Propagating terraces in a proof of the Gibbons conjecture and related results

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Dedicated to Paul Rabinowitz.

Abstract. The Gibbons conjecture stating the one-dimensional symmetry of certain solutions of semilinear elliptic equations has been proved by several authors. We show how attractivity properties of minimal propagating terraces of one-dimensional parabolic problems can be used in a proof of a version of this result and related statements.

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1 Introduction and statement of the results

Consider the elliptic equation

$$\Delta v + f(v) = 0, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$ and f is a C^1 function on $\mathbb{R}$. Writing the spatial variable as $x = (x_1, x')$, with $x_1 \in \mathbb{R}$, we are interested in solutions whose limits as $x_1 \to \pm\infty$ exist, uniformly in x', and are equal to distinct zeros of f. Without loss of generality, using dilation and translation if necessary, we place these zeros at ± 1:

$$f(\pm 1) = 0. \tag{1.2}$$

The solutions we consider are assumed to satisfy the following conditions:

(U) $|v| \leq 1$ and $\lim_{x_1 \to \pm\infty} v(x_1, x') = \mp 1$ uniformly in $x' \in \mathbb{R}^{N-1}$.

Note that with the change of variable $x_1 \to -x_1$ one can simultaneously examine solutions with the uniform limits $\lim_{x_1 \to \pm\infty} v(x_1, x') = \pm 1$.

Several authors have proved that under suitable assumptions on f, any solution of (1.1) satisfying (U) is necessarily a function of x_1 only, that is, it is independent of x'. Proofs of this result, often referred to as the Gibbons conjecture, can be found in [1, 4, 7, 8, 11, 14] (see also [15] and references therein for related results on the De Giorgi conjecture, in which $f(u) = u(1-u^2)$ and the solution v is monotone in x_1, but the uniformity requirement in (U) is dropped). In these papers, it is typically assumed that f satisfies the relations $f'(\pm 1) < 0$ or the following weaker conditions:

(M−) $f' \leq 0$ in $[-1, -1+\delta]$ for some $\delta > 0$.

(M+) $f' \leq 0$ in $[1-\delta, 1]$ for some $\delta > 0$.

We quote here the following result of [4].

Theorem BHM. *Assuming that* (M−), (M+) *hold, if* v *is a solution of* (1.1) *satisfying* (U)*, then* v *is a function of* x_1 *only. Specifically,* $v(x_1, x') = \varphi(x_1)$ *where* φ *is a solution of*

$$\varphi'' + f(\varphi) = 0 \quad \text{in } \mathbb{R} \tag{1.3}$$

$$\varphi(\pm\infty) = \mp 1. \tag{1.4}$$

In particular, the existence of a solution v of (1.1) *satisfying* (U) *implies the existence of a solution φ of* (1.3)*,* (1.4)*.*

(Actually, in [4], f is assumed to be Lipschitz, rather than C^1, and conditions (M) are phrased as monotonicity properties of f.)

In this paper (assuming $f \in C^1$), we give a new proof of this result and relax its hypotheses a bit. Our proof reveals a connection of the one-dimensional symmetry property of solutions of (1.1) to propagating terraces, or, stacked families of traveling waves, of parabolic equations on the real line, which we think is also interesting.

We assume throughout the paper that the following conditions are satisfied:

(N) there is $\delta_0 > 0$ such that $f \le 0$ in $[-1, -1+\delta_0]$ and $f \ge 0$ in $[1-\delta_0, 1]$.

To formulate our next hypotheses, which are weaker than (M$-$), (M$+$), we need to introduce some notation. Set

$$F(u) := \int_{-1}^{u} f(s)\, ds \quad (u \in [-1, 1]). \tag{1.5}$$

Hypothesis (N) implies that there is $\delta \in (0, \delta_0]$ such that either $f \equiv 0$ in $[-1, -1+\delta]$ or

$$F(u) < 0 \quad (u \in (-1, -1+\delta]). \tag{1.6}$$

We define a function E^- on $(-1, -1+\delta]$ by $E^- \equiv 0$ if $f \equiv 0$ in $(-1, -1+\delta]$, and by

$$E^-(u) = \int_{u}^{-1+\delta} \frac{ds}{\sqrt{-2F(s)}}\, ds \tag{1.7}$$

if (1.6) holds.

Making δ smaller if necessary, we also have either $f \equiv 0$ in $[1-\delta, 1]$ or

$$F(u) < F(1) \quad (u \in [1-\delta, 1)). \tag{1.8}$$

In the former case, we set $E^+ \equiv 0$ in $[1-\delta, 1)$; in the latter case, we define E^+ on $[1-\delta, 1)$ by

$$E^+(u) = \int_{1-\delta}^{u} \frac{ds}{\sqrt{2F(1) - 2F(s)}}\, ds. \tag{1.9}$$

Finally, we introduce functions $g^\pm$ on $[-1,1]$ defined by

$$g^-(u) := \max_{s\in[-1,u]} f'(s), \quad g^+(u) := \max_{s\in[u,1]} f'(s). \tag{1.10}$$

One or both of the following conditions will be assumed in our main results.

(H−) $\limsup_{u\to -1}(E^-(u))^2 g^-(u) < \frac{1}{4}$,

(H+) $\limsup_{u\to 1}(E^+(u))^2 g^+(u) < \frac{1}{4}$.

Notice that conditions (N), (H−), (H+) are trivially satisfied if conditions (M−), (M+) hold.

Our first result shows in particular the validity of the last statement of Theorem BHM under weaker conditions on f.

Proposition 1.1. *Assuming that $f \in C^1$ and conditions* (1.2), (N) *hold, let v be a solution of* (1.1) *satisfying* (U). *Then, necessarily, $F(1) = 0$, $F \le 0$ in $[-1,1]$, and there are solution φ^-, φ^+ of* (1.3) *such that $|\varphi^\pm| < 1$, $\varphi^\pm_x < 0$, and*

$$\varphi^-(\infty) = -1, \quad \varphi^+(-\infty) = 1. \tag{1.11}$$

If one of the conditions (H−), (H+) *is satisfied, then also $\varphi^-(-\infty) = 1$ (so φ^- is a solution of* (1.3), (1.4)*) and $\varphi^+(\infty) = -1$.*

Remark 1.2. The last two conclusions, $\varphi^-(-\infty) = 1$ and $\varphi^+(\infty) = -1$, remain valid if instead of (H−) or (H+) one assumes the generic condition that 0 is a regular value of the function $F\big|_{(-1,1)}$. Indeed, according to the first statement, $F(1) = F(-1) = 0$, $F \le 0$ in $[-1,1]$. If 0 is not a critical value of $F\big|_{(-1,1)}$, then $F < 0$ in $(-1,1)$, which is a necessary and sufficient condition for the existence of a solution φ of (1.3), (1.4) (and $\varphi^\pm$ is then necessarily equal to a shift of φ). The generic condition is a global condition on the function f, whereas the conditions (H−), (H+) only concern the behavior of f near ± 1.

It is well known and easy to prove by elementary phase plane arguments that the solution φ of (1.3), (1.4) is unique up to translations. Of course, $\varphi(x_1)$ can be viewed as a solution of (1.1) independent of x'. The next theorem gives sufficient conditions for the translations of φ to exhaust all solutions of (1.1) satisfying (U).

Theorem 1.3. *Assume that $f \in C^1$, and that* (1.2), (N), (H−), *and* (H+) *hold. Let v be a solution of* (1.1) *satisfying* (U)*. Then there exists a solution φ of* (1.3), (1.4) *such that*

$$v(x_1, x') = \varphi(x_1) \quad ((x_1, x') \in \mathbb{R}^N). \tag{1.12}$$

As mentioned above, hypothesis (H±) is weaker than the monotonicity assumption (M±). The nonlinearity in the following example satisfies (N), (M+), and (H−), but not (M−). Clearly, one can extend this nonlinearity in such a way that $F(1) = F(-1) = 0$ and $F < 0$ in $(-1, 1)$, which means that (1.1) admits a solution satisfying (U) (namely, a solution of (1.3) (1.4)).

Example 1.4. *Given any $p > 1$, define $f(u)$ for $u \approx 1$ by $f(u) = 1 - u$ and for $u \approx -1$ by*

$$f(u) = (u + 1)^p(-1 + b \cos \log(u + 1)).$$

Here b is a constant, which we choose such that the following relations hold:

$$(1 + \frac{1}{p^2})^{-\frac{1}{2}} < b < 1, \tag{1.13}$$

$$\frac{b(1 + \frac{1}{p^2})^{\frac{1}{2}} - 1}{1 - b} < \frac{(p - 1)^2}{8p(p + 1)} \tag{1.14}$$

(note that (1.14) *holds if b is sufficiently close to $(1+1/p^2)^{-1/2}$. Then* (H−) *is satisfied, but* (M−) *is not satisfied. See Section 4 for a justifying computation.*

2 Outline of the proofs and the meaning of conditions (H−), (H+)

Let us outline briefly how we prove the above results. Doing so, we will also demystify conditions (H−), (H+).

First, we explain how traveling waves of the one-dimensional parabolic problem

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t > 0, \tag{2.1}$$

are used in the proof of Proposition 1.1. Recall that a traveling wave is a solution U of (2.1) of the form $U(x, t) = \phi(x - ct)$, where c, the *speed* of the wave, is a real number, and ϕ, the *profile* of the wave, is a C^2 function.

Clearly, the profile function must satisfy the ordinary differential equation (ODE)

$$\phi_{xx} + c\phi_x + f(\phi) = 0. \tag{2.2}$$

When $\phi' \neq 0$ everywhere (usually, we assume $\phi' < 0$), $U(x,t) = \phi(x - ct)$ is also called a traveling front connecting the limits $\phi(-\infty)$, $\phi(\infty)$. A standing front is a traveling front with the zero speed; it is a steady state of (2.1). Thus the last statement of Proposition 1.1 says that there is a standing front of (2.1) connecting 1 and -1. Given a nonlinearity f, a priori the existence of a standing front or, for that matter, any traveling front connecting 1 and -1, is not guaranteed. What we do know, however, is that there always exists a stacked family of traveling fronts, or, a minimal propagating terrace, connecting 1 and -1 (see Section 3 for details). A key to our proof of Proposition 1.1 is the observation that the existence of a solution v of (1.1) satisfying (U) implies that the minimal propagating terrace necessarily consists of standing fronts. The reason for this is, roughly, as follows. Set

$$v^-(x_1) := \inf_{x' \in \mathbb{R}^{N-1}} v(x_1, x'), \quad v^+(x_1) := \sup_{x' \in \mathbb{R}^{N-1}} v(x_1, x').$$

These are, respectively, time-independent supersolution and subsolution of (2.1). Now take the solutions u^- and u^+ of (2.1) with the initial data v^- and v^+, respectively. As $t \to \infty$, each of the solutions $u^-(\cdot, t)$, $u^-(\cdot, t)$ approaches in a suitable way the minimal propagating terrace. At the same time, one has $u^- \leq v^- \leq v^+ \leq u^+$, and this is easily shown to imply that there can be no traveling front with nonzero speed in the terrace.

From the previous conclusion, we obtain solutions φ^-, φ^+ as in Proposition 1.1. If condition (H$-$) is satisfied, we show by an application of the sliding method [5] that $\varphi^-(-\infty) < 1$ is impossible, hence $\varphi^-(-\infty) = 1$. Likewise, if condition (H$+$) holds, then it can be shown that $\varphi^+(\infty) = -1$. Thus we obtain the last conclusion of Proposition 1.1.

If (H$-$) and (H$+$) are both satisfied, then it can also be proved that if φ is a suitable shift of φ^-, then there is $\xi \in \mathbb{R}$ such that

$$\varphi(x_1 + \xi) \leq v(x_1, x') \leq \varphi(x_1) \quad ((x_1, x') \in \mathbb{R}^N). \tag{2.3}$$

Hence, v is sandwiched between two shifts of φ. Let us now explain how this is used in our proof of Theorem 1.3. The proof uses another sliding technique, which we borrow from [4]. It consists in comparing v with its shifts: one starts with a suitable shift which is above v, and slides it in the direction of the x_1-axis as long as this relation is preserved. This method is powered by a version of the maximum principle for unbounded domains. Specifically, the difference w of any two shifts of v satisfies a linear equation of the form

$$\Delta w + a(x)w = 0, \quad x \in \mathbb{R}^N, \tag{2.4}$$

where $a(x) = f'(\zeta(x))$ and ζ is between the two shifts. In the sliding method, one needs to apply the maximum principle to equation (2.4) considered in the halfspaces $\{(x_1, x') : x_1 > \rho\}$, $\{(x_1, x') : x_1 < -\rho\}$ with a sufficiently large $\rho > 0$. As is well known, the maximum principle does apply if $a(x) \leq 0$ for $|x_1| \geq \rho$ and this is true, thanks to condition (U), for all sufficiently large ρ if the hypotheses (M$-$), (M$+$) are assumed [4]. Now, as is also well known, for the maximum principle to apply the condition $a \leq 0$ can be replaced by a suitable decay condition on a; a transformation of (2.4) then has a negative zero order coefficient (see Section 3 for details).

Relations (2.3) and hypotheses (H$-$), (H$+$), provide such a bound on the coefficient $a(x)$ for large $|x_1|$, as we now show. The existence of a solution φ of (1.3), (1.4) implies that $F(1) = F(-1) = 0$ and $F < 0$ in $(-1, 1)$. This follows from the fact that the function $(\varphi')^2/2 + F(\varphi)$ is identical to 0, as one shows easily by computing the derivative and using (1.3), (1.4). In particular, (1.6) and (1.8) hold and the functions E^- and E^+ are defined by the integrals (1.7), (1.9). Now, since $\varphi' < 0$ and $(\varphi')^2/2 + F(\varphi) \equiv 0$, for some $\eta \in \mathbb{R}$ the function φ is the solution of

$$\dot{\theta} = -\sqrt{-2F(\theta)}, \quad \theta(\eta) = -1 + \delta. \tag{2.5}$$

Integrating this ODE, we find

$$E^-(\varphi(x_1)) = \int_{\varphi(x_1)}^{-1+\delta} \frac{du}{\sqrt{-2F(u)}} = x_1 - \eta. \tag{2.6}$$

Taking $u = \varphi(x_1)$ in (H$-$) and substituting (2.6), we obtain

$$\limsup_{x_1 \to \infty} x_1^2 g^-(\varphi(x_1)) = \limsup_{x_1 \to \infty} \, (x_1 - \eta)^2 g^-(\varphi(x_1)) < \frac{1}{4}. \tag{2.7}$$

By analogous arguments, choosing ϑ so that $\varphi(\vartheta) = 1 - \delta$, we obtain from (H$+$) that

$$\limsup_{x_1 \to -\infty} x_1^2 g^+(\varphi(x_1)) < \frac{1}{4}. \tag{2.8}$$

Recalling the meaning of $g^{\pm}$ (see (1.10)), these relations indeed yield an upper estimate on the potential $a(x)$ in (2.4). The constant 1/4 in these estimates is a sharp constant for the validity of the maximum principle.

The proof sketched above works under more general assumptions on v. It is sufficient to assume that, in place of (U), the following conditions are satisfied:

$$\liminf_{x_1\to-\infty,\ x'\in\mathbb{R}^{N-1}} v(x_1,x') \in D_1 \qquad \limsup_{x_1\to\infty,\ x'\in\mathbb{R}^{N-1}} v(x_1,x') \in D_{-1}. \tag{2.9}$$

Here $D_{\pm 1}$ is the domain of attraction of the equilibrium ± 1 for the equation $\dot\theta = f(\theta)$ in $[-1,1]$ (it may coincide with $\{\pm 1\}$ if ± 1 is not asymptotically stable). However, it is not difficult to show directly by considering locally uniform limits of functions $v(x_1+x_{1,k},x')$ with $x_{1,k}\to\pm\infty$, that any solution v satisfying (2.9) along with $|v|\le 1$ must actually satisfy (U).

We also remark that our proofs apply, with minor modifications, when v, rather than being a solution of (1.1), is a solution of the parabolic equation $v_t = \Delta v + f(v)$ defined for all $t\in\mathbb{R}$ (that is, $v = v(x,t)$ is an entire solution). Condition (U) is modified accordingly so that it also includes the uniformity with respect to $t\in\mathbb{R}$. Under the hypotheses of Theorem 1.3, for example, one can then show that v is independent of x' and t (see [3] for related results).

3 Proofs of the main results

Throughout this section we assume that f is a C^1 function satisfying (1.2) and (N). We give the proofs of Proposition 1.1 and Theorem 1.3 in Subsections 3.3, 3.4. This is preceded by preliminary sections on propagating terraces and a comparison principle.

3.1 The approach to the minimal propagating terrace for one-dimensional parabolic equations

Here, we recall the definition and an attractivity property of the minimal propagating terrace of the equation

$$u_t = u_{xx} + f(u), \quad x\in\mathbb{R},\ t>0. \tag{3.1}$$

(Note that x is a one-dimensional variable in this subsection).

We first recall the notion of a minimal system of waves, as given in [18]. If ϕ is a solution of (2.2), we denote by $\tau(\phi)$ its planar trajectory:

$$\tau(\phi) = \{(\phi(x), \phi_x(x) : x \in \mathbb{R}\}. \tag{3.2}$$

Definition 3.1. A *system of waves* (or, more precisely, $[-1,1]$-system of waves) is a continuous function R on $[-1,1]$ with the following properties:

(i) $R(-1) = R(1) = 0, \quad R(u) \leq 0 \ \ (u \in [-1,1])$;

(ii) If $I = (a,b) \subset [-1,1]$ is a nodal interval of R, that is, a connected component of the set $R^{-1}(-\infty,0)$, then there is $c \in \mathbb{R}$ and a decreasing solution ϕ of (2.2) such that $\phi(-\infty) = b$, $\phi(\infty) = a$, and

$$\{(u, R(u)) : u \in (a,b)\} = \tau(\phi). \tag{3.3}$$

Thus the graph of R between its successive zeros coincides with the trajectory of the profile function of a traveling front.

Definition 3.2. A system of waves R_0 is said to be *minimal* if for an arbitrary system of waves R one has

$$R_0(u) \leq R(u) \quad (u \in [-1,1]).$$

By definition, the minimal system of waves is unique. As shown in [18, Theorem 1.3.2], for any f satisfying (1.2), a minimal system of waves exists and the following relation holds:

$$R_0^{-1}(0) \subset f^{-1}(0). \tag{3.4}$$

Many other properties of the minimal system of waves can be found in [18] and [12].

We denote by $\mathcal{N}$ the (countable) set of all nodal intervals of R_0. This set is nonempty, unless $R_0 \equiv 0$, hence, unless $f \equiv 0$ on $[-1,1]$. Since R_0 is single valued, for each $I \in \mathcal{N}$ the speed $c = c_I$ and the solution $\phi = \phi_I$ in Definition 3.1(ii) are determined uniquely if we postulate

$$\phi(0) = \frac{a+b}{2}. \tag{3.5}$$

This way we obtain the *families of speeds and profile functions* corresponding to R_0:

$$\{c_I : I \in \mathcal{N}\}, \ \{\phi_I : I \in \mathcal{N}\}. \tag{3.6}$$

The set $\mathcal{N}$ can in general be infinite and the set $\{c_I : I \in \mathcal{N}\}$ may include positive, negative, as well as zero values. The following result, which is a part of [12, Proposition 3.10(ii)], concerns the intervals $I \in \mathcal{N}$ with $c_I = 0$.

Lemma 3.3. *If $I = (a, b) \in \mathcal{N}$ and $c_I = 0$, then a and b are global maximizers of the function F in $[-1, 1]$.*

Consider now the family of traveling fronts $U_I(x, t) = \phi_I(x - c_I t)$, $I \in \mathcal{N}$. As in [6, 12], we refer to this family as the $[-1, 1]$-*minimal propagating terrace* or simply the *minimal propagating terrace*. We now state a theorem on the global attractivity property of the minimal propagating terrace with respect to a class of solutions of (3.1). Namely, we consider solutions of the Cauchy problem

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.7}$$

where u_0 is a continuous function satisfying

$$|u_0| \le 1, \quad \lim_{x \to \pm\infty} u_0(x) = \mp 1. \tag{3.8}$$

The following statement is a part of [12, Theorem 2.9] (for earlier related results see [6, 9, 10, 13, 18]). We note that functions satisfying (3.8) form a special class of admissible initial data for statements (i)-(ii) of [12, Theorem 2.9] and that the following stability condition, which is an assumption of [12, Theorem 2.9], is satisfied due to hypothesis (N):

- **(S)** For the ODE $\dot{\eta} = f(\eta)$, the equilibrium -1 is stable from above (not necessarily asymptotically) and the equilibrium 1 is stable from below.

Theorem 3.4. *Assume that $u_0 \in C(\mathbb{R})$ satisfies (3.8) and let u be the solution of (3.1), (3.7). Then for each $I = (a, b) \in \mathcal{N}$ there is a C^1 function ζ_I defined on some interval (s_I, ∞) such that the following statements are valid:*

(a) $\lim_{t\to\infty} \zeta_I'(t) = 0 \quad (I \in \mathcal{N})$;

(b) $\big((a + b)/2 - u(x + c_I t + \zeta(t), t)\big)x > 0 \quad (x \in \mathbb{R} \setminus \{0\},\ t > s_I)$;

(c) $\lim_{t\to\infty} u(\cdot + c_I t + \zeta_I(t), t) - \phi_I = 0$, *locally uniformly on* $\mathbb{R}$;

(d) *if* $I_1, I_2 \in \mathcal{N}$, $I_1 < I_2$, *and* $c_{I_1} = c_{I_2}$, *then* $\zeta_{I_1}(t) - \zeta_{I_2}(t) \to \infty$ *as* $t \to \infty$.

We remark, although we will not need this below, that if the set $R_0^{-1}\{0\}$ is finite, say

$$R_0^{-1}\{0\} = \{a_1, \dots, a_{k+1}\}, \text{ with } 0 = a_1 < a_2 < \cdots < a_{k+1} = \gamma,$$

then the attractivity property of the minimal propagating terrace can be made more precise as follows. One has $\mathcal{N} = \{I_1, \dots, I_k\}$ with $I_j = (a_j, a_{j+1})$, $j = 1, \dots, k$, and, as $t \to \infty$,

$$u(x,t) - \Big(\sum_{j=1,\dots,k} \phi_{I_j}(x - c_{I_j} t - \zeta_{I_j}(t)) - \sum_{1 \le j \le k-1} a_{j+1} \Big) \to 0 \quad (x \in \mathbb{R}),$$

uniformly on $\mathbb{R}$.

3.2 A comparison principle

At several places below we use the following comparison principle for half-spaces ($g^{\pm}$ are as in (1.10)).

Lemma 3.5. *Let v, $\tilde{v}$ be solutions of* (1.1) *with ranges in* $[-1,1]$ *and* $\vartheta \in \mathbb{R}$.

(i) Assume that

$$\tilde{v}(\vartheta, x') \le v(\vartheta, x') \quad (x' \in \mathbb{R}^{N-1}), \tag{3.9}$$

$$g^-(\tilde{v}(x_1, x')) \le \frac{1}{4(x_1 - \vartheta + 1)^2} \quad (x_1 \ge \vartheta,\ x' \in \mathbb{R}^{N-1}). \tag{3.10}$$

Then $\tilde{v} \le v$ *in* $[\vartheta, \infty) \times \mathbb{R}^{N-1}$.

(ii) Assume that

$$\tilde{v}(-\vartheta, x') \le v(-\vartheta, x') \quad (x' \in \mathbb{R}^{N-1}),$$

$$g^+(v(x_1, x')) \le \frac{1}{4(x_1 + \vartheta - 1)^2} \quad (x_1 \le -\vartheta,\ x' \in \mathbb{R}^{N-1}).$$

Then $\tilde{v} \le v$ *in* $(-\infty, -\vartheta] \times \mathbb{R}^{N-1}$.

This is very similar to comparison principles found in [2, 4], although those results do not apply here directly because we do not assume monotonicity of f near ± 1 (see also [16, 17] for related maximum principles for unbounded domains).

Proof of Lemma 3.5. As these are analogous statements, we only prove statement (i). We start by observing that the function $w := v - \tilde{v}$ solves a linear equation (2.4) with

$$a(x) = \int_0^1 f'(\tilde{v}(x) + s(v(x) - \tilde{v}(x)))\, ds. \tag{3.11}$$

Suppose, for a contradiction, that $w < 0$ somewhere in $(\vartheta, \infty) \times \mathbb{R}^{N-1}$. Consider the function $\tilde{w}(x_1, x') := w(x_1, x')(x_1 - \vartheta + 1)^{-1/2}$. It satisfies the following equation in $(\vartheta, \infty) \times \mathbb{R}^{N-1}$

$$\Delta \tilde{w} + \frac{\tilde{w}_{x_1}}{x_1 - \vartheta + 1} + (a(x) - \frac{1}{4(x_1 - \vartheta + 1)^2})\tilde{w} = 0. \tag{3.12}$$

Recalling the meaning of g_- (see (1.10)), we see that $a(x) \le g_-(\tilde{v}(x))$ for any point x with $v(x) \le \tilde{v}(x)$. Condition (3.10) implies that the coefficient of $\tilde{w}$ in (3.12) is nonpositive at such points. Therefore, by the strong maximum principle, the function $\tilde{w}$ does not assume its infimum over the set $(\vartheta, \infty) \times \mathbb{R}^{N-1}$; this infimum—$m$, say—is negative by what we are assuming about w. Hence, there is an unbounded sequence $\{(x_{1,k}, x'_k)\}$ in $(\vartheta, \infty) \times \mathbb{R}^{N-1}$ such that $\tilde{w}(x_{1,k}, x'_k) \to m$. By the definition of $\tilde{w}$, the sequence $\{x_{1,k}\}$ is bounded; hence, passing to a subsequence, we may assume that $x_{1,k} \to x_{1,0} \in [\vartheta, \infty)$. Using standard regularity estimates and passing to another subsequence if necessary, we obtain that $v(x_1, x' + x'_k) \to \bar{v}(x_1, x')$, $\tilde{v}(x_1, x' + x'_k) \to \underline{v}(x_1, x')$, locally uniformly on $\mathbb{R}^N$, where $\bar{v}$, $\underline{v}$ are solutions of (1.1). Moreover, for the function

$$\hat{w}(x_1, x') = (\bar{v}(x_1, x') - \underline{v}(x_1, x'))(x_1 - \vartheta + 1)^{-1/2}$$

one has

$$\hat{w}(x_{1,0}, 0) = m = \inf_{(x_1, x') \in (\vartheta, \infty) \times \mathbb{R}^{N-1}} \hat{w}(x_1, x'). \tag{3.13}$$

One verifies easily that the assumptions of Lemma 3.5 remain valid if v is replaced by $\bar{v}$ and $\tilde{v}$ by $\underline{v}$. Repeating what we said above, the function $\hat{w}$ cannot achieve its infimum $m < 0$ in $(\vartheta, \infty) \times \mathbb{R}^{N-1}$ (and on $\{\vartheta\} \times \mathbb{R}^{N-1}$ one has $\hat{w} \ge 0$). This is contradicted by (3.13). This contradiction proves that $\tilde{v} \le v$ in $(\vartheta, \infty) \times \mathbb{R}^{N-1}$, as desired. □

Remark 3.6. In statement (i) of Lemma 3.5, once we know that $\tilde{v} \le v$, the following conclusion can be added. Either

$$\inf_{(x_1, x') \in [\vartheta + \frac{1}{\ell}, \vartheta + \ell] \times \mathbb{R}^{N-1}} (v(x_1, x') - \tilde{v}(x_1, x')) > 0 \quad (\ell = 1, 2, \dots)$$

or there is a sequence $\{x'_k\}$ in $\mathbb{R}^{N-1}$ such that

$$v(x_1, x'_k + x') - \tilde{v}(x_1, x'_k + x') \to 0 \quad (x_1 \geq \vartheta,\, x' \in \mathbb{R}^{N-1}).$$

This can be proved by limiting arguments as in the proof of the next lemma. An analogous remark applies to statement (ii).

Lemma 3.7. *Assume that v, $\tilde{v}$ are solutions of* (1.1) *with ranges in* $[-1, 1]$ *such that $\tilde{v} \leq v$. Then either*

$$\inf_{(x_1,x')\in[-\ell,\ell]\times\mathbb{R}^{N-1}} (v(x_1, x') - \tilde{v}(x_1, x')) > 0 \quad (\ell = 1, 2, \dots) \tag{3.14}$$

or there is a sequence $\{x'_k\}$ in $\mathbb{R}^{N-1}$ such that

$$v(x_1, x'_k + x') - \tilde{v}(x_1, x'_k + x') \to 0 \quad ((x_1, x') \in \mathbb{R}^{N-1}). \tag{3.15}$$

Proof. We use the fact that $w := v - \tilde{v}$ is a solution of the linear equation (2.4). Since $w \geq 0$, we will not need any sign condition on the coefficient $a(x)$.

Suppose that for some ℓ relation (3.14) is not valid. Then there is a sequence $\{(x_{1,k}, x'_k)\}$ in $[-\ell, \ell] \times \mathbb{R}^{N-1}$ such that $v(x_{1,k}, x'_k) - \tilde{v}(x_{1,k}, x'_k) \to 0$ and, for some $x_{1,0} \in [-\ell, \ell]$, $x_{1,k} \to \bar{x}_{1,0}$. As in the proof of Lemma 3.5, passing to a subsequence we obtain $v(x_1, x'+x'_k) \to \bar{v}(x_1, x')$, $\tilde{v}(x_1, x'+x'_k) \to \underline{v}(x_1, x')$, locally uniformly on $\mathbb{R}^N$, where $\bar{v}$, $\underline{v}$ are solutions of (1.1) such that

$$\bar{v} - \underline{v} \geq 0 \text{ on } \mathbb{R}^N \quad \text{and} \quad \bar{v}(\bar{x}_{1,0}, 0) - \underline{v}(x_{1,0}, 0) = 0. \tag{3.16}$$

Applying the strong maximum principle (or Harnack inequality) to $\tilde{w} = \bar{v} - \underline{v}$, a nonnegative solution of a linear equation, we obtain $\bar{v} \equiv \underline{v}$. In particular, we have found a sequence such that (3.15) holds. □

3.3 Proof of Proposition 1.1

Assume that v is a solution of (1.1) satisfying hypothesis (U). By the strong comparison principle, $|v| < 1$.

We show first of all that $f \not\equiv 0$ on any interval of the form $[-1, -1 + \delta]$ or $[1 - \delta, 1]$ with $\delta > 0$. Indeed, suppose that, say, $f \equiv 0$ on $[-1, -1 + \delta]$. Let D be a connected component of the set $\{x \in \mathbb{R}^N : v(x) < -1 + \delta\}$. Then the function $w := -1 + \delta - v$ is bounded and harmonic in D, and $w = 0$

on ∂D. Moreover, by (U), the complement $\mathbb{R}^N \setminus D$ contains a half plane, which is known to be sufficient for the maximum principle to apply to $\pm w$ (see [2, 16, 17], for example). This gives $w \equiv 0$ in D, which is a contradiction to (U).

It follows from the property just established that the set $\mathcal{N}$ of the nodal intervals of the function R_0 is nonempty (cp. (3.4)) and the function F satisfies conditions (1.6), (1.8) for some $\delta > 0$.

As indicated in Section 2, for the proof of Proposition 1.1 we take the functions

$$v^-(x_1) := \inf_{x' \in \mathbb{R}^{N-1}} v(x_1, x'), \quad v^+(x_1) := \sup_{x' \in \mathbb{R}^{N-1}} v(x_1, x')$$

as initial data for (3.1). We denote by $u^\pm$ the solution of (3.1), (3.7) with $u_0 = \pm v^\pm$. Note that the solutions have the following properties:

$$u^-(x_1, t) \le v^-(x_1) \quad (x_1 \in \mathbb{R},\ t > 0), \tag{3.17}$$

$$u^+(x_1, t) \ge v^+(x_1) \quad (x_1 \in \mathbb{R},\ t > 0). \tag{3.18}$$

Indeed, we can view u^- as a solution of the multidimensional equation

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^N,\ t > 0, \tag{3.19}$$

independent of x'. By the comparison principle, $u^-(x_1, t) \le v(x_1, x')$. Taking the infimum over $x' \in \mathbb{R}^{N-1}$, we obtain (3.17). Relation (3.18) is proved similarly.

Next, observe that hypothesis (U) implies that (3.8) is satisfied if $u_0 = v^\pm$. Thus Theorem 3.4 applies to the solutions $u^\pm$. We use this to prove that all traveling fronts in the minimal propagating terrace are standing waves, that is,

$$c_I = 0 \quad (I \in \mathcal{N}). \tag{3.20}$$

Indeed, suppose that $c_I > 0$ for some interval $I = (a, b) \in \mathcal{N}$ (the case $c_I <$ can be ruled out in an analogous way). From Theorem 3.4(b) and (3.17) we obtain

$$v^-(c_I t + \zeta_I(t)) \ge u^-(c_I t + \zeta_I(t), t) = (a + b)/2 \in (-1, 1) \quad (t > s_I).$$

By Theorem 3.4(a), $c_I t + \zeta_I(t) \to \infty$ as $t \to \infty$, and we have a contradiction to $v^-(\infty) = 0$. Relations (3.20) are proved.

Remembering that f does not vanish identically on any interval $[-1,-1+\delta]$, $\delta > 0$, we obtain from (3.4) that one of the following two possibilities occurs. Either $I_0 := (-1,b) \in \mathcal{N}$ for some $b > 0$, or there is a sequence of intervals $I_k = (a_k, b_k) \in \mathcal{N}$, $k = 1, 2, \ldots$ such that $b_k \to -1$. However, (3.20) and Lemma 3.3 imply that if the latter occurs, then the points b_k, $k = 1, 2, \ldots$ are global maximizers of F in $[-1, 1]$, thus $F(b_k) \geq F(-1) = 0$, in contradiction to (1.6). Therefore, the former possibility must occur. Similarly one shows, using (1.8), that $\mathcal{N}$ contains an interval $I_1 := (a, 1)$, with $a < 1$. Clearly, the standing waves $\varphi^- := \phi_{I_0}$ and $\varphi^+ := \phi_{I_1}$ satisfy all the relations in the first statement of Proposition 1.1. Moreover, according Lemma 3.3, the points -1, 1 are global maximizers of F in $[-1, 1]$, hence $F \leq F(1) = F(-1) = 0$. The first statement of Proposition 1.1 is proved.

The second statement of Proposition 1.1 is a part of the following lemma.

Recall that we are assuming that v is a solution of (1.1) satisfying (U).

Lemma 3.8. *Let φ be a solution of* (1.3) *such that $|\varphi| < 1$ and $\varphi' < 0$.*

If $\varphi(\infty) = -1$ and (H−) *holds, then $\varphi(-\infty) = 1$ and there is $\xi \in \mathbb{R}$ such that*

$$\varphi(x_1 + \xi) \leq v(x_1, x') \quad ((x_1, x') \in \mathbb{R}^N). \tag{3.21}$$

If $\varphi(-\infty) = 1$ and (H+) *holds, then $\varphi(\infty) = -1$ and there is $\theta \in \mathbb{R}$ such that*

$$\varphi(x_1 + \theta) \geq v(x_1, x') \quad ((x_1, x') \in \mathbb{R}^N). \tag{3.22}$$

Proof. We only prove the first statement, the proof of the second statement being analogous.

As noted in Section 2, for suitable η the function φ coincides on $[\eta, \infty)$ with the solution of (2.5) and this yields estimate (2.7). Using this and hypothesis (U), we find constants $\vartheta \in \mathbb{R}$ and $\xi < \vartheta - 1$ such that

$$\varphi(\vartheta - \xi) \leq v(\vartheta, x') \quad (x' \in \mathbb{R}^{N-1}), \tag{3.23}$$

$$g^-(\varphi(x_1 - \xi)) < \frac{1}{4(x_1 - \xi)^2} < \frac{1}{4(x_1 - \vartheta + 1)^2} \quad (x_1 > \vartheta). \tag{3.24}$$

Applying Lemma 3.5 with $\tilde{v} = \varphi(\cdot - \xi)$, we obtain

$$\varphi(x_1 - \xi) \leq v(x_1, x') \quad (x_1 \geq \vartheta,\, x' \in \mathbb{R}^{N-1}). \tag{3.25}$$

We next show that $b := \varphi(-\infty)$ must be equal to 1. Assume not: $b < 1$. Then, in view of the monotonicity of φ and hypothesis (U), making ξ smaller if necessary, we achieve that $\varphi(\cdot - \xi) \le v$ everywhere in $\mathbb{R}^N$. Set

$$\varsigma := \sup\{\xi \in \mathbb{R} : \varphi(\cdot - \xi) \le v\}.$$

Obviously, $\varsigma \in \mathbb{R}$ and $\varphi(\cdot - \varsigma) \le v$. In view of the assumption $b < 1$ and hypothesis (U), Lemma 3.7 implies that relation (3.14) must hold with $\tilde{v} = \varphi(\cdot - \varsigma)$ for all $\ell = 1, 2, \dots$. Using this (and assumptions $b < 1$ and (U) again), we obtain that for each $\bar{\vartheta}$ one has

$$\inf_{(x_1,x')\in(-\infty,\bar{\vartheta}]\times\mathbb{R}^{N-1}} (v(x_1, x') - \varphi(\cdot - \varsigma)) > 0. \tag{3.26}$$

Using hypothesis (H−), we find $\bar{\vartheta}$ such that

$$g^-(\varphi(x_1 - \xi)) < \frac{1}{4(x_1 - \bar{\vartheta} + 1)^2} \quad (x_1 > \bar{\vartheta}) \tag{3.27}$$

for all $\xi \in [\varsigma, \varsigma + 1]$. From (3.26) we infer that if $\xi > \varsigma$ is sufficiently close to ς, then $\varphi(\cdot - \xi) < v$ in $\{(x_1, x') : x_1 \le \bar{\vartheta}\}$. Applying Lemma 3.5 with $\tilde{v} = \varphi(\cdot - \xi)$, we obtain that $\varphi(\cdot - \xi) < v$ holds in $\{(x_1, x') : x_1 > \bar{\vartheta}\}$ as well. This is a contradiction to the definition of ς.

This contradiction shows that $b < 1$ is impossible, hence $b = \varphi(-\infty) = 1$.

It remains to prove (3.21). We already know—see (3.25)—that this relation holds for all $x_1 \ge \vartheta$ if ξ, $\vartheta \in \mathbb{R}$ are chosen suitably. Clearly, in view of the monotonicity of φ and hypothesis (U), we can make ξ smaller so as to achieve that

$$\varphi(x_1 - \xi) \le v(x_1, x') \quad (x_1 \ge \xi,\, x' \in \mathbb{R}^{N-1}). \tag{3.28}$$

Consider now the solution of the Cauchy problem

$$z_{yy} + f(z) = 0, \quad y \in \mathbb{R}, \tag{3.29}$$
$$z(\xi) = \varphi(0),\; z'(\xi) = \nu, \tag{3.30}$$

where $\nu \in (\varphi'(0), 0)$. We have $\sup_{y\in\mathbb{R}} z(y) < 1$. To show this, first note that the phase-plane trajectory of (z, z_y) is trapped inside the homoclinic loop consisting of the equilibria $(-1, 0)$, $(1, 0)$, and the trajectories of $(\varphi, \pm\varphi_y)$. This gives $|z| < 1$. Now $z_y^2/2 + F(z) \equiv c$, where the constant c satisfies $c < 0 = \varphi_y^2/2 + F(\varphi)$, due to the choice of initial conditions. Thus (z, z_y) stays at a positive distance from the equilibrium $(1, 0)$, which gives the desired relation.

Let z^ν stand for the solution of (3.29), (3.30). Fixing some $\nu_0 \in (\varphi'(0), 0)$, the relation $\sup_{y\in\mathbb{R}} z^{\nu_0}(y) < 1$ just established and hypothesis (U) imply that—possibly upon making ξ smaller, which has no effect on (3.28)—the following holds for $\nu = \nu_0$:

$$z^\nu(x_1) \le v(x_1, x') \quad (x_1 \le \xi,\ x' \in \mathbb{R}^{N-1}). \tag{3.31}$$

Let

$$\nu_1 := \inf\{\nu \in (\varphi'(0), \nu_0] : (3.31) \text{ holds}\}.$$

We claim that $\nu_1 = \varphi'(0)$. Indeed, if not then (3.31) holds with $\nu = \nu_1$ and $\sup_{y\in\mathbb{R}} z^{\nu_1}(y) < 1$. Since v satisfies (U), the strict inequality holds in (3.31) and, moreover, limiting arguments as in the proof of Lemma 3.5 (cp. Lemma 3.7) show that $\nu = \nu_1$ can be properly decreased while preserving relation (3.31). This contradiction to the definition of ν_1 proves our claim. Due to the continuity of solutions with respect to the initial conditions, we have $z^\nu \to \varphi(\cdot - \xi)$ uniformly on compact subintervals of $(-\infty, \xi]$. Therefore, (3.31) yields

$$\varphi(x_1 - \xi) \le v(x_1, x') \quad (x_1 \le \xi,\, x' \in \mathbb{R}^{N-1}). \tag{3.32}$$

This and (3.28) give (3.21). □

3.4 Proof of Theorem 1.3

Assume that the hypotheses of Theorem 1.3 are satisfied. Using Proposition 1.1, we find a solution φ of (1.3), (1.4) and a positive number ξ such that $\varphi' < 0$ and

$$\varphi(x_1 + \xi) \le v(x_1, x') \le \varphi(x_1) \quad ((x_1, x') \in \mathbb{R}^N). \tag{3.33}$$

With an arbitrary $\sigma \in \mathbb{R}^{N-1}$ fixed, consider the family of functions $v^\eta(x_1, x') := v(x_1 + \eta, x' + \sigma)$, $\eta \in \mathbb{R}$ (the idea to use this family in a sliding argument comes from [4]). Taking $\eta \in [0, \xi]$, we have, by (3.33) and the monotonicity of φ,

$$\varphi(x_1 + 2\xi) \le \varphi(x_1 + \xi + \eta) \le v^\eta(x_1, x') \le \varphi(x_1 + \eta) \le \varphi(x_1). \tag{3.34}$$

As shown in Section 2, hypotheses (H−), (H+) give estimates (2.7), (2.8). Since the function g^- is increasing and the function g^+ is decreasing (cp.

(1.10)), estimates (2.7), (2.8) in conjunction with (3.34), (3.33) imply that if $\vartheta > 0$ is sufficiently large, then

$$\begin{aligned} g^-(v^\eta(x_1,x')) &< \frac{1}{4(x_1-\vartheta+1)^2} \quad (x_1 \geq \vartheta,\, x' \in \mathbb{R}^{N-1};\, \eta \in [0,\xi]), \\ g^+(v(x_1,x')) &< \frac{1}{4(x_1+\vartheta-1)^2} \quad (x_1 \leq -\vartheta,\, x' \in \mathbb{R}^{N-1}). \end{aligned} \tag{3.35}$$

Notice that for $\eta = \xi$ we have, by (3.33),

$$v^\xi(x_1,x') \leq \varphi(x_1+\xi) \leq v(x_1,x'). \tag{3.36}$$

Let now

$$\eta_0 := \inf\{\eta \in (0,\xi] : v^\eta \leq v\}. \tag{3.37}$$

We want to prove that $\eta_0 = 0$. Suppose for a contradiction that $\eta_0 > 0$. Then $v^{\eta_0} \leq v$ and, by Lemma 3.7, either

$$\inf_{(x_1,x')\in[-\vartheta,\vartheta]\times\mathbb{R}^{N-1}} (v(x_1,x') - v^{\eta_0}(x_1,x')) > 0 \tag{3.38}$$

or there is a sequence $\{x'_k\}$ in $\mathbb{R}^{N-1}$ such that

$$v(x_1,x'_k+x') - v^{\eta_0}(x_1,x'_k+x') \to 0 \quad (x_1 \in \mathbb{R},\, x' \in \mathbb{R}^{N-1}). \tag{3.39}$$

If the latter holds, we follow an argument of [4]. By standard regularity estimates, passing to a subsequence, we may assume that $v(x_1,x'+x'_k) \to \bar{v}(x_1,x')$, locally uniformly on $\mathbb{R}^N$, where $\bar{v}$ is a solution of (1.1). From (3.39) we obtain that $\bar{v}(x_1+\eta_0,x'+\sigma) = \bar{v}(x_1,x')$ for all $(x_1,x') \in \mathbb{R}^N$. It follows that the function $\bar{v}$ is periodic with period $(\eta_0,\sigma) \in \mathbb{R}^N$. However, it is clear that $\bar{v}$ satisfies hypothesis (U) and, since $\eta_0 > 0$, this is not consistent with the periodicity.

Having ruled out (3.39), relation (3.38) implies that for $\eta < \eta_0$ sufficiently close to η_0 we have $v > v^\eta$ in $[-\vartheta,\vartheta] \times \mathbb{R}^{N-1}$. Applying Lemma 3.5 with $\tilde{v} = v^\eta$–which is legitimate by (3.35)–we obtain $v > v^\eta$ on $\mathbb{R}^N$, contradicting the definition of η_0.

Thus, indeed, $\eta_0 = 0$, which gives

$$v(x_1,x') \geq v(x_1,x'+\sigma) \quad ((x_1,x') \in \mathbb{R}^N).$$

Since $\sigma \in \mathbb{R}^{N-1}$ is arbitrary, this clearly implies that $v(x_1,x')$ is independent of x'. The proof of the theorem is complete.

4 Details for Example 1.4

Consider the nonlinearity f defined for $u \approx -1$ as in Example 1.4:

$$f(u) = (u+1)^p(-1 + b\cos\log(u+1)),$$

where $p > 1$,

$$(1 + \frac{1}{p^2})^{-\frac{1}{2}} < b < 1, \tag{4.1}$$

and

$$\frac{b(1+\frac{1}{p^2})^{\frac{1}{2}} - 1}{1-b} < \frac{(p-1)^2}{8p(p+1)}. \tag{4.2}$$

We have

$$\begin{aligned} f'(u) &= p(u+1)^{p-1}\big(-1 + b\cos\log(u+1) - \frac{b}{p}\sin\log(u+1)\big) \\ &= p(u+1)^{p-1}\big(-1 + b(1+\frac{1}{p^2})^{\frac{1}{2}}\cos(\alpha + \log(u+1))\big) \end{aligned}$$

for some constant $\alpha = \alpha(p)$. Since $b(1+1/p^2)^{1/2} > 1$ (cp. (4.1)), f' changes sign in any interval $(-1, -1+\epsilon)$, $\epsilon > 0$. Hence, (M$-$) is not satisfied.

We next show that (H$-$) is satisfied. We have

$$g^-(u) = \max_{s\in[-1,u]} f'(s) \le p(u+1)^{p-1}(b(1+1/p^2)^{1/2} - 1).$$

Further, the obvious relation $f(u) \le (b-1)(u+1)^p$ gives $F(u) \le (b-1)(u+1)^{p+1}/(p+1)$ and

$$(-2F(u))^{-\frac{1}{2}} \le \Big(\frac{p+1}{2(1-b)}\Big)^{\frac{1}{2}}(u+1)^{-\frac{p+1}{2}}.$$

Hence, for $u \in (-1, -1+\delta)$,

$$E^-(u) = \int_u^{-1+\delta} \frac{ds}{\sqrt{-2F(s)}}\,ds \le c + \Big(\frac{p+1}{2(1-b)}\Big)^{\frac{1}{2}}\frac{2}{p-1}(u+1)^{\frac{1-p}{2}},$$

where c is a constant. Combining the above estimates, we obtain

$$\limsup_{u\to-1}(E^-(u))^2 g^-(u) \le \frac{b(1+1/p^2)^{1/2} - 1}{1-b}\,\frac{2p(p+1)}{(p-1)^2} < \frac{1}{4}$$

(we have used (4.2) for the last inequality). Thus, (H$-$) is satisfied.

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