A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation

Peter Poláčik*
School of Mathematics, University of Minnesota
Minneapolis, MN 55455

Pavol Quittner†
Department of Applied Mathematics and Statistics, Comenius University,
Mlynská dolina, 84248 Bratislava, Slovakia

Abstract. We consider the semilinear parabolic equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$, where the power nonlinearity is subcritical. We first address the question of existence of entire solutions, that is, solutions defined for all $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Our main result asserts that there are no positive radially symmetric bounded entire solutions. Then we consider radial solutions of the Cauchy problem. We show that if such a solution is global, that is, defined for all $t \geq 0$, then it necessarily converges to 0, as $t \to \infty$, uniformly with respect to $x \in \mathbb{R}^N$.

Key words: Subcritical semilinear heat equation, entire solutions, Liouville theorem, decay of global solutions.

1 Introduction

In this paper we study nonnegative solutions of parabolic equations of the form

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{1.1}$$

*Supported in part by NSF Grant DMS-0400702
†Supported in part by VEGA Grant 1/0259/03

where $p > 1$. We are particularly interested in classical solutions defined for all $t \in \mathbb{R}$, that is, solutions contained in $C^{2,1}(\mathbb{R}^N \times \mathbb{R})$. Below we refer to such solutions as *entire solutions*. We address the question of existence of positive bounded entire solutions. In case $p \geq p_S$, where p_S is the Sobolev critical exponent,

$$p_S := \begin{cases} \dfrac{N+2}{N-2}, & \text{if } N \geq 3, \\ \infty, & \text{if } N \in \{1,2\}, \end{cases} \tag{1.2}$$

such solutions exist. In fact, there are positive bounded steady states, or time-independent solutions. On the other hand, it has been known for some time that no positive bounded steady states exist for $p < p_S$ (see [8, 4]). The question of existence of more general entire positive solutions has not been settled yet in the entire subcritical range. The following partial results are available. For $p \leq 1 + 2/N$, classical Fujita-type results rule out positive bounded solutions even on $(0, \infty)$. A result of Bidaut-Véron [2] implies nonexistence of positive bounded entire solutions in the larger range $p < N(N+2)/(N-1)^2$. As consequence of a result of Merle and Zaag [12], one obtains that for any $p < p_S$ there are no positive entire solutions satisfying the extra condition

$$\limsup_{t \to -\infty} |t|^{\frac{1}{p-1}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} < \infty. \tag{1.3}$$

More specifically, they proved that any such solution is necessarily constant in space, thus, being a positive solution of $u_t = u^p$, it blows up in finite time. For dimensions $N \leq 3$ (and any $p < p_S$), the nonexistence of positive entire solutions which are radial and radially decreasing has been proved by Matos and Souplet [11]. In this paper we prove the nonexistence in the radial case without any monotonicity assumption and without restrictions on the dimension.

Theorem 1.1. *Let $p < p_S$. If $u = u(|x|, t)$ is a nonnegative radial entire solution of* (1.1) *such that u is bounded in $\mathbb{R}^N \times (-\infty, T]$ for some $T \in \mathbb{R}$ then $u \equiv 0$.*

Our proof is based on intersection comparison arguments (therefore the restriction to the radial case) and on properties of (sign changing) steady states of (1.1). Whenever we consider sign changing solutions the nonlinearity u^p is interpreted as $|u|^{p-1}u$.

Liouville-type results for elliptic and parabolic equations have proved very useful in many applications. For example, combined with scaling arguments they yield a priori bounds on positive steady states or time-dependent solutions of the problem

$$\begin{aligned} u_t - \Delta u &= \lambda u + a(x)u^p, && x \in \Omega,\ t > 0, \\ u &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x) \geq 0, && x \in \Omega, \end{aligned}$$

on bounded domains Ω. Such bounds play a crucial role in the study of equilibria, heteroclinic orbits and blow-up. See [6, 11, 13] and the references therein for a discussion of these and more general results.

As another application, let us now derive the following result on asymptotic behavior of global positive radial solutions of the Cauchy problem

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^N,\ t > 0, \tag{1.4}$$

$$u(x,0) = u_0(|x|). \tag{1.5}$$

A global solution refers to a (classical) solution defined for all $t \in (0, \infty)$.

Theorem 1.2. *Let $p < p_S$ and let $u_0 \in L^\infty(0,\infty)$ be nonnegative. If the solution u of* (1.4), (1.5) *is global, then $u(\cdot,t) \to 0$ as $t \to \infty$ uniformly in $\mathbb{R}^N$.*

Theorem 1.2 significantly improves a result by Souplet [16] concerning the radial case. Its proof consists of three steps in which we prove the following.

Step 1. $u(\cdot,t)$ is bounded, uniformly with respect to t.

Step 2. $u(\cdot,t) \to 0$ as $t \to \infty$ in $C_{loc}(\mathbb{R}^N)$.

Step 3. $u(\cdot,t) \to 0$ as $t \to \infty$ uniformly in $\mathbb{R}^N$.

The last two steps are based, respectively, on Theorem 1.1 and a result on nonexistence of positive bounded entire solution of (1.1) (without the symmetry restriction) for $N = 1$. This one-dimensional nonexistence result follows from [2], but we provide an independent, more elementary proof using similar arguments as in the proof of Theorem 1.1 (see Proposition 3.1). As for Step 1, we give two independent proofs of the boundedness of u. The first one is based on a result by Galaktionov and Lacey [7] which says that $u(x,t)$

becomes increasing in t at any point x where it attains a sufficiently large value. The second proof relies on the nonexistence results in Theorem 1.1 and Proposition 3.1, and certain rescaling arguments. These rescaling arguments have broader applicability. In particular, we use them in Proposition 3.3 to show that Theorem 1.2 remains valid for non-radial solutions, provided (1.1) does not possess positive bounded entire (radial or non-radial) solutions. This more general Liouville-type result is already known, as remarked above, for $p < N(N+2)/(N-1)^2$. It is most likely valid for the remaining subcritical values of p as well, but this for now remains open.

2 Preliminaries

From now on we assume that $p < p_S$. As we deal with radial solutions only, they can be viewed as functions of $x \in \mathbb{R}^N$ or of $r = |x| \in \mathbb{R}$. It should cause no confusion that we shall indifferently write $u(x,t)$ or $u(r,t)$ for the same solution.

2.1 Radial steady states

Let φ_1 be the solution of the equation

$$\varphi'' + \frac{N-1}{r}\varphi' + |\varphi|^{p-1}\varphi = 0, \quad r > 0, \tag{2.1}$$

satisfying $\varphi(0) = 1$, $\varphi'(0) = 0$. Obviously $\varphi_1''(0) < 0$. It is well known that the solution is defined on some interval and it changes sign due to $p < p_S$. We denote by $r_1 > 0$ its first zero. By uniqueness for the initial-value problem, $\varphi_1'(r_1) < 0$. We thus have

$$\varphi_1(r) > 0 \text{ in } [0, r_1) \text{ and } \varphi_1(r_1) = 0 > \varphi_1'(r_1). \tag{2.2}$$

Clearly, $\varphi_\alpha(r) := \alpha\varphi_1(\alpha^{\frac{p-1}{2}} r)$ is the solution of (2.1) with $\varphi(0) = \alpha$, $\varphi'(0) = 0$, and with the first positive zero $r_\alpha = \alpha^{-\frac{p-1}{2}} r_1$. As an elementary consequence of the properties of φ_1 we obtain the following (cf. Figure 1).

Lemma 2.1. *Given any $m > 0$, we have*

$$\lim_{\alpha\to\infty} \left(\sup\{\varphi_\alpha'(r) : r \in [0, r_\alpha] \textit{ is such that } \varphi_\alpha(r) \le m\}\right) = -\infty. \tag{2.3}$$

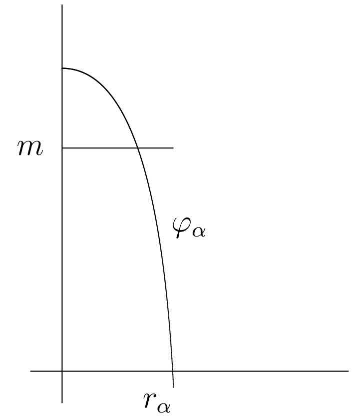

Figure 1: Graph of φ_α

2.2 Zero number

We use the notation φ_α, r_α introduced above.

Let $u = u(r,t)$, $r = |x|$, be a radial solution of (1.1) on a time interval (s,T). For $\alpha > 0$ we denote by $z_{[0,r_\alpha]}(u(\cdot,t) - \varphi_\alpha)$ the number of zeros of the function $u(\cdot,t) - \varphi_\alpha$ in $[0,r_\alpha]$. If u is a positive solution, then $u(r_\alpha,t) - \varphi_\alpha(r_\alpha) \neq 0$ for all t and in this case we have the following (see [5]).

Lemma 2.2. *Let $u = u(r,t)$ be a positive radial solution of* (1.1) *on a time interval (s,T). Then*

1. *$t \mapsto z_{[0,r_\alpha]}(u(\cdot,t) - \varphi_\alpha)$ takes finite values and it is nonincreasing,*
2. *if $t_0 \in (s,T)$ is such that $u(\cdot,t_0) - \varphi_\alpha$ has a multiple zero in $[0,r_\alpha)$, then $z_{[0,r_\alpha]}(u(\cdot,t) - \varphi_\alpha)$ drops at $t = t_0$:*

$$z_{[0,r_\alpha]}(u(\cdot,t_1) - \varphi_\alpha) > z_{[0,r_\alpha]}(u(\cdot,t_2) - \varphi_\alpha) \quad (s < t_1 < t_0 < t_2 < T).$$

   In particular, $u(\cdot,t) - \varphi_\alpha$ can have a multiple zero only for isolated values of t.

Note that since $u_r(0,t) - \varphi_\alpha'(0) = 0$, by symmetry, $u(\cdot,t) - \varphi_\alpha$ has a multiple zero (hence its zero muber drops) whenever $u(0,t) - \varphi_\alpha(0) = 0$.

Below we also consider non-symmetric solutions of (1.1) with $N = 1$. In the one-dimensional case it is more convenient to view φ_α as an even function of x with zero points $\pm x_\alpha$, where $x_\alpha = \alpha^{-\frac{p-1}{2}} r_1$ is the first positive zero. For any (not necessarily symmetric) solution $u(x,t)$ of (1.1) on a time interval

(s,T), the zero number $z_{[-x_\alpha,x_\alpha]}(u(\cdot,t)-\varphi_\alpha)$ has the properties stated in Lemma 2.2, with natural replacements of r by x and $[0,r_\alpha]$ by $[-x_\alpha,x_\alpha]$ (see [1]).

2.3 Compactness and limit sets

If u is a bounded solution of (1.1) on the interval $(-\infty,T)$, for some $T\le\infty$, then standard parabolic estimates [9] imply that u is bounded in $C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[T_1,T_2])$ for any ball $\Omega\subset\mathbb{R}^N$, $T_1<T_2<T$, $\theta\in(0,1)$ and the bound does not depend on Ω,T_1,T_2. It follows, by compactness of suitable imbeddings and a standard diagonalization, that each sequence $\tilde{t}_k\to-\infty$ has a subsequence t_k such that for some v one has

$$u(x,t_k+t)\to v(x,t)\quad(x\in\mathbb{R}^N,t\in\mathbb{R}),\tag{2.4}$$

and the convergence takes place in $C^{2+\theta,1+\theta/2}(\overline{\Omega}\times[-T_0,T_0])$ for each ball $\Omega\subset\mathbb{R}^N$, $T_0\in\mathbb{R}$ and $\theta\in[0,1)$. Consequently, v is a bounded entire solution of (1.1). Clearly, v is nonnegative or radial if u is such. Denoting by $\alpha(u)$ the α-limit set of u,

$$\alpha(u)=\{\psi\in C(\mathbb{R}^N):u(\cdot,t_k)\to\psi \text{ in } C_{loc}(\mathbb{R}^N) \text{ for some } t_k\to-\infty\},$$

the above remarks imply that $\alpha(u)$ is nonempty and for each $\psi\in\alpha(u)$ there is an entire bounded solution v of (1.1) such that $v(\cdot,0)=\psi$.

Similar remarks apply to any bounded solution defined on (T,∞) and its *ω-limit set,*

$$\omega(u)=\{\psi\in C(\mathbb{R}^N):u(\cdot,t_k)\to\psi \text{ in } C_{loc}(\mathbb{R}^N) \text{ for some } t_k\to\infty\}.$$

3 Proofs of the theorems

Proof of Theorem 1.1. Let $u=u(r,t)$ satisfy the hypotheses. The proof is by contradiction. Assume $u\not\equiv0$. Then $u>0$ everywhere by the maximum principle. By the boundedness assumption and parabolic estimates, u and u_r are bounded on $[0,\infty)\times(-\infty,T]$ for some $T\in\mathbb{R}$. Without loss of generality we may assume $T=0$. It follows from Lemma 2.1 that if α is sufficiently large and $t\le0$, then $u(\cdot,t)-\varphi_\alpha$ has exactly one zero and the zero is simple.

We next claim that

$$z_{[0,r_\alpha]}(u(\cdot,t)-\varphi_\alpha)\ge1\quad(t\le0,\alpha>0).\tag{3.1}$$

Indeed, if not then $u(\cdot,t_0) > \varphi_\alpha$ in $[0, r_\alpha]$ for some t_0. It is well known (see for example [10, 3]) that each solution of the Dirichlet problem

$$\begin{aligned} \overline{u}_t &= \Delta\overline{u} + \overline{u}^p, && |x| < r_\alpha,\ t > 0,\\ \overline{u} &= 0, && |x| = r_\alpha,\ t > 0,\\ \overline{u}(x,t_0) &= \overline{u}_0(x), && |x| < r_\alpha, \end{aligned}$$

blows up in finite time provided $\overline{u}_0 > \varphi_\alpha$ in $[0, r_\alpha]$. Choosing the initial function $\overline{u}_0$ between φ_α and $u(\cdot,t_0)$ we conclude, by comparison, that $\overline{u}$ and u both blow up in finite time, in contradiction to the global existence assumption on u. This proves the claim.

Set

$$\alpha_0 := \inf\{\beta > 0 : z_{[0,r_\alpha]}(u(\cdot,t) - \varphi_\alpha) = 1 \text{ for all } t \le 0 \text{ and } \alpha \ge \beta\}.$$

In view of the above remark on large α, we have $\alpha_0 < \infty$. Also $\alpha_0 > 0$. Indeed, for small $\alpha > 0$ we have $\varphi_\alpha(0) < u(0,t)$ for $t = 0$ and for $t \approx 0$. We can choose $t \approx 0$, $t < 0$, such that $\varphi_\alpha(0) - u(\cdot,t)$ has only simple zeros (cf. Lemma 2.2(ii)) and then, by (3.1), $z_{[0,r_\alpha]}(u(\cdot,t) - \varphi_\alpha) \ge 2$.

By definition of α_0 (and (3.1)), there are sequences $\alpha_k \nearrow \alpha_0$ and $t_k \le 0$ such that

$$z_{[0,r_{\alpha_k}]}(u(\cdot,t_k) - \varphi_{\alpha_k}) \ge 2 \quad (k = 1,2,\dots).$$

By the nonincrease of the zero number, we have

$$z_{[0,r_{\alpha_k}]}(u(\cdot,t_k + t) - \varphi_{\alpha_k}) \ge 2 \quad (t \le 0,\ k = 1,2,\dots). \tag{3.2}$$

This in particular allows us to assume, choosing different t_k if necessary, that $t_k \to -\infty$. Passing to a subsequence, we may further assume that (2.4) holds with convergence in $C^{2,1}_{loc}(\mathbb{R}^n \times \mathbb{R})$. Clearly then, there is $\delta > 0$ such that for each fixed t,

$$u(\cdot,t_k + t) - \varphi_{\alpha_k} \to v(\cdot,t) - \varphi_{\alpha_0}$$

in $C^1[0, r_{\alpha_0} + \delta]$. This and (3.2) imply that for each $t \le 0$, $v(\cdot,t) - \varphi_{\alpha_0}$ has at least two zeros or a multiple zero in $[0, r_{\alpha_0})$. Referring to Lemma 2.2(ii), we choose $t < 0$ so that $v(\cdot,t) - \varphi_{\alpha_0}$ has only simple zeros (and, hence at least two of them). Since $u(\cdot,t_k + t) - \varphi_{\alpha_0}$ is close to $v(\cdot,t) - \varphi_{\alpha_0}$ in $C^1[0, r_{\alpha_0}]$, if k is large, it has at least two simple zeros in $[0, r_{\alpha_0})$ as well. But then, for $\alpha > \alpha_0$, $\alpha \approx \alpha_0$, the function $u(\cdot,t_k + t) - \varphi_\alpha$ has at least two zeros in $[0, r_\alpha)$, contradicting the definition of α_0.

We have thus shown that the assumption $u \not\equiv 0$ leads to a contradiction, which proves the theorem. □

In the proof of Theorem 1.2 we shall need the following result.

Proposition 3.1. *Let* $N = 1$*. If* u *is a nonnegative entire solution of* (1.1) *such that* $\limsup_{t\to-\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} < \infty$ *then* $u \equiv 0$*.*

Proof. The proof is similar to the proof of Theorem 1.1, with some modifications that we now indicate.

Assuming $u(x,t)$ is an entire positive solution satisfying

$$\limsup_{t\to-\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} < \infty,$$

we consider the zero number $z_{[-x_\alpha,x_\alpha]}(u(\cdot,t) - \varphi_\alpha)$ for various values of α. First we observe, using Lemma 2.1, that for large α

$$z_{[-x_\alpha,x_\alpha]}(u(\cdot+b,t) - \varphi_\alpha) = 2 \quad (t \le 0,\ b \in \mathbb{R}).$$

Further, by the same argument as in the previous proof, we have

$$z_{[-x_\alpha,x_\alpha]}(u(\cdot+b,t) - \varphi_\alpha) \ge 2 \quad (t \le 0,\ b \in \mathbb{R},\ \alpha > 0). \tag{3.3}$$

Next we claim that for some $b \in \mathbb{R}$,

$$z_{[-x_\alpha,x_\alpha]}(u(\cdot+b,0) - \varphi_\alpha) > 2 \tag{3.4}$$

if α is sufficiently small. Indeed, choose α such that $u(0,0) > \varphi_\alpha(0)$. In view of (3.3) and the fact that $u(x,0) \ge \varphi_\alpha(x)$ ($x \in [-x_\alpha, x_\alpha]$) cannot hold, we either have (3.4) for $b = 0$ or else $u(\cdot,0) - \varphi_\alpha$ has no zeros in $[-x_\alpha, 0]$ or no zeros in $[0, x_\alpha]$, cf. Figure 2. Assuming that $u(\cdot,0) - \varphi_\alpha$ has no zeros in

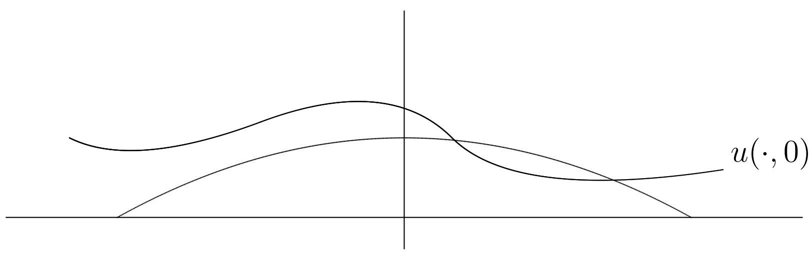

Figure 2: Graphs of $u(\cdot,0)$ and φ_α

$[-x_\alpha, 0]$ (the latter possibility is analogous), we start shifting the graph of $u(\cdot,0)$ to the right. It is clear that one of the following holds

(i) for some $b < 0$, $u(\cdot + b, 0) - \varphi_\alpha$ has at least two positive zeros and at least one negative zero,

(ii) $u(\cdot + b, x) \geq \varphi_\alpha(x)$ $(x \in [-x_\alpha, x_\alpha])$.

As noted in the previous proof, (ii) is impossible. Hence (i) holds which proves (3.4).

Now replacing $u(x,t)$ by $u(x+b,t)$, we can proceed analogously as in the radial case. Define

$$\alpha_0 = \inf\{\beta > 0 : z_{[-x_\alpha, x_\alpha]}(u(\cdot, t) - \varphi_\alpha) = 2 \text{ for all } t \leq 0,\ \alpha \geq \beta\}.$$

We have $\alpha_0 \in (0, \infty)$ and a contradiction is derived similarly as in the proof of Theorem 1.1. □

The first proof of Theorem 1.2. Let the hypotheses of the theorem be satisfied. We first prove that the solution u is bounded. Since there exists $\epsilon > 0$ such that u is bounded for $t \in [0, \epsilon]$ and the spatial derivative of $u(\cdot, \epsilon)$ is bounded as well, we may assume that u_0 is differentiable with bounded derivative. Arguing by contradiction, we distinguish two cases:

Case 1. $u(0,t)$ is unbounded.

Case 2. u is unbounded, but there is a constant c_0 such that

$$u(0,t) \leq c_0 \quad (t \geq 0). \tag{3.5}$$

To find a contradiction in Case 1, we use an argument of [7]. Since u_0 is bounded together with its derivative, by Lemma 2.1, we can choose α so large that $u(0,0) < \varphi_\alpha(0)$ and $z_{[0,r_\alpha]}(u(\cdot,0) - \varphi_\alpha) = 1$. Since $u(0,t)$ is unbounded, we have $u(0,t_1) > \varphi_\alpha(0)$ for some $t_1 > 0$. Then there exists $t_0 \in (0, t_1)$ such that $u(0, t_0) = \varphi_\alpha(0)$, hence $z_{[0,r_\alpha]}(u(\cdot,t) - \varphi_\alpha)$ drops at t_0. This implies that $u(r,t) > \varphi_\alpha(r)$ for each $r \in [0, r_\alpha)$ and hence u blows up in finite time (see the argument following (3.1)), a contradiction.

Consider Case 2. We use the following result of [7]. There exists $M_0 > 0$ (depending on $\|u_0\|_{L^\infty(0,\infty)} + \|u_0'\|_{L^\infty(0,\infty)}$) such that for each $M \geq M_0$ and $\rho > 0$, the relation $u(\rho, t_0) > M$ implies $u(\rho, t) > M$ (and $u_t(\rho,t) > 0$) for all $t > t_0$.

Choose $M > \max\{c_0, M_0\}$. Since u is unbounded, there is $\rho > 0$ such that $u(\rho, t) > M$ for some $t = t_0$ and consequently for all $t > t_0$. Hence u satisfies

$$u_t = \Delta u + u^p, \quad |x| < \rho,\ t > t_0, \tag{3.6}$$

$$u(x,t) > M, \quad |x| = \rho,\ t > t_0, \tag{3.7}$$

$$u(x,t_0) > 0, \quad |x| \le \rho. \tag{3.8}$$

Let ξ be the solution of

$$\dot{\xi} = \xi^p, \quad \xi(t_0) = \epsilon_0 := \min_{|x|\le\rho} u(x,t_0),$$

and let $T > t_0$ be such that $\xi(T) = M$ (note that $0 < \epsilon_0 \le c_0 < M$). Then, by comparison, $u(x,t) \ge \xi(t)$ for $|x| < \rho$ and $t_0 \le t \le T$. In particular, $u(0,T) \ge M > c_0$, contradicting the assumption in Case 2.

We have thus proved that u is bounded. Next we prove that $u(\cdot,t) \to 0$ in $C_{loc}(\mathbb{R}^N)$. By standard parabolic estimates, $\{u(\cdot,t) : t \ge 1\}$ is relatively compact in $C_{loc}(\mathbb{R}^N)$. We denote by $\omega(u)$ the ω-limit set of u in this space:

$$\omega(u) = \{\psi \in C(\mathbb{R}^N) : u(\cdot,t_k) \to \psi \text{ in } C_{loc}(\mathbb{R}^N) \text{ for some } t_k \to \infty\}.$$

Consider any function ψ in $\omega(u)$. By the invariance principle (see Section 2.1), there is an entire bounded solution v of (1.1) such that $v(\cdot,0) = \psi$ and $v(\cdot,t) \in \omega(u)$ for all t. Now, being in $\omega(u)$, the solution v is nonnegative radial and bounded, therefore, by Theorem 1.1, $v \equiv 0$. This shows that $\omega(u) = \{0\}$ and thus, by compactness, $u(\cdot,t) \to 0$ in $C_{loc}(\mathbb{R}^N)$.

Finally we prove that $u(\cdot,t) \to 0$ uniformly in $\mathbb{R}^N$. Assume on the contrary that there exist $c_1 > 0$, $t_k \to \infty$ and $r_k \to \infty$ such that

$$u((r_k, 0, 0, \dots, 0), t_k) \ge c_1.$$

Set

$$v_k(r,t) := u((r + r_k, 0, 0, \dots, 0), t + t_k),$$

where $(r,t) \in D_k := \{(r,t) : r > -r_k,\ t > -t_k + 1\}$. Notice that v_k are uniformly bounded in $C^{2+\theta,1+\theta/2}(D_k)$ for any $\theta \in [0,1)$ and satisfy

$$\partial_t v_k = \partial_{rr} v_k + \frac{N-1}{r + r_k}\partial_r v_k + v_k^p \quad \text{in } D_k, \qquad v_k(0,0) \ge c_1.$$

Passing to a suitable subsequence we obtain $v_k \to v$ in $C^{2,1}_{loc}(\mathbb{R}\times\mathbb{R})$, where v is a global positive solution of (1.1) with $N = 1$, which contradicts Proposition 3.1. □

The following lemma is a variant of [15, Lemma 2.2] and will be used in the second proof of Theorem 1.2.

Lemma 3.2. *Let the hypotheses of Theorem 1.2 be satisfied and $K > 0$. Let $r, t \geq 0$ and $u(r,t) > 0$. Set*

$$Q(r,t,K) := \{(\rho,\tau) \in [0,\infty)^2 : \rho \leq r + Ku(r,t)^{-(p-1)/2}, \\ \tau \leq t + Ku(r,t)^{-(p-1)}\}.$$

Then there exist $\tilde{r}, \tilde{t} \geq 0$ such that $u(\tilde{r},\tilde{t}) \geq u(r,t)$ and $u(\rho,\tau) \leq 2u(\tilde{r},\tilde{t})$ for any $(\rho,\tau) \in Q(\tilde{r},\tilde{t},K)$.

Proof. Set $r_0 := r$, $t_0 := t$ and $M_0 := u(r,t)$. If $u \leq 2M_0$ on $Q(r_0,t_0,K)$ then we may choose $\tilde{r} := r_0$, $\tilde{t} := t_0$. Otherwise there exists $(r_1,t_1) \in Q(r_0,t_0,K)$ such that $M_1 := u(r_1,t_1) > 2M_0$. If $u \leq 2M_1$ on $Q(r_1,t_1,K)$ then we choose $\tilde{r} := r_1$, $\tilde{t} := t_1$. Otherwise we find $(r_2,t_2) \in Q(r_1,t_1,K)$ such that $M_2 := u(r_2,t_2) > 2M_1$, etc. Notice that

$$r_{j+1} - r_j \leq KM_j^{-(p-1)/2} \leq K(2^j M_0)^{-(p-1)/2}, \\ t_{j+1} - t_j \leq KM_j^{-(p-1)} \leq K(2^j M_0)^{-(p-1)},$$

so that the sequence $\{(r_j,t_j)\}$ is bounded. Since u is bounded on bounded sets and $u(r_j,t_j) \geq 2^j M_0$, the inequality $u \leq 2M_j$ on $Q(r_j,t_j,K)$ has to be satisfied for some j and we may set $\tilde{r} := r_j$, $\tilde{t} := t_j$. $\square$

The second proof of Theorem 1.2. Let the hypotheses of the theorem be satisfied and $|u_0| \leq M$. We will prove that u is bounded; the rest of the proof is the same as in the first proof of Theorem 1.2. Assume on the contrary that there exist $r_k, t_k \geq 0$ such that $M_k := u(r_k,t_k) \to \infty$. We may assume $M_k \geq 2M$, hence $t_k \geq t_0$ for some $t_0 > 0$. Due to Lemma 3.2 we may also assume that $u \leq 2M_k$ on

$$Q_k := \{(r,t) \in [0,\infty)^2 : r \leq r_k + kM_k^{-(p-1)/2},\ t \leq t_k + kM_k^{-(p-1)}\}.$$

Set $\lambda_k := M_k^{-(p-1)/2}$. We distinguish two cases:

(i) the sequence $\{r_k/\lambda_k\}$ is bounded;

(ii) the sequence $\{r_k/\lambda_k\}$ is unbounded.

First assume (i). Set $v_k(\rho,s) := \frac{1}{M_k}u(r,t)$, where $\rho := r/\lambda_k$ and $s := (t-t_k)/\lambda_k^2$. Then v_k solves the equation $v_s - v_{\rho\rho} - \frac{N-1}{\rho}v_\rho = v^p$ in

$$D_k := \{(\rho,s) : 0 \le \rho \le r_k/\lambda_k + k,\ -t_0/\lambda_k^2 \le s \le k\},$$

$v_k(r_k/\lambda_k, 0) = 1$, $0 \le v_k \le 2$ in D_k. Passing to a subsequence we may assume $r_k/\lambda_k \to \rho_\infty \ge 0$ and $v_k \to v$ in $C^{2,1}_{loc}([0,\infty)\times\mathbb{R})$, where v is a positive radial bounded entire solution of (1.1) which contradicts Theorem 1.1.

Next consider case (ii). Without loss of generality we may assume $r_k/\lambda_k \to \infty$. Set $v_k(y,s) := \frac{1}{M_k}u(r,t)$, where $y := (r-r_k)/\lambda_k$, $s := (t-t_k)/\lambda_k^2$. Then v_k solves the equation

$$v_s - v_{yy} - \frac{N-1}{y + r_k/\lambda_k}v_y = v^p$$

in

$$D_k := \{(y,s) : -r_k/\lambda_k \le y \le k,\ -t_0/\lambda_k^2 \le s \le k\},$$

$v_k(0,0) = 1$, $0 \le v_k \le 2$ in D_k. Passing to a subsequence we may assume $v_k \to v$ in $C^{2,1}_{loc}(\mathbb{R}\times\mathbb{R})$, where v is a positive bounded entire solution of (1.1) with $N = 1$, which contradicts Proposition 3.1. $\square$

In the following proposition we state a non-radial analogue to Theorem 1.2.

Proposition 3.3. *Let p be such that the equation* (1.1) *does not possess positive bounded entire solutions. Let $u_0 \in L^\infty(\mathbb{R}^N)$ be nonnegative. If the solution u of* (1.4) *satisfying the initial condition $u(\cdot,0) = u_0$ is global, then $u(\cdot,t) \to 0$ as $t \to \infty$ uniformly in $\mathbb{R}^N$.*

Sketch of the proof. The proof follows the lines of the second proof of Theorem 1.2. To prove the boundedness by contradiction, choose (x_k,t_k) such that $M_k := u(x_k,t_k) \to \infty$ and $u \le 2M_k$ on

$$Q_k := \{(x,t) : |x| \le |x_k| + kM_k^{-(p-1)/2},\ t \le t_k + kM_k^{-(p-1)}\}.$$

Then use the rescaling $v_k(y,s) := \frac{1}{M_k}u(r,t)$, where $y := (x-x_k)/\lambda_k$, $s := (t-t_k)/\lambda_k^2$. The rest of the proof is a straightforward modification of arguments in the proofs of Theorem 1.2. $\square$

References

[1] S. Angenent, The zero set of a solution of a parabolic equation, *J. reine angew. Math.* **390** (1988), 79-96.

[2] M.-F. Bidaut-Véron, Initial blow-up for the solutions of a semilinear parabolic equation with source term, In: *Equations aux dérivées partielles et applications, articles dédiés à Jacques-Louis Lions*, Gauthier-Villars, Paris (1998), 189-198.

[3] T. Cazenave and P.-L. Lions, Solutions globales d'équations de la chaleur semi linéaires, *Comm. Part. Diff. Eqns.* **9** (1984), 955-978.

[4] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.* **63** (1991), 615-622.

[5] X.-Y. Chen and P. Poláčik, Asymptotic periodicity of positive solutions of reaction diffusion equations on a ball, *J. reine angew. Math.* **472** (1996), 17-51.

[6] Y. Du and S. Li, Nonlinear Liouville theorems and a priori estimates for indefinite superlinear elliptic equations, *Adv. Differential Equations* **10** (2005), 841-860.

[7] V.A. Galaktionov and A.A. Lacey, Monotonicity in time of large solutions to a nonlinear heat equation, *Rocky Mountain J. Math.* **28** (1998), 1279-1301.

[8] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* **34** (1981), 525-598.

[9] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Urall'ceva, Linear and quasilinear equations of parabolic type, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1968, Russian Original: "Nauka", Moscow 1967.

[10] P.-L. Lions, Asymptotic behavior of some nonlinear heat equations, *Physica D* **5** (1982), 293-306.

[11] J. Matos and Ph. Souplet, Universal blow-up rates for a semilinear heat equation and applications, *Adv. Differential Equations* **8** (2003), 615-639.

[12] F. Merle and H. Zaag, Optimal estimates for blowup rate and behavior for nonlinear heat equations, *Comm. Pure Appl. Math.* **51** (1998), 139-196.

[13] P. Poláčik and P. Quittner, Liouville type theorems and complete blow-up for indefinite superlinear parabolic equations, Progress in Nonlinear Differential Equations and Their Applications, Vol. 64, Birkhäuser, Basel, 2005, to appear.

[14] P. Quittner and F. Simondon, A priori bounds and complete blow-up of positive solutions of indefinite superlinear parabolic problems, *J. Math. Anal. Appl.* **304** (2005), 614-631.

[15] P. Quittner and Ph. Souplet, A priori estimates of global solutions of superlinear parabolic problems without variational structure, *Discrete Contin. Dyn. Systems* **9** (2003), 1277-1292.

[16] Ph. Souplet, Sur l'asymptotique des solutions globales pour une équation de la chaleur semi-linéaire dans des domaines non bornés, *C. R. Acad. Sci. Paris* **323** (1996), 877-882.