$. By Lemma 3.1, there exist positive constants ν and C, both independent of z, such that $|U(x,t)| \le Ce^{-\nu|x|}$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Taking $t = 0$ we obtain (4.5).

In order to prove that (4.6) holds (adjusting C and ν if necessary), we show that for each $i \in \{1, \cdots, N\}$ the function $V := U_{x_i}$ satisfies an exponential decay estimate. First we note that parabolic regularity and boundedness of U imply that V is bounded by a constant independent of z. Moreover, V is a bounded solution of

$$V_t = \Delta V + f'(U)V, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

and $V(\cdot, 0) = z_{x_i}$. Using similar arguments as above, one shows that

$$|V(x,t)| \le \tilde{C} e^{-\tilde{\nu}|x|}$$

for some $\tilde{\nu}, \tilde{C} > 0$ independent of z.

Finally, using the above bounds on U and ∇U, we obtain a C^1 bound on $f(U)$. Standard estimates for the heat equation then imply that (4.7) holds (again C may have to be adjusted). □

Lemma 4.2. *Under the assumptions of Theorem 2.2, $\omega(u)$ is chain transitive under $S\big|_{\omega(u)}$.*

We shall derive this lemma from the following abstract result, which is a continuous-time analog of [6, Lemma 7.5]. It's proof is similar to that of [6, Lemma 7.5], with one or two extra arguments. For the reader's convenience, we include the proof in Appendix A. Similar results can be found in [32].

Lemma 4.3. *Let (Y,d) be a metric space, $G : Y \times [0,\infty) \to Y$ a continuous map, and $v : [s,\infty) \to Y$ a uniformly continuous map, for some $s \geq 0$, such that $\{v(t) \in Y : t \geq s\}$ is relatively compact in Y. Using the notation $G(t)y = G(y,t)$, assume that for each $\tau > 0$ one has $d(G(\tau)v(t), v(t+\tau)) \to 0$ as $t \to \infty$. Then the ω-limit set*

$$\omega(v) := \{\xi \in Y : v(t_k) \to \xi \text{ for some sequence } t_k \to \infty\}$$

is chain transitive under G: for any $\phi, \psi \in \omega(v)$ and any $\varepsilon > 0$, $T > 0$ there exist an integer $k \geq 1$, real numbers $t_1, \cdots, t_k \geq T$, and points $\phi_0, \phi_1, \cdots, \phi_k \in \omega(v)$ with $\phi_0 = \phi$, $\phi_k = \psi$, such that

$$d(G(t_{i+1})\phi_i, \phi_{i+1}) < \varepsilon \qquad (0 \leq i < k)\,. \tag{4.9}$$

Proof of Lemma 4.2. All we need to do is to verify the hypotheses of Lemma 4.3 for $v(t) = u(\cdot,t)$, $G = S$, and the distance d given by the norm of the space $Y = C_0(\mathbb{R}^N)$. The Hölder estimates on u shown in Section 2 in conjunction with (1.3) imply the uniform continuity of v on $[1,\infty)$.

Next, for any $t, \tau > 0$ consider the function $w(x,\tau) := u(x,t+\tau) - U(x,\tau)$, where $U(\cdot,\tau) := S(\tau)u(\cdot,t)$. Note that $U(x,\tau)$ is the solution of $U_\tau = \Delta U + f(U)$ with $U(\cdot,0) = u(\cdot,t)$. Hadamard's formula shows that w solves the problem

$$\begin{aligned} w_\tau &= \Delta w + c(x,\tau)w + h(x,t+\tau), \qquad && (x,\tau) \in \mathbb{R}^N \times (0,\infty), \\ w(\cdot,0) &= 0\,, \end{aligned}$$

where c is bounded by β_0, the Lipschitz constant of f. Applying a version of [30, Theorem 2.11] to w, we obtain

$$\|w(\cdot,\tau)\|_{L^\infty(\mathbb{R}^N)} \leq C \sup_{s\in(t,t+\tau)} \|h(\cdot,s)\|_X \quad (t \geq 0)\,,$$

where C is determined by N, β_0, and τ. Consequently, keeping τ fixed and taking $t \to \infty$, we obtain from (2.1) that

$$\lim_{t\to\infty} \|u(\cdot,t+\tau) - S(\tau)u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = 0\,.$$

This completes the verification of the hypotheses of Lemma 4.3. □

5 Chain recurrence and the energy functional

The standing assumptions in this section are the same as in the previous one.

Denote by E the set of all nonnegative steady states of (1.2). As mentioned above, E consists of ground states (positive steady states that decay to 0 at $|x| = \infty$) and the trivial steady state. Our goal in this section is to prove that $\omega(u) \subset E$. Naturally, we want to make use of the fact that the limit equation admits a Lyapunov functional given by the usual energy functional

$$V(v) = \int_{\mathbb{R}^N} |\nabla v(x)|^2 - F(v(x))\, dx\,,$$

where $F(y) := \int_0^y f(s)ds$.

There are two difficulties we need to deal with. First, the natural space to consider the semiflow of the limit equation on is $C_0(\mathbb{R}^N)$. Indeed, this is the space for which we can verify the hypotheses of Lemma 4.3; better spaces would require stronger decay assumptions on u. Of course, V is not defined on the whole space Y. Fortunately, by Lemma 4.1, V is defined on $\omega(u)$ and this will be sufficient for our purposes, see Lemma 5.1 below. The second difficulty is that the presence of a Lyapunov functional for a semiflow does not automatically guarantee that all chain recurrent points are steady states (see [24] for counterexamples). However, a sufficient condition is, as shown in [6], that the values of V at the ground states form a set of measure zero. We verify this condition in Lemma 5.3.

Lemma 5.1. *The functional V is well defined on $\omega(u)$ and, equipping $\omega(u)$ with the induced topology from $Y = C_0(\mathbb{R}^N)$, V is continuous on $\omega(u)$.*

Proof. The fact that V is well defined on $\omega(u)$ follows directly from Lemma 4.1. To prove the continuity of V, we first consider a different topology on $\omega(u)$. Namely, the topology σ induced on $\omega(u)$ from the Banach space $C_b^1(\mathbb{R}^N)$ of all C^1 functions z which are bounded on $\mathbb{R}^N$ together with their first order derivatives (the norm is given by the maximum of the supremum norms of z and its derivatives). Using (4.5)-(4.7), it is straightforward to verify that $\omega(u)$ is compact and $V : \omega(u) \to \mathbb{R}$ is continuous in this topology. However, σ coincides with the topology induced on $\omega(u)$ from Y. This follows easily from the compactness of these two metrizable topological spaces and the fact that the convergence of a sequence in either of these topologies implies the converges of the sequence in $L^\infty(\mathbb{R}^N)$ (cp. Remark 5.2 below). $\square$

Remark 5.2. It will be useful below to compare yet different topologies on $\omega(u)$. It is a simple exercise to show that if $(\omega(u), \sigma_1)$ and $(\omega(u), \sigma_2)$ are two compact metrizable topological spaces, then they coincide, provided the following condition is satisfied for some $1 \le p \le \infty$. If $i \in \{1, 2\}$ and a sequence converges to some $\varphi \in \omega(u)$ with respect to σ_i, then it has a subsequence which converges to φ in $L^p_{\rm loc}(\mathbb{R}^N)$ (convergence in a Hausdorff space $(\omega(u), \sigma^*)$, in place of the convergence in $L^p_{\rm loc}(\mathbb{R}^N)$, is also sufficient).

Lemma 5.3. *The set $V(\omega(u) \cap E) = \{V(z) : z \in \omega(u) \cap E\} \subset \mathbb{R}$ has measure zero.*

For the proof of this result we need the following lemma. The symbol $H^1_{\rm rad}(\mathbb{R}^N)$ stands for the closed subspace of $H^1(\mathbb{R}^N)$ consisting of functions that are radially symmetric (around $x = 0$); the inner product and norm on $H^1_{\rm rad}(\mathbb{R}^N)$ are those of $H^1(\mathbb{R}^N)$. We use a similar notation and convention for spaces $H^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$.

Lemma 5.4. *Let ϕ be a radially symmetric ground state of* (1.4)*. Then there exist a neighborhood U of ϕ in $H^2_{\rm rad}(\mathbb{R}^N)$, a positive number ϵ, and a C^1 function $m : (-\epsilon, \epsilon) \to H^2_{\rm rad}(\mathbb{R}^N)$ such that*

$$E \cap U \subset \{m(s) : s \in (-\epsilon, \epsilon)\}. \tag{5.1}$$

The same statement holds if $H^2_{\rm rad}(\mathbb{R}^N)$ is replaced with $H^1_{\rm rad}(\mathbb{R}^N)$.

Proof. Set

$$Y_1 := H^1_{\rm rad}(\mathbb{R}^N), \qquad Y_2 := H^2_{\rm rad}(\mathbb{R}^N), \qquad Z := L^2_{\rm rad}(\mathbb{R}^N).$$

Let $\Phi : Y_2 \to Z$ be the map defined by $\Phi(v) = \Delta v + f(v)$. As we show in Appendix B, Φ is of class C^1. Of course, $\Phi(v) = 0$ for each $v \in E \cap Y_2 \subset E \cap Z$.

Fix any $\phi \in E \cap Y_2$. If the linear map $\Phi'(\phi) : Y_2 \to Z$ is an isomorphism, then, by the inverse function theorem, there exists a neighborhood U of ϕ in Y_2 such that $\Phi^{-1}(0) \cap U = \{\phi\}$. We can then choose any C^1 curve passing through ϕ to complete the proof of the first statement. This applies in particular to $\phi = 0$ (as $f'(0) < 0$), so we can further assume that $\phi > 0$.

Assume $\Phi'(\phi)$ is not an isomorphism. In this case, $\Phi'(\phi) = \Delta + f'(\phi)$, viewed as an unbounded self-adjoint operator on Z with domain Y_2, has 0 in its spectrum. As is well known (see for example [15]), the radial symmetry of ϕ (and all functions in Z) together with the condition $f'(0) < 0$ imply that the kernel of $\Phi'(\phi)$ is one-dimensional and its range is a closed subspace of Z with codimension 1.

The rest of the proof goes by a standard Lyapunov-Schmidt reduction. Fix ψ with $\|\psi\|_{Y_2} = 1$ such that $\ker(\Phi'(\phi)) = \operatorname{span}\{\psi\}$. Let P be the orthogonal projection of Z onto the kernel of $\Phi'(\phi)$ (hence $I - P$ is the orthogonal projection of Z onto the range of $\Phi'(\phi)$). Of course, the restriction of P to Y_2 is still a continuous projection. Writing any $v \in Y_2$ as $v = \phi + s\psi + w$, where $s\psi := P(v - \phi)$ and $w := (I - P)(v - \phi)$, the equation $\Phi(v) = 0$, is equivalent to the following system of equations for s and w:

$$P\Phi(\phi + s\psi + w) = 0\,, \qquad (I - P)\Phi(\phi + s\psi + w) = 0\,. \tag{5.2}$$

Since $(I - P)\Phi'(\phi)$ is an isomorphism from $(I - P)(Y_2)$ onto $(I - P)(Z)$, using the implicit function theorem we arrive at the following conclusion. There exist $\epsilon > 0$, a neighborhood G of the origin in $(I - P)(Y_2)$, and a C^1 function $s \mapsto w(s) : (-\epsilon, \epsilon) \to G$ such that all solutions (s, w) of the second equation in (5.2) that are contained in $(-\epsilon, \epsilon) \times G$ are also contained in the set $\{(s, w(s)) : s \in (-\epsilon, \epsilon)\}$. It is now easy to verify that the first statement of Lemma 5.4 holds with $U = \{\phi + s\psi + w : (s, w) \in (-\epsilon, \epsilon) \times G\}$ and $m(s) = \phi + s\psi + w(s)$.

To prove the second statement, we first note that since $Y_2 \hookrightarrow Y_1$, the function m can also be viewed as a Y_1-valued function and it is still of class C^1. Next, we claim that for the Y_2-neighborhood U found above, there exists a Y_1-neighborhood $\tilde{U}$ such that $E \cap \tilde{U} \subset U$. Indeed, using a regularity estimate for (1.4) and the continuity of the map $u \mapsto f(u) : Y_1 \to Z$ (see Lemma 9.1), one easily shows that if $v_j \in E \cap Y_1$ and $v_j \to \phi$ in Y_1, then $v_j \to \phi$ in Y_2. This readily implies the claim, completing the proof of the second statement. $\square$

Proof of Lemma 5.3. Since $E \cap \omega(u)$ is closed, hence compact, in $\omega(u)$, it is sufficient to prove that each $\phi \in E \cap \omega(u)$ has a neighborhood G in $\omega(u)$ such that the set $\{V(z) : z \in E \cap G\}$ has measure zero. Also observe that the topology induced on $\omega(u)$ from Y is the same as the topology induced from $H^1(\mathbb{R}^N)$. This follows from Remark 5.2 and Lemma 4.1. So a neighborhood in $\omega(u)$ can refer to any of these topologies.

Fix any $\phi \in E \cap \omega(u)$. Without loss of generality, using a translation if needed, we may assume that ϕ is radially symmetric (about the origin). Let U and m be as in Lemma 5.4 and consider the function $q(s) = V(m(s))$. The functional V is C^1 on $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. This follows from Lemma 9.1 (a different argument not using radial symmetry can be found in the appendix of [2]). Therefore, $q : (-\epsilon, \epsilon) \to \mathbb{R}$ is of class C^1. By Sard's theorem, the set of critical values of q has measure zero.

Using the fact that each ground state has a radially symmetric and radially decreasing translate, one easily finds a neighborhood G of ϕ in $\omega(u)$ such that

$$G \cap E \subset \{z(\cdot + a) : z \in U \cap E,\ a \in \mathbb{R}^N\}.$$

Since $V(z(\cdot + a)) = V(z)$ for any $a \in \mathbb{R}^N$ and $z \in E$, that is, $V(G \cap E) = V(U \cap E)$, the proof will be complete once we show that the set $K := \{V(z) : z \in E \cap U\}$ has measure zero. To show this, we claim that K is contained in the set of critical values of the function q. Indeed, for each $z \in E \cap U$ there is $s \in (-\epsilon, \epsilon)$ such that $z = m(s)$, hence $V(z) = V(m(s)) = q(s)$. An elementary computation using integration by parts shows that $q'(s) = 0$, which proves the claim. □

Remark 5.5. Using similar arguments, one can prove that, in fact, the whole set $V(E)$ has measure zero. Indeed, one has $V(E) = V(E \cap H^1_{\mathrm{rad}}(\mathbb{R}^N))$ and, although $E \cap H^1_{\mathrm{rad}}(\mathbb{R}^N)$ in not compact in general, it can always be covered by a countable union of compact sets. Then the local argument as above can be used.

Lemma 5.6. *$\omega(u) \subset E$.*

Proof. We derive this conclusion from the abstract result of [6, Lemma 6.4]. First we extend the semiflow $S\big|_{\omega(u)}$ to a flow on $\omega(u)$ (cp. Lemma 4.1). For each $t \geq 0$, the map $S(t)$ is one-to-one, thanks to the backward uniqueness for parabolic equations. This and the invariance of the compact set $\omega(u)$ (cp. Lemma 4.1) imply that $S(t)\big|_{\omega(u)}$ is a homeomorphism. Setting, for any $t \geq 0$, $\mathcal{S}(t) = S(t)$ and $\mathcal{S}(-t) = (S(t))^{-1}$, we obtain a continuous flow $\mathcal{S}$ on $\mathcal{A} = \omega(u)$, as needed in [6, Lemma 6.4].

Lemmas 5.1 and 5.3 verify two other hypotheses of [6, Lemma 6.4], the continuity of V on $\omega(u)$ and the fact that $V(E \cap \omega(u))$ is nowhere dense. The last hypothesis of [6, Lemma 6.4] requires that for any $z \in \omega(u) \setminus E$ there be $t_1 < 0 < t_2$ such that $V(\mathcal{S}(t_1)z) > V(z) > V(\mathcal{S}(t_2)z)$. This follows readily from the well known fact that the energy functional V is strictly decreasing along any nonstationary entire solution U satisfying the exponential decay estimates as in Lemma 4.1.

We have verified all hypotheses of [6, Lemma 6.4]. The assertion of that lemma is that the set of all chain recurrent points of $\mathcal{S}$ (which is the same as the set of all chain recurrent points of $S\Big|_{\omega(u)})$ is contained in E. Using Lemma 4.2, we conclude that $\omega(u) \subset E$. $\square$

6 Proof of Theorem 2.2

From Lemma 5.6, we already know that $\omega(u) \subset E$. Hence, $\omega(u)\setminus\{0\}$ consists of ground states of (1.4). We next show that if $0 \in \omega(u)$, then $\omega(u) = \{0\}$. Indeed, if it was not true, then, by the connectedness of $\omega(u)$ in $C_0(\mathbb{R}^N)$, we could find ground states with arbitrarily small maximum. This, however, is easily ruled out by the maximum principle and the fact that $f(u) < 0$ for $u > 0$, $u \approx 0$. Since $\omega(u) = \{0\}$ implies $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$, Theorem 2.2 is proved.

7 Proof of Theorem 2.1

Assume the hypotheses Theorem 2.1 and also assume that the trivial case $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ does not occur. Then, by Theorem 2.2, $\omega(u)$ consists of ground states of (1.4). Under the exponential decay assumption (2.2), we can also apply [18, Remark 2.5], to conclude that all elements of $\omega(u)$ share their center of radial symmetry, that is, there is $x_0 \in \mathbb{R}^N$ such that for each $z \in \omega(u)$, the function $z(\cdot - x_0)$ is radially symmetric.

Using a translation, we may assume without loss of generality, that all $z \in \omega(u)$ are radially symmetric around $x_0 = 0$.

Applying Theorem 2.4, we want to show that $\omega(u)$ is a singleton. To set up (1.1) as an abstract equation (2.14), take $X = L^\infty(\mathbb{R}^N)$ and let A be the X-realization of the Laplacian. Specifically, $Au = \Delta u$ for $u \in D(A)$, and the domain $D(A)$ of A is given by

$$D(A) := \{u \in L^\infty(\mathbb{R}^N) : \Delta u \in L^\infty(\mathbb{R}^N)\},$$

where the Laplacian is considered in the distributional sense. By standard regularity results,

$$D(A) = \{u \in \bigcap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N) : u, \Delta u \in L^\infty(\mathbb{R}^N)\},$$

and, by [31, Section 3.1.2], A is a sectorial operator on X. We take $\beta = 0$, so that $X_\beta = X = L^\infty(\mathbb{R}^N)$. It is well known that the Nemytskii operator of f is a C^1 map on X (see [36, Theorem X.1.20]). Hence hypotheses (A) and (F) of Theorem 2.4 are satisfied. Also, with $\alpha > 0$ as in (2.2), one has $C^\alpha(\mathbb{R}^N) \hookrightarrow D(\alpha', p)$ for each sufficiently small $\alpha' > 0$. Then the function $t \mapsto h(\cdot, t)$ satisfying (2.2) induces an abstract function satisfying hypothesis (H).

The solution $u(\cdot, t)$ has a relatively compact trajectory $\{u(\cdot, t) : t \geq 1\}$ in X and its ω-limit set, as defined in (2.3), is the same as the one defined in (2.16) (with $\beta = 0$).

Assume now that $\omega(u)$ is not a singleton. Arguing as in [4, Section 2.2], we now show that some $\phi \in \omega(u)$ satisfies the normal hyperbolicity condition. Since $\omega(u)$ is a connected subset of Y_{rad}, the subspace of $C_0(\mathbb{R}^N)$ consisting of radial functions, a standard argument shows that $\omega(u)$ contains a C^1 curve (a one-dimensional submanifold) J in Y_{rad}. Let us recall briefly how that can be shown (for details see [20, 3], for example). Consider equation (1.2) on Y_{rad} and pick any $\tilde{\phi} \in \omega(u)$. The C^1-center manifold of $\tilde{\phi}$ is one-dimensional (in the radial space) and contains all radial steady states near $\tilde{\phi}$. Since $\omega(u) \subset Y_{\mathrm{rad}}$ is connected and consists of steady states, a relatively open nonempty subset of the center manifold consists of steady states, that are all elements of $\omega(u)$. This part of the center manifold gives the sought C^1 curve.

Now

$$M := \{w(x - a) : w \in J, a \in \mathbb{R}^N\}$$

is a C^1 submanifold of X of dimension $N + 1$, consisting of steady states of (1.2).

Take any ground state $\phi \in J \subset \omega(u)$. The linearization of the right hand side of (1.2) at ϕ is the operator $L = A + a$, where a is the multiplication operator given by the continuous bounded function $f'(\phi(x))$; clearly a is a bounded operator on X and $D(L) = D(A)$. It is known that the spectrum of the L^p-realization of the Schrödinger operator L is independent of $1 \leq p \leq \infty$, and so are the multiplicities of the eigenvalues above the top of the essential spectrum (see for example [21]). The following is well known about the spectrum of L in L^2 (see [15]): L has zero as an eigenvalue of multiplicity N or $N + 1$ and the rest of the spectrum is (real and) in a positive distance from the imaginary axis (this uses the fact that $f'(\phi(x)) < 0$ for large $|x|$ which follows from the assumption $f'(0) < 0$). In the presence of the $(N+1)$-dimensional manifold of steady states containing ϕ, the multiplicity of the eigenvalue zero is necessarily $N + 1$. This shows that ϕ satisfies the normal hyperbolicity condition.

Theorem 2.4 now implies that $\omega(u) = \{\phi\}$, which contradicts the assumption that $\omega(u)$ contains a continuum. This contradiction shows that $\omega(u)$ is a singleton and Theorem 2.1 is proved.

8 Appendix A: Proof of Lemma 4.3

Proof. We first claim that for any compact interval $I \subset [0,\infty)$, the convergence $d(G(\tau)v(t), v(t+\tau)) \to 0$ as $t \to \infty$ is uniform with respect to $\tau \in I$. Indeed, if it is not true, then there are sequences t_j, τ_j, such that $t_j \to \infty$, $\tau_j \to \tau$ for some $\tau \in I$, and

$$\epsilon_0 := \inf_{j=1,2\dots} d(G(\tau_j)v(t_j), v(t_j+\tau_j)) > 0. \tag{8.1}$$

However,

$$\begin{aligned} &d(G(\tau_j)v(t_j), v(t_j+\tau_j)) \\ &\leq d(G(\tau_j)v(t_j), G(\tau)v(t_j)) + d(G(\tau)v(t_j), v(t_j+\tau)) + d(v(t_j+\tau), v(t_j+\tau_j)). \end{aligned} \tag{8.2}$$

Since G is uniformly continuous on the compact set

$$K := \overline{\{v(t) \in Y : t \geq s\}} \times I = (\{v(t) \in Y : t \geq s\} \cup \omega(v)) \times I, \tag{8.3}$$

and v is uniformly continuous on $[s,\infty)$, the first and the last terms on the right hand side of (8.2) converge to zero. The second term converges to zero by assumption and we have a contradiction. The claim is proved.

Fix any $\varepsilon, T > 0$ and $\phi, \psi \in \omega(u)$. Take $I = [T, 2T]$ in (8.3). Using the uniform continuity of G on K, we find $\delta \in (0, \varepsilon/3)$ such that

$$d(G(\tau)\xi, G(\tau)\eta) < \frac{\varepsilon}{3} \quad (\tau \in [T,2T], \xi, \eta \in K, d(\xi,\eta) < \delta)\,. \tag{8.4}$$

Next, by the above claim, there is $T_0 \geq s$ such that

$$d(G(\tau)v(t), v(t+\tau)) < \frac{\varepsilon}{3} \quad (t \geq T_0, \tau \in [T,2T])\,. \tag{8.5}$$

Fix $T_1 \geq T_0$ with

$$\operatorname{dist}(v(t), \omega(v)) < \delta < \frac{\varepsilon}{3} \quad (t \geq T_1)\,. \tag{8.6}$$

Since $\phi, \psi \in \omega(v)$, there are $s_2' > s_1' \geq T_1$ with $s_2' - s_1' > T$, $d(v(s_1'), \phi) < \delta$, and $d(v(s_2'), \psi) < \delta$. Clearly, there exist $k \in \mathbb{N}$ and an increasing finite sequence $(s_i)_{i=0}^k$ with $s_0 = s_1'$, $s_k = s_2'$, and $2T \geq s_{i+1} - s_i \geq T$. As $s_i \geq s_1' \geq T_1$, (8.6) implies the existence of points $\phi_i \in \omega(v)$, $i = 0, \ldots, k$, with $\phi_0 = \phi$, $\phi_k = \psi$, and $d(\phi_i, v(s_i)) < \delta$ for each $j = 0, \cdots, k$. We show that these points satisfy (4.9) with $t_i := s_i - s_{i-1} \in [T, 2T]$. Indeed,

$$d(G(t_{i+1})\phi_i, \phi_{i+1}) \leq d(G(t_{i+1})\phi_i, G(t_{i+1})v(s_i)) + d(G(t_{i+1})v(s_i), v(s_{i+1})) + d(v(s_{i+1}), \phi_{i+1}) < \varepsilon\,,$$

where we used (8.4), (8.5), and the relations $d(\phi_i, v(s_i)) < \delta < \frac{\varepsilon}{3}$ for $i = 0, \ldots, k$. □

9 Appendix B: Differentiability of a Nemytskii operator

To prove that the map $v \mapsto \Delta v + f(v)$ belongs to $C^1(H^2_{\rm rad}(\mathbb{R}^N), L^2_{\rm rad}(\mathbb{R}^N))$, it is obviously sufficient to prove that the map $u \mapsto f(u)$ belongs to that space. The next lemma gives a slightly stronger result.

Lemma 9.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 map such that $f(0) = 0$ and f' is bounded. Then the Nemytskii operator $\tilde{f} : u \mapsto f(u)$ takes $H^1_{rad}(\mathbb{R}^N)$ to $L^2_{rad}(\mathbb{R}^N)$ and it belongs to $C^1(H^1_{rad}(\mathbb{R}^N), L^2_{rad}(\mathbb{R}^N))$.*

Note that the lemma does not follow from standard results as $H^1(\mathbb{R}^N)$ is not imbedded in $L^\infty(\mathbb{R}^N)$ for $N > 1$. The radial symmetry is important here.

Before we proceed to the proof, we recall the following imbedding relations:

$$H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N) \quad (p \in [2, p*)), \tag{9.1}$$

$$H^1_{\rm rad}(\mathbb{R}^N) \hookrightarrow C_b(\mathbb{R}^N \setminus B), \tag{9.2}$$

where $B := B(0,1)$, $p^* = \infty$ for $N \in \{1,2\}$, and $p^* = 2N/(N-2)$ for $N > 2$. The first relation is the standard Sobolev imbedding. The second relation is meant to say that the restriction of any function $v \in H^1_{\rm rad}(\mathbb{R}^N)$ to $\mathbb{R}^N \setminus B$ is continuous and bounded and for some constant $C = C(N)$ one has

$$\sup_{|x|\geq 1} |v(x)| \leq C\|v\|_{H^1(\mathbb{R}^N)} \quad (v \in H^1_{\rm rad}(\mathbb{R}^N)). \tag{9.3}$$

These properties (and more) are proved in [2, Lemma A.II].

Proof of Lemma 9.1. We use the notation as in the proof of Lemma 5.4:

$$Y_1 := H^1_{\mathrm{rad}}(\mathbb{R}^N), \qquad Z := L^2_{\mathrm{rad}}(\mathbb{R}^N).$$

Fix $p \in (2, p^*)$.

Take an arbitrary $u \in Y_1$ and define $g(x) = f'(u(x))$. As $f(0) = 0$ and f' is bounded, $\tilde{f}(u) \in Z$ (hence $\tilde{f}$ takes Y_1 to Z) and $g \in L^\infty$. Therefore the multiplication operator $\tilde{g} : v \mapsto gv$ belongs to $L(Z, Z)$, hence also to $L(Y_1, Z)$. We prove that $\tilde{g} = \tilde{f}'(u)$ (the Frechet derivative), that is, we prove that

$$\lim_{\|v\|_{Y_1} \to 0} \frac{\|\tilde{f}(u+v) - \tilde{f}(u) - \tilde{g}v\|_Z}{\|v\|_{Y_1}} = 0\,. \tag{9.4}$$

We have, for almost all x,

$$\begin{aligned}
&|f(u(x)+v(x)) - f(u(x)) - f'(u(x))v(x)|^2 \\
&\qquad = \Big|\int_0^1 (f'(u(x)+sv(x)) - f'(u(x)))v(x)\,ds\Big|^2 \\
&\qquad \le \int_0^1 |(f'(u(x)+sv(x)) - f'(u(x)))v(x)|^2\,ds,
\end{aligned}$$

by the Hölder inequality. Therefore, by Fubini's theorem,

$$\begin{aligned}
&\|\tilde{f}(u+v) - \tilde{f}(u) - \tilde{g}v\|_Z^2 \\
&\quad \le \int_0^1 \int_{\mathbb{R}^N} |f'(u(x)+sv(x)) - f'(u(x))|^2 |v(x)|^2\,dx\,ds \\
&\quad = \int_0^1 \int_B |f'(u(x)+sv(x)) - f'(u(x))|^2 |v(x)|^2\,dx\,ds \\
&\qquad\qquad + \int_0^1 \int_{\mathbb{R}^N \setminus B} |f'(u(x)+sv(x)) - f'(u(x))|^2 |v(x)|^2\,dx\,ds \\
&\quad =: I_1(v) + I_2(v).
\end{aligned}$$

It is sufficient to prove that if $v_n \in Y_1 \setminus \{0\}$ is any sequence such that $v_n \to 0$ in Y_1, then passing to a subsequence one achieves

$$\frac{(I_i(v_n))^{\frac{1}{2}}}{\|v_n\|_{Y_1}} \to 0 \qquad (i = 1, 2). \tag{9.5}$$

Take any such sequence v_n and choose a subsequence (still denoted by v_n) such that $v_n \to 0$ almost everywhere. By the Hölder inequality

$$I_1(v_n) \leq \left(\int_0^1 \int_B |f'(u(x) + sv_n(x)) - f'(u(x))|^{\frac{2p}{p-2}} \, dx \, ds \right)^{\frac{p-2}{p}} \|v_n\|^2_{L^p(B)}.$$

Since $\|v_n\|_{L^p(B)} \leq \|v_n\|_{L^p(\mathbb{R}^N)} \leq C_1 \|v_n\|_{Y_1}$, we have

$$k_n^1 := \frac{(I_i(v_n))^{\frac{1}{2}}}{\|v_n\|_{Y_1}} \leq C_1 \left(\int_0^1 \int_B |f'(u(x) + sv_n(x)) - f'(u(x))|^{\frac{2p}{p-2}} \, dx \, ds \right)^{\frac{p-2}{2p}}.$$

Since $|f'|$ is bounded by some constant β_0, the integrand is bounded by $(2\beta_0)^{\frac{2p}{p-2}}$. Using the Lebesgue dominated convergence theorem, we conclude that $k_n^1 \to 0$.

Next,

$$I_2(v_n) \leq \|v_n\|_Z^2 \sup_{|x| \geq 1} |f'(u(x) + sv_n(x)) - f'(u(x))|^2.$$

Since $u \in C_b(\mathbb{R}^N \backslash B)$ and $v_n \to 0$ in $C_b(\mathbb{R}^N \backslash B)$ (by (9.2)), and f' is uniformly continuous on each compact interval, we obtain

$$\frac{(I_2(v_n))^{\frac{1}{2}}}{\|v_n\|_{Y_1}} \leq \frac{(I_2(v_n))^{\frac{1}{2}}}{\|v_n\|_Z} \to 0.$$

This proves that $\tilde{g} = \tilde{f}'(u)$.

To prove that the derivative is continuous at any fixed $u \in Y_1$, we need to show that if $v_n \to 0$ in Y_1, then $\|(\tilde{f}'(u + v_n) - \tilde{f}'(u))w\|_Z \to 0$ uniformly with respect to $w \in Y_1$ with $\|w\|_{Y_1} \leq 1$. This amounts to estimating $\int_{\mathbb{R}^N} |f'(u(x) + v_n(x)) - f'(u(x))|^2 |w(x)|^2 \, dx$ in pretty much the same way as $\int_0^1 \int_{\mathbb{R}^N} |f'(u(x) + sv_n(x)) - f'(u(x))|^2 |v_n(x)|^2 \, dx \, ds$ was estimated above and we omit the details. □

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