Some generic properties of Schrödinger operators with radial potentials

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June 19, 2018

Abstract. We consider a class of Schrödinger operators on $\mathbb{R}^N$ with radial potentials. Viewing them as self-adjoint operators on the space of radially symmetric functions in $L^2(\mathbb{R}^N)$, we show that the following properties are generic with respect to the potential:

(P1) the eigenvalues below the essential spectrum are nonresonant (that is, rationally independent) and so are the square roots of the moduli of these eigenvalues;

(P2) the eigenfunctions corresponding to the eigenvalues below the essential spectrum are algebraically independent on any nonempty open set.

*Supported in part by the NSF Grant DMS-1565388

†Supported in part by CONICYT-Chile Becas Chile, Convocatoria 2010

The genericity means that in suitable topologies the potentials having the above properties form a residual set. As we explain, (P1), (P2) are prerequisites for some applications of KAM-type results to nonlinear elliptic equations. Similar properties also play a role in optimal control and other problems in linear and nonlinear partial differential equations.

Key words: Schrödinger operators, radial potentials, rational independence of eigenvalues, algebraic independence of eigenfunctions, generic properties.

AMS Classification: 35J10, 35P99, 47A75, 47A55.

Contents

1 Introduction

In this paper we consider Schrödinger operators $-\Delta + V$, where Δ is the Laplacian in N variables $(x_1, \dots, x_N) =: x$ and V is a radial function on $\mathbb{R}^N$; namely, $V(x) = a(|x|)$ for some function $a : [0, \infty) \to \mathbb{R}$ bounded from below. Assuming that $a \in C^m[0, \infty)$ for an integer $m \geq 0$, we show the genericity, with respect to a C^m-topology, of certain properties of the eigenvalues and eigenfunctions of the self-adjoint operator A defined by $-\Delta + V$ on $X := L^2_{rad}(\mathbb{R}^N)$, the closed subspace of $L^2(\mathbb{R}^N)$ consisting of radially symmetric functions. Specifically, denoting by $\beta \in (-\infty, \infty]$ the bottom of the essential spectrum of A, $\beta := \inf \sigma_{ess}(A)$, let $\mu_1 < \mu_2 < \dots$ be the eigenvalues of A in $(-\infty, \beta)$ (there are finitely or countably many of them). We show that, generically, the sets of numbers $\{\mu_j\}$ and $\{\sqrt{|\mu_j|}\}$ are rationally independent and the eigenfunctions corresponding to $\mu_1, \mu_2 \dots$ are algebraically independent (the precise meaning of this is explained below). Note that if A has only discrete spectrum (a sufficient condition for this is that $\lim_{|x|\to\infty} V(x) = \infty$), then $\beta = \infty$ and the eigenvalues $\mu_1 < \mu_2 < \dots$ exhaust the whole spectrum of A.

Similar properties of eigenvalues and eigenfunctions of Schrödinger operators arise in several different contexts. The absence of any nontrivial rational relation between the eigenvalues plays a role in optimal control problems involving the Schrödinger and heat equations (see [5, 16, 31] and references therein). The quadratic independence of a set of eigenfunctions, in particular, the linear independence of the squares of the eigenfunctions, is also relevant in optimal control problems and in results on generic properties of eigenvalues. Indeed, the derivatives of simple eigenvalues with respect to parameters involve the squares of the eigenfunctions, and their linear independence often allows one to move the eigenvalues into preferred positions by adjusting the potential or the parameters (see the previous references or [11, 23], for example; see also [12, 28] and references therein for related perturbation results on bounded domains).

Higher powers of eigenfunctions and more general algebraic expressions involving eigenfunctions of Schrödinger operators on bounded domains occur frequently in realization problems in nonlinear parabolic equations. The goal in such problems is to show that equations of a given structure can generate complicated—in some sense arbitrary—dynamics on finite dimensional center manifolds. The algebraic formulas arise in the computation of the Taylor expansion of the center manifold reduction; their linear independence allows one to "prescribe" the Taylor expansion arbitrarily up to any finite order (see [7, 9, 20, 24, 25] or the survey [21]).

The square roots of the eigenvalues of Schrödinger operators on bounded or unbounded domains appear naturally as frequencies in second order evolution equations, such as the wave equation $u_{tt} = \Delta u + au$. The rational independence of (some of) the frequencies gives rise to quasiperiodic solutions of the linear equations. In nonlinear perturbations of such equations, the rational independence of the frequencies plays an important role in the normal-form computations [2, 3] as well in some KAM-type results (see [6, 15, 30], for example).

Our main motivation to consider the genericity of the properties outlined above stems from our investigation of nonlinear elliptic equations of the form

$$\Delta u + u_{yy} + a(x)u + f(x,u) = 0, \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where Δ is the Laplacian in x, as above, $a : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are sufficiently smooth functions, a is bounded, and $f(\cdot,0) \equiv f_u(\cdot,0) \equiv 0$.

We considered such equations in [22] with the main goal to give sufficient conditions, in terms of a and f, for the existence of solutions which decay to 0 as $|x| \to \infty$ and are quasiperiodic in $y \in \mathbb{R}$ (related results for elliptic equations on unbounded strips were previously obtained in [27, 29]). We were especially interested in the case where a and f are radially symmetric in x and the sought-after y-quasiperiodic solutions are also required to be radially symmetric in x. In this case, the sufficient conditions found in [22] include the following properties of the Schrödinger operator $A := -\Delta + a(x)$, considered as a self-adjoint operator on $L^2_{rad}(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N) \cap L^2_{rad}(\mathbb{R}^N)$:

(i) The bottom of the essential spectrum of A is positive.

(ii) For a given integer $n \geq 2$, A has exactly n negative eigenvalues $\mu_1 < \cdots < \mu_n$, and the square roots $\sqrt{|\mu_j|}$, $j = 1, \ldots, n$, of these eigenvalues are rationally independent.

(iii) If $\varphi_1, \ldots, \varphi_n$ are (radial) eigenfunctions of A corresponding to the eigenvalues $\mu_1, \ldots, \mu_n$, then the functions $\varphi_i^2 \varphi_j^2$, $1 \leq i \leq j \leq n$ are linearly independent on some nonempty open set.

In [22], we gave examples of radial potentials satisfying these conditions. One of the objectives of this paper is to show that these properties are generic in a suitable sense. This will come out as a special case of our results below. Of course, for (ii) and (iii) to make sense for a given n, one needs to take potentials for which A has at least n eigenvalues below the essential spectrum (the set of such potentials is nonempty and open in "reasonable" topologies).

Our main results are stated in detail in the next section. Their proofs are given in Section 3.

2 Statement of the main results

To consider generic properties of the operators $-\Delta + V$, we choose the following setup:

(A1) $V(x) = a_0(r) + b(r)$, where $r = |x|$ and, for some integer $m \geq 0$, $a_0 \in C^m[0, \infty)$, $a_0 \geq 0$, and $b \in C_b^m[0, \infty)$.

Here $C_b^m[0, \infty)$ stands for the space of all functions on $[0, \infty)$ which are continuous and bounded on $[0, \infty)$ together with their derivatives up to order

m. This space is equipped with the usual norm

$$\|f\|_{C_b^m[0,\infty)} = \sum_{j=0}^{m} \|f^{(j)}\|_{L^\infty(0,\infty)}.$$

(We generally work with spaces of real-valued functions and real operators, unless it is necessary to consider complexifications in the spectral theory.)

We say that some statement is *generic* (with respect to b, the function in (A1)), or, that a property of $-\Delta + V$ is generic, if there is a residual set $\mathcal{F}$ in $C_b^m[0,\infty)$ such that the statement holds (that is, $-\Delta + V$ has the property) for each $b \in \mathcal{F}$. Recall that a residual set is the intersection of a countable collection of open and dense sets. In particular, a residual set is dense, since $C_b^m[0,\infty)$ is a Banach space.

Using our techniques, we could also study more general potentials. For example, some singularities of a_0 at $r = 0$ could be allowed. For the sake of simplicity, we refrain from considering such operators.

Given a nonempty open set $U \subset \mathbb{R}^N$, let $\mathscr{D}(U)$ be the space of smooth (real-valued) functions on U with compact support. We write $\mathscr{D}$ for $\mathscr{D}(U)$ if $U = \mathbb{R}^N$.

Recall that the $L^2(\mathbb{R}^N)$-valued operator $A_0 : u \mapsto -\Delta u + a_0 u$, with domain $\mathscr{D}$, is essentially self-adjoint (see [13, Sect. 8.6]). This means that the closure of this operator, which we denote by $\tilde{A}_0$, is self-adjoint on $L^2(\mathbb{R}^N)$. Also, $\tilde{A}_0$ is the unique self-adjoint extension of A_0; in particular, it coincides with the Friedrichs extension of A_0 [8, 14]. For a_0 bounded (in the L^∞-norm), the domain of $\tilde{A}_0$ is given by $D(\tilde{A}_0) = H^2(\mathbb{R}^N)$, but in general the domain depends on a_0 and it is not always the "natural domain" $\{u \in H^2(\mathbb{R}^N) : a_0 u \in L^2(\mathbb{R}^N)\}$ [17, Example 3.7]. (For sufficient conditions for $D(\tilde{A}_0)$ to be equal to the natural domain see [17, 18] and references therein.) Nevertheless, from the assumption that $a_0 \geq 0$ and the characterization of $\tilde{A}_0$ as the Friedrichs extension it follows that

$$D(\tilde{A}_0) \subset D(\tilde{A}_0^{\frac{1}{2}}) = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} a_0 u^2 < \infty \right\}, \tag{2.1}$$

the set on the right being the domain of the closure of the quadratic form $u \mapsto (u, A_0 u)_{L^2}$ (the standard inner product of $L^2(\mathbb{R}^N)$). Moreover, for any $u \in D(\tilde{A}_0)$ there is $f \in L^2(\mathbb{R}^N)$ (namely, $f = \tilde{A}_0 u$) such that u is a weak solution of the elliptic equation

$$-\Delta u + a_0(x) u = f(x), \tag{2.2}$$

in the sense that

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + a_0 uv - fv) = 0 \quad (v \in \mathscr{D}). \tag{2.3}$$

We denote by $\tilde{A}$ the operator defined on $D(\tilde{A}) = D(\tilde{A}_0)$ as the sum of A_0 and the multiplication operator $u \mapsto bu$. For $b \in C_b^0$, the multiplication operator is bounded and self-adjoint on $L^2(\mathbb{R}^N)$, hence $\tilde{A}$ is self-adjoint. Also, $\tilde{A}$ is bounded from below: for all $u \in D(\tilde{A})$ one has $(u, \tilde{A}u)_{L^2} \geq c\|u\|_{L^2}$, where c is a constant and $(\cdot,\cdot)_{L^2}$, $\|\cdot\|_{L^2}$ stand for the usual inner product and norm of $L^2(\mathbb{R}^N)$, respectively.

Let now $X := L^2_{rad}(\mathbb{R}^N)$, the closed subspace of $L^2(\mathbb{R}^N)$ consisting of all radially symmetric functions, that is, the common fixed points of the bounded linear maps $u \mapsto u \circ R$, $R \in O(n)$. We denote by A the restriction of $\tilde{A}$ to X (with the domain $X \cap D(\tilde{A})$). Viewing X as a Hilbert space with the induced inner product, A is self-adjoint and bounded from below. As usual, $\sigma(A)$ and $\sigma_{ess}(A)$ denote the spectrum and the essential spectrum of A, respectively. Recall that $\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A)$, where $\sigma_d(A)$ (the discrete spectrum) is the set of all eigenvalues of A of finite multiplicity which are isolated in $\sigma(A)$. Set

$$\beta = \inf \sigma_{ess}(A), \tag{2.4}$$

with the understanding that $\beta = \infty$ if $\sigma_{ess}(A) = \emptyset$. The fact that A is bounded from below gives $\beta \geq \inf \sigma(A) > -\infty$ [14].

We are concerned with the eigenvalues in $\sigma(A) \cap (-\infty, \beta)$, that is, the eigenvalues below the bottom of the essential spectrum. Since they are isolated, they form a finite or countable sequence, which we label in an increasing manner:

$$\mu_1 < \mu_2 < \ldots; \qquad \sigma(A) \cap (-\infty, \beta) = \{\mu_j : j \in \mathbb{N},\ j < \nu + 1\}, \tag{2.5}$$

where ν is the cardinality of $\sigma(A) \cap (-\infty, \beta)$. By the radial symmetry, the eigenvalues μ_j are simple [26]. For each $j < \nu + 1$, we denote by φ_j the eigenfunction of A corresponding to μ_j, normalized in the $L^2(\mathbb{R}^N)$-norm. The normalization determines φ_j uniquely up to a sign; we select φ_j such that φ_j is positive for all sufficiently large $|x|$. This requirement makes sense, since the nodal set of φ_j is confined to a compact set (see [26]). Abusing the notation slightly, we view φ_j and V as functions of $x \in \mathbb{R}^N$ or $r = |x| \in [0, \infty)$, depending on the context. This should cause no confusion.

As a function of r, φ_j solves the following ordinary differential equation:

$$\varphi_j'' + \frac{N-1}{r}\varphi_j' - V\varphi_j + \mu_j\varphi_j = 0, \quad r > 0, \tag{2.6}$$

and $\varphi_j'(0) = 0$. We need to spend some words about the regularity of φ_j here. A priori, the eigenfunction φ_j, as a function of $x \in \mathbb{R}^N$, is a weak solution of

$$-\Delta u + (V(x) - \mu_j)u = 0$$

(cp. (2.3)). Since V is continuous on $\mathbb{R}^N$, the elliptic interior L^p estimates imply that $\varphi_j \in W^{2,p}_{loc}(\mathbb{R}^N)$ for any $p \in (1,\infty)$. In particular, $\varphi_j \in C^1(\mathbb{R}^N)$. This implies that φ_j, as a function of r, is in $C^1[0,\infty)$, $\varphi_j'(0) = 0$, and the above equation (2.6) is satisfied in the classical sense (and $\varphi_j \in C^2(0,\infty)$).

Definition 2.1. Given positive integers n, k, a finite set of numbers $\omega_1, \dots, \omega_n$ is said to be *nonresonant up to order k* if $\omega := (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ satisfies

$$\omega \cdot \alpha \neq 0 \quad (\alpha \in \mathbb{Z}^n \setminus \{0\},\ |\alpha| \leq k). \tag{2.7}$$

(Here $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $\omega \cdot \alpha$ is the usual dot product.) If (2.7) holds for all $k = 1, 2, \dots$, we say that $\omega_1, \dots, \omega_n$ are *nonresonant*, or, *rationally independent*. An infinite sequence of numbers is said to be rationally independent (or, nonresonant) if all its finite subsequences are nonresonant.

It will be convenient to define that the empty set is rationally independent.

Definition 2.2. Given positive integers n, k, a set of n continuous functions $\psi_1, \dots, \psi_n$ on an interval I is said to be *algebraically independent on I up to order k* if for any nonzero polynomial Q on $\mathbb{R}^n$ of degree at most k the function $y \mapsto Q(\psi_1(y), \dots, \psi_n(y))$ is not identical to zero on I. If this is true for all $k = 1, 2, \dots$, we say that $\psi_1, \dots, \psi_n$ are *algebraically independent on I*. An infinite sequence of continuous functions on I is said to be algebraically independent on I if all its finite subsequences are algebraically independent on I.

In other words, $\psi_1, \dots, \psi_n$ are rationally independent on I if the functions

$$\psi^\alpha, \quad \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq k, \tag{2.8}$$

are linearly independent on I (that is, the only linear combination of these functions which vanishes identically on I is the trivial one). Here, $\psi = (\psi_1, \dots, \psi_n)$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$,

$$\psi^\alpha = \psi_1^{\alpha_1} \psi_2^{\alpha_2} \dots \psi_n^{\alpha_n}. \tag{2.9}$$

Similarly as with rational independence, if a given set of functions is empty, we consider it to be algebraically independent.

With the notation introduced above, consider the following properties of the operator A:

(p1) The (finite or countable) sequence $\mu_1, \mu_2, \dots$ is rationally independent.

(p2) The sequence $\sqrt{|\mu_1|}, \sqrt{|\mu_2|}, \dots$ is rationally independent.

(p3) The sequence $\varphi_1, \varphi_2, \dots$ is algebraically independent on any interval $I \subset (0, \infty)$ (here, $\varphi_1, \varphi_2, \dots$ are viewed as functions of $r = |x|$).

It is understood here that (p1)–(p3) are trivially satisfied if the sets in question are void (i.e., if $\sigma(A) \cap (-\infty, \beta) = \emptyset$).

Our main result is that (p1)–(p3) are generic with respect to $b \in C_b^m[0, \infty)$:

Theorem 2.3. *Given any integer $m \geq 0$ and any nonnegative function $a_0 \in C^m[0, \infty)$, there is a residual set $\mathcal{F}$ in $C_b^m[0, \infty)$ such that if $b \in \mathcal{F}$, then* (p1)–(p3) *hold for the operator $A = -\Delta + V$, with V as in* (A1).

It is reasonable to ask if the theorem remains valid if one puts additional restrictions on the function b. For example, one may wish to work with potentials decaying to 0 as $r \to \infty$ (this guarantees that the operator A is a relatively compact perturbation of $-\Delta$ [13, 14]). We state one genericity result of this sort in the following theorem. Let Y_m^0 be the closed subspace of $C_b^m[0, \infty)$ consisting of all functions $b \in C_b^m[0, \infty)$ with $\lim_{r \to \infty} b(r) = 0$. We equip it with the induced norm.

Theorem 2.4. *The statement of Theorem 2.3 remains valid if the space $C_b^m[0, \infty)$ is replaced by Y_m^0.*

In this paper, we restrict our attention to Schrödinger operators with radial potentials and we consider them in the space of radially symmetric L^2-functions. The restriction is essential: in the full space $L^2(\mathbb{R}^N)$, for a nonempty open set of potentials the operator has eigenvalues below the essential spectrum which are not simple (each eigenvalue with a nonsymmetric eigenfunction is a multiple eigenvalue). For such operators, some eigenvalues in the sequence $\mu_1 < \mu_2 \le \mu_3 \le \dots$ are repeated, rendering the sequence resonant.

3 Proofs

For the whole section, we fix an integer $m \ge 0$ and a function a_0 as in (A1). Set $Y_m := C_b^m[0,\infty)$.

We use the notation introduced in the previous section. To stress the dependence on $b \in Y_m$, we sometimes use b as an argument of the operator A, writing $A(b)$, as well as for related quantities and functions defined above: $\beta = \beta(b)$—the bottom of the essential spectrum, $\nu = \nu(b)$—the cardinality of $\sigma(A) \cap (-\infty,\beta)$, $\mu_j = \mu_j(b)$ and $\varphi_j = \varphi_j(b)$—the eigenvalues in $(-\infty,\beta)$ and the corresponding eigenfunctions, respectively.

Given $q \in \mathbb{N} = \{1,2,\dots\}$ and $b \in Y_m$, we define

$$\beta_q := \min\{q, \beta - 1/q\}, \quad J_q := \{j \in \mathbb{N} : j < \nu + 1 \text{ and } \mu_j \le \beta_q\}.$$

Again, we will write $\beta_q(b)$, $J_q(b)$ when the dependence on b is to be explicitly indicated. By the definition of the essential spectrum, the set J_q is finite, whether ν is finite or infinite. Of course, the set may be empty.

For $q = 1,2,\dots;\ k = 1,2,\dots,$ denote by $\mathcal{G}_{q,k}$ the set of all $b \in Y_m$ such that the operator $A = A(b)$ has the following properties:

(p1qk) The set $\{\mu_j : j \in J_q\}$ is nonresonant up to order k.

(p2qk) The set $\{\sqrt{|\mu_j|} : j \in J_q\}$ is nonresonant up to order k.

(p3qk) The set of functions $\{\varphi_j : j \in J_q\}$ is algebraically independent up to order k on any interval $I \subset [1/q, q]$ of length at least $1/(2q)$.

By definition, (p1qk)–(p3qk) are trivially satisfied if the sets in question are empty, that is, if $J_q = \emptyset$.

Obviously, for any $b \in \cap_{q=1}^{\infty} \cap_{k=1}^{\infty} \mathcal{G}_{q,k}$ the operator $A(b)$ has the properties (p1)–(p3). To prove Theorem 2.3, it is therefore sufficient to show that the sets $\mathcal{G}_{q,k}$ are open and dense in Y_m. For the proof of Theorem 2.4, it will then suffice to prove that $Y_m^0 \cap \mathcal{G}_{q,k}$ is dense in Y_m^0.

In the proof of the openness, we use results from the perturbation theory for linear operators. Note that for $b \in Y_m$ the multiplication operator $u \mapsto bu$ is bounded on $X = L^2_{rad}(\mathbb{R}^N)$ and its operator norm is bounded from above by $\|b\|_{Y_m}$. Moreover, the map associating $b \in Y_m$ with the multiplication operator in $\mathcal{L}(L^2_{rad}(\mathbb{R}^N))$ is linear and bounded, therefore, standard results from the analytic perturbation theory apply. We recall some results we will need later on.

Lemma 3.1. *Assume that $b \in Y_0$ and $\nu(b) > 0$ (that is, $A(b)$ has some eigenvalues below $\beta(b)$). Given any $n \in \mathbb{N}$ with $n < \nu(b) + 1$, let $\vartheta \in (\mu_n(b), \beta(b))$ be such that*

$$\sigma(A(b)) \cap (-\infty, \vartheta] = \{\mu_j(b) : j = 1, \dots, n\}.$$

Then there is $\epsilon > 0$ such that the following statements hold.

(i) For each $\tilde{b} \in \mathcal{V} := \{\tilde{b} \in Y_0 : \|b - \tilde{b}\|_{Y_0} < \epsilon\}$ one has $n < \nu(\tilde{b}) + 1$ (so the eigenvalues $\mu_1(\tilde{b}) < \cdots < \mu_n(\tilde{b})$ are defined) and

$$\sigma(A(\tilde{b})) \setminus \{\mu_j(\tilde{b}) : j = 1, \dots, n\} \subset (\vartheta, \infty). \tag{3.1}$$

(ii) For $j = 1, \dots, n$, the function $\tilde{b} \mapsto \mu_j(\tilde{b})$ is analytic on $\mathcal{V}$ (the set defined in (i)) and one has the following formula for its derivative:

$$\mu_j'(\tilde{b})\bar{b} = \int_0^\infty \varphi_j^2(r)\bar{b}(r) r^{N-1}\, dr. \tag{3.2}$$

(iii) For $j = 1, \dots, n$, the function $\tilde{b} \mapsto \varphi_j(\tilde{b})$ is analytic on $\mathcal{V}$ as a $C[R_1, R_2]$-valued function for any $R_2 > R_1 > 0$ (more precisely, here we are taking the restriction of the radial function $\varphi_j(\tilde{b})$ to the interval $[R_1, R_2]$).

Obviously, the maps $\tilde{b} \mapsto \mu_j(\tilde{b})$, $\tilde{b} \mapsto \varphi_j(\tilde{b})$ in statements (i) and (ii) remain analytic when restricted to $Y_m \cap \mathcal{V}$, $m = 1, 2, \dots$.

Proof of Lemma 3.1. Statements (i), (ii) combine well known results on the the continuity and analyticity properties of simple eigenvalues (see [13, 14]) and the upper semicontinuity of the spectrum (in fact, for the specific perturbations considered here, the spectrum is both upper and lower semicontinuous, see [14, Theorem V.4.10]).

The analyticity of the eigenfunctions $\varphi_j(\tilde{b})$, as $L^2(\mathbb{R}^N)$-valued functions of $\tilde{b}$, is also a well-known consequence of the results in [14, Sections iv.3.3-iv.3.5]. This result can be improved using the resolvent of the operator $\tilde{A}_0$ (see the paragraph containing (2.1)). Since $\tilde{A}_0$ is positive and $\tilde{A} = \tilde{A}_0 + b$ (viewing b as a multiplication operator), the eigenfunction-eigenvalue relation gives

$$\varphi_j(\tilde{b}) = (\tilde{A}_0 - I)^{-1}((\mu_n(\tilde{b}) - 1 - \tilde{b})\varphi_j(\tilde{b})).$$

It follows that $\tilde{b} \mapsto \varphi_j(\tilde{b})$ is in fact analytic as a function taking values in the space $D(\tilde{A}_0)$ equipped with the graph norm $v \mapsto \|\tilde{A}_0 v\|_{L^2} + \|v\|_{L^2}$. As one easily verifies (cp. (2.1)), this space is continuously imbedded in $H^1(\mathbb{R}^N)$. Since we are restricting all operators to the radial space X and $H^1(\mathbb{R}^N) \cap X$ is continuously imbedded in $C[R_1, R_2]$ for any $R_2 > R_1 > 0$ (see [4, Lemma A.II]), we obtain the conclusion in (iii).

Finally, equation (3.2) can be found by differentiating (2.6) (here $V = a_0 + \tilde{b}$) with respect to $\tilde{b}$, using the differentiability of the maps $\tilde{b} \mapsto \mu_j(\tilde{b})$ and $\tilde{b} \mapsto \varphi_j(\tilde{b})$, multiplying the resulting equation by $r^{N-1}\varphi_j(\tilde{b})$ and integrating by parts. We remark that the integral in (3.2) makes sense since the eigenfunctions decay exponentially as $|x| \to \infty$ [1, 13]. □

We shall also use the fact that the bottom of the essential spectrum depends continuously on b. This is surely a well-known property, but we were unable to locate it in the literature. We give a proof here for completeness.

Lemma 3.2. *The function $b \mapsto \beta(b) \in (-\infty, \infty]$ is continuous on Y_0 (hence also on Y_m).*

The continuity is understood in the usual sense: given $b_0 \in Y_0$ and a neighborhood U of $\beta(b_0)$, one has $\beta(b) \in U$ if b is close enough to b_0 in Y_0—taking the intervals (θ, ∞), $\theta \in \mathbb{R}$, as neighborhoods of ∞.

Proof of Lemma 3.2. We use Persson's characterization of $\beta(b)$:

$$\beta(b) = \sup_{R>0} \inf \left\{ \frac{(\psi, A(b)\psi)_{L^2}}{\|\psi\|^2_{L^2}} : \psi \in X \cap \mathscr{D}(\mathbb{R}^N \setminus \bar{B}_R), \quad \psi \neq 0 \right\}, \qquad (3.3)$$

where $\bar{B}_R := \{x \in \mathbb{R}^N : |x| \leq R\}$. This is proved in [19] (see also [1, 10, 13]). Strictly speaking, the result in [19] is formulated for the Schrödinger operator on the full space $L^2(\mathbb{R}^N)$, without restricting it to the radial space X, but the proof works the same in the radial setting. Also, it is not difficult to prove—using the spherical harmonics expansion, for example—that the infimum in

(3.3) remains unchanged if $\psi \neq 0$ is allowed to vary in the whole space $\mathscr{D}(\mathbb{R}^N \setminus \bar{B}_R)$—in other words, the bottom of the essential spectrum in the full space and in the radial space are the same.

If b, $\tilde{b} \in Y_0$ and $\|b - \tilde{b}\|_{Y_0} < \epsilon$, then for any $\psi \in \mathscr{D} \supset \mathscr{D}(\mathbb{R}^N \setminus \bar{B}_R)$, $\psi \neq 0$, the Cauchy-Schwarz inequality gives

$$\left| \frac{(\psi, A(b)\psi)_{L^2} - (\psi, A(\tilde{b})\psi)_{L^2}}{\|\psi\|^2_{L^2}} \right| \leq \left\| (A(b) - A(\tilde{b})) \frac{\psi}{\|\psi\|_{L^2}} \right\|_{L^2} < \epsilon. \tag{3.4}$$

This, (3.3), and an elementary consideration yield the continuity of $b \mapsto \beta(b)$. □

Henceforth we fix an arbitrary pair $(q, k) \in \mathbb{N}^2$.

Proof of the openness of $\mathcal{G}_{q,k}$. Fix any $b \in \mathcal{G}_{q,k}$. We need to find a neighborhood $\mathcal{V} \subset Y_m$ of b contained in $\mathcal{G}_{q,k}$.

If $J_q(b) = \emptyset$, then, using the definition of $J_q = J_q(b)$ (note in particular the nonstrict inequality $\mu_j \leq \beta_q$ in the definition) and Lemmas 3.1, 3.2, one shows easily that $J_q(\tilde{b}) = \emptyset$ for all $\tilde{b}$ in a small enough neighborhood $\mathcal{V} \subset Y_m$ of b. Thus, (p1qk)–(p3qk) are trivially satisfied for all $\tilde{b} \in \mathcal{V}$, and in this case we are done.

We proceed assuming that $J_q(b) \neq \emptyset$; that is, $J_q(b) = \{1, \dots, n\}$ for some $n \in \mathbb{N}$. Since there are only finitely many vectors $\alpha \in \mathbb{Z}^n$ with $|\alpha| \leq k$, we obtain from Lemma 3.1 that there is a neighborhood $\mathcal{V} \subset Y_m$ of b such that for each $\tilde{b} \in \mathcal{V}$ the sets $\{\mu_1(\tilde{b}), \dots, \mu_n(\tilde{b})\}$, $\left\{ \sqrt{|\mu_j(\tilde{b})|}, \dots, \sqrt{|\mu_n(\tilde{b})|} \right\}$ are both nonresonant up to order k. Note that the eigenvalues $\mu_j(\tilde{b})$ are defined, due to Lemma 3.1, although it is not necessarily true that $J_q(\tilde{b}) = \{1, \dots, n\}$. The latter holds, possibly after shrinking the neighborhood $\mathcal{V}$, if $\mu_n(b) < \beta(b) - 1/q$. This follows from the continuity of $\mu_n(\tilde{b})$ and $\beta(\tilde{b})$ with respect to $\tilde{b}$ (Lemmas 3.1, 3.2). If $\mu_n(b) = \beta(b) - 1/q$, then Lemmas 3.1, 3.2 imply that for $\tilde{b} \approx b$, either $J_q(\tilde{b}) = \{1, \dots, n\}$ or $J_q(\tilde{b}) = \{1, \dots, n-1\}$ (the latter set is empty if $n = 1$). Since the nonresonance of a set obviously implies the nonresonance of any subset, we conclude that if the neighborhood $\mathcal{V}$ is small enough, then (p1qk), (p2qk) hold for any $\tilde{b} \in \mathcal{V}$.

We now claim that if $\tilde{b} \in Y_m$ is sufficiently close to b, then the functions

$$(\varphi(\tilde{b}))^\alpha, \quad \alpha \in \mathbb{N}_0^n, \ |\alpha| \leq k, \tag{3.5}$$

are linearly independent on any interval $I \subset [1/q, q]$ of length at least $1/(2q)$. Here $(\varphi(\tilde{b})) = (\varphi_1(\tilde{b}), \dots, \varphi_n(\tilde{b}))$ and we use the multi-index notation as in (2.8). Since $J_q(\tilde{b})$ is equal to either $\{1, \dots, n\}$ or $\{1, \dots, n-1\}$, the linear independence of the functions (3.5) implies that (p3qk) holds for $\tilde{b}$. Hence, the proof of the openness of $\mathcal{G}_{q,k}$ will be complete once we prove the claim.

We use the Gram-determinant criterion for the linear independence on an interval I. Specifically, arrange the multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ in a finite sequence $\{\alpha^j\}_{j=1}^{\kappa}$ ($\kappa = \kappa(n, k)$ is the number of all such multi-indices) and consider the following symmetric $\kappa \times \kappa$ matrix:

$$\left(\int_I (\varphi(\tilde{b}))^{\alpha^i} (\varphi(\tilde{b}))^{\alpha^j} r^{N-1} \, dr \right)_{i,j=1,\dots,\kappa}. \tag{3.6}$$

The functions (3.5) are linearly independent on I if and only if the determinant of this matrix is nonzero.

Suppose that our claim is not true. Then there is a sequence $\{\tilde{b}_\ell\}$ in Y_m converging to b satisfying the following. For each $\ell = 1, 2, \dots,$ there is an interval $I_\ell \subset [1/q, q]$, of length $1/(2q)$, such that the determinant of the matrix (3.6), with $I = I_\ell$, is equal to zero. Passing to a subsequence, we may assume that the centers of the intervals I_ℓ approach a number in $[5/(4q), q - 1/(4q)]$. Using this and the continuity of the map $\tilde{b} \mapsto \varphi_j(\tilde{b}) \in C[1/q, q]$ (cp. Lemma 3.1(iii)), we conclude that there is an interval $I \subset [1/q, q]$ of length $1/(2q)$ such that the determinant in (3.6) vanishes for $\tilde{b} = b$, contradicting the assumption that $b \in \mathcal{G}_{q,k}$. This contradiction proves the claim and completes the proof of the openness of $\mathcal{G}_{q,k}$. □

In the proof of the density, we consider radial potentials which are constant on an interval. On such an interval, the eigenvalue equation (2.6) yields expressions for the eigenfunctions in terms of the eigenvalues. Using these expressions, we can relate algebraic independence properties of the eigenfunctions to rational independence properties of the eigenvalues (the idea to use such relations goes back to [9], where it was exploited in a study of nonlocal parabolic equations in one space dimension).

In the following lemma, $b \in Y_m$ is fixed, thus the argument b in $\beta(b)$ and related quantities is suppressed for notational simplicity.

Lemma 3.3. *Assume that $b \in Y_m$ is such that for some constant c one has $V = a_0 + b \equiv c$ on an interval $I = (R_1, R_2) \subset (0, \infty)$. Assume further that for a positive integer $n < \nu + 1$ one has $\mu_n(b) - c < 0$. Then the following statements are valid.*

(i) The functions

$$\varphi_j^2, \quad j = 1, \ldots, n, \tag{3.7}$$

are linearly independent on I.

(ii) If for some $k \in \mathbb{N}$ the set $\{\sqrt{|\mu_j - c|} : j = 1, \ldots, n\}$ is nonresonant up to order k, then the set of functions $\{\varphi_j : j = 1, \ldots, n\}$ is algebraically independent up to order k on the interval I.

Proof. First we prove statement (ii) and then explain why statement (i) holds (without any nonresonance condition involving the μ_j). Statement (ii) is a generalization of a result in [22, Section 2], and is proved by similar arguments.

Assume first that $N \geq 2$. For $j = 1, \ldots, n$, consider the eigenfunction equation (2.6). On the interval (R_1, R_2) the equation simplifies, due to $V \equiv c$:

$$\varphi_j'' + \frac{N-1}{r}\varphi_j' + (\mu_j - c)\varphi_j = 0. \tag{3.8}$$

Since $\mu_j - c \leq \mu_n - c < 0$, the general solution of this equation, and therefore also the solution φ_j on (R_1, R_2), can be expressed in terms of modified Bessel functions rescaled by $\omega_j := \sqrt{|\mu_j - c|}$. More specifically, for some constants C_{j1}, C_{j2} one has $\varphi_j \equiv \tilde{\varphi}_j$ on (R_1, R_2), where

$$\tilde{\varphi}_j(r) := C_{j1} r^{1-N/2} I_{N/2-1}(\omega_j r) + C_{j2} r^{1-N/2} K_{N/2-1}(\omega_j r) \tag{3.9}$$

and $I_{N/2-1}$ and $K_{N/2-1}$ are modified Bessel functions of the first and second kind, respectively. Note that these functions are defined for all $r \in (0, \infty)$ and are analytic in this interval (of course, the eigenfunctions φ_j themselves may not be analytic outside (R_1, R_2)). For each $j \in \{1, \ldots, n\}$, the constants C_{j1}, C_{j2} cannot be both equal to zero: otherwise, $\varphi_j \equiv 0$ on $[R_1, R_2]$, hence φ_j, as a solution of a second order equation, vanishes identically on $(0, \infty)$, which is impossible for an eigenfunction.

We now recall the asymptotics of the modified Bessel functions as $r \to \infty$. For $j = 1, \ldots, n$, we have:

$$\begin{aligned} I_{N/2-1}(\omega_j r) &= C_j e^{\omega_j r} r^{-1/2}(1 + \mathcal{O}(1/r)), \\ K_{N/2-1}(\omega_j r) &= C_j' e^{-\omega_j r} r^{-1/2}(1 + \mathcal{O}(1/r)), \end{aligned} \tag{3.10}$$

with some nonzero constants C_j, C_j'.

For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0$ with $|\alpha| \le k$, define

$$\gamma(\alpha) = \sum_{j=1}^{n} s_j \, \alpha_j \, \omega_j, \tag{3.11}$$

where, for each j,

$$s_j = \begin{cases} 1 & \text{if } C_{j1} \neq 0, \\ -1 & \text{if } C_{j1} = 0 \ \ (\text{and } C_{j2} \neq 0). \end{cases} \tag{3.12}$$

Note that, as $r \to \infty$, we have, by (3.9), (3.10),

$$\tilde{\varphi}^{\alpha}(r) \sim r^{\frac{-|\alpha|}{2}} e^{\gamma(\alpha) r}. \tag{3.13}$$

Here $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_n)$ and we are using the multi-index notation, as above.

Since, by assumption, the set $\{\omega_1, \dots, \omega_n\}$ is nonresonant up to order k, one has $\gamma(\alpha) \neq \gamma(\alpha')$ if $\alpha \neq \alpha'$. We can thus arrange the multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$ in a finite sequence α^ℓ, $\ell = 1, \dots, \kappa$ (as above, $\kappa = \kappa(n, k)$ is the number of all multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$), such that

$$\gamma(\alpha^1) > \gamma(\alpha^2) > \dots > \gamma(\alpha^\kappa).$$

We now show that, on (R_1, R_2), the functions

$$\varphi^{\alpha^\ell} \equiv \tilde{\varphi}^{\alpha^\ell}, \quad \ell = 1, \dots, \kappa, \tag{3.14}$$

are linearly independent, thereby concluding the proof of (ii) for $N \ge 2$. To that aim, assume that for some constants c_ℓ, $\ell = 1, \dots, \kappa$, one has

$$\sum_{\ell=1}^{\kappa} c_\ell \tilde{\varphi}^{\alpha^\ell}(r) = 0 \tag{3.15}$$

for all $r \in (R_1, R_2)$. By the analyticity of $\tilde{\varphi}_\ell$, (3.15) then holds for all $r \in (0, \infty)$. Divide the identity by $r^{-|\alpha|/2} e^{\gamma(\alpha^1)}$ to obtain

$$\sum_{\ell=1}^{\kappa} c_\ell \frac{\tilde{\varphi}^{\alpha^\ell}(r)}{r^{-|\alpha|/2} e^{\gamma(\alpha^1) r}} = 0 \quad (r > 0). \tag{3.16}$$

Since $\gamma(\alpha^1) > \gamma(\alpha^\ell)$ for all $\ell \in \{2, \dots, \kappa\}$, using (3.13) we obtain

$$\lim_{r \to \infty} \frac{\tilde{\varphi}^{\alpha^\ell}(r)}{r^{-|\alpha|/2} e^{\gamma(\alpha^1) r}} \begin{cases} = 0 & \text{for } \ell \in \{2, \dots, \kappa\}, \\ \neq 0 & \text{for } \ell = 1. \end{cases}$$

Thus, taking $r \to \infty$ in (3.16), we deduce that $c_1 = 0$. We then successively divide by $r^{-|\alpha|/2} e^{\gamma(\alpha^\ell)}$, $\ell = 2, \dots, \kappa$, and take $r \to \infty$ to conclude that $c_\ell = 0$ for $\ell = 2, \dots, \kappa$. Hence, all the coefficients in (3.15) must vanish, which proves the desired linear independence.

The case $N = 1$ can be treated similarly. This time, for $r \in (R_1, R_2)$ the eigenfunctions φ_j, $j = 1, \dots, n$, satisfy

$$\varphi_j'' + (\mu_j - c)\varphi_j = 0.$$

Letting again $\omega_j = \sqrt{|\mu_j - c|}$, it follows that, on (R_1, R_2), one has $\varphi_j \equiv \tilde{\varphi}_j$, where

$$\tilde{\varphi}_j(r) = C_{j1} e^{\omega_j r} + C_{j2} e^{-\omega_j r}$$

with C_{j1}, C_{j2} not both equal to 0. The conclusion in (ii) can one be proved by a growth-analyticity argument very similar to the one used above.

We now prove statement (i). We argue as above, except now we need to consider the functions (3.7) instead of the functions (3.14). For the linear independence of these functions on (R_1, R_2), where they coincide with the functions

$$\tilde{\varphi}_j^2, \quad j = 1, \dots, n, \tag{3.17}$$

the quantities to consider in place of the $\gamma(\alpha)$ (cp. (3.11)) are

$$s_j 2\omega_j, \quad j = 1, \dots, n, \tag{3.18}$$

where the s_j are as in (3.12). Since the sequence $\omega_j := \sqrt{|\mu_j - c|}$, $j = 1, \dots, n$, is (strictly) decreasing, the numbers in (3.18) are mutually distinct. Therefore, applying the above growth-analyticity arguments, one proves that the functions $\varphi_j^2 = \tilde{\varphi}_j^2$ on (R_1, R_2) are linearly independent. □

It is well known that the linear independence of the squares of the eigenfunctions allows one to control the (simple) eigenvalues locally, by adjusting the potential. We state this precisely as follows.

Lemma 3.4. *Suppose that $b \in Y_0$, $n \in \mathbb{N}$, $n < \nu(b) + 1$, and the functions $\varphi_j^2(b)$, $j = 1, \dots, n$, are linearly independent on an open bounded interval $I \subset (0, \infty)$. Then there is a neighborhood $\mathcal{U}_0$ of the origin in Y_0 such that the map*

$$\tilde{b} \mapsto \mu(\tilde{b}) := (\mu_1(\tilde{b}), \dots, \mu_n(\tilde{b})) : b + (\mathcal{U}_0 \cap \mathscr{D}(I)) \to \mathbb{R}^n \tag{3.19}$$

is (defined and) locally surjective on $b + (\mathcal{U}_0 \cap \mathscr{D}(I))$.

The meaning of the conclusion is that for any neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of 0 in Y_0, the image of $b+(\mathcal{U}\cap\mathscr{D}(I))$ under the map μ contains a neighborhood of $\mu(b)$. We take the intersection of $\mathcal{U}$ with $\mathscr{D}(I)$ (the space of smooth functions on I with compact support, extended trivially to $[0,\infty)$) to guarantee that we can choose smooth and localized perturbations of b.

Proof of Lemma 3.4. Set $\mathcal{U}_0 := \mathcal{V} - b$, where $\mathcal{V}$ is a neighborhood of $b \in Y_0$ as in Lemma 3.1. Then the map μ is defined on $\mathcal{U}_0$ and is smooth (even analytic) on $\mathcal{U}_0$. We show that there is an n-dimensional subspace $\mathcal{H}$ of $\mathscr{D}(I)$ such that the restriction of the derivative $\mu'(b)$ to $\mathcal{H}$ is an isomorphism of $\mathcal{H}$ onto $\mathbb{R}^n$. The conclusion of the lemma is then a direct consequence of the implicit function theorem.

Using Lemma 3.1(ii) and writing the integrals in spherical coordinates, we obtain

$$\mu'(b)\bar{b} = \int_0^\infty \varphi_j^2(r)\bar{b}(r)r^{N-1}\,dr.$$

We want to find functions $\bar{b}_i \in \mathscr{D}(I)$, $i = 1,\dots,n$, such that the matrix

$$\left(\int_I \varphi_j^2(r)\bar{b}_i(r)r^{N-1}\,dr\right)_{j,i=1}^n \tag{3.20}$$

in nonsingular. This can be done in two steps. First, take $\bar{b}_i \in L^2(I)$ such that

$$\int_I \bar{b}_i(r)\varphi_j^2(r)r^{N-1}\,dr = \delta_{ij} \quad (i,j = 1,\dots,n),$$

where δ_{ij} is the Kronecker delta. Such $\bar{b}_i$ exist due to the independence condition on φ_j^2, $j = 1,\dots,n$. In fact, one can find each $\bar{b}_i$ as a linear combination of the functions $\varphi_j^2(b)$, whose coefficients are determined from a regular system of n equations (the matrix of the system is a Gram matrix of linearly independent functions). In the second step, we replace the $\bar{b}_i$ by some functions in $\mathscr{D}(I)$. Due to the density of $\mathscr{D}(I)$ in $L^2(I)$, this can be done in such a way that the matrix (3.20) is close to the identity matrix, and is thus nonsingular. With $\bar{b}_i \in \mathscr{D}(I) \subset \mathscr{D}$, one can replace I by $(0,\infty)$ in the integrals in (3.20). Hence, taking $\mathcal{H} = \operatorname{span}\{\bar{b}_1,\dots,\bar{b}_n\}$, we see that $\mu'(b)\big|_{\mathcal{H}}$ is an isomorphism, as desired. □

The foregoing lemma has the following consequence.

Corollary 3.5. *Suppose that the assumptions of Lemma 3.4 are satisfied, and let $c > \mu_n(b)$, $k \in \mathbb{N}$. Then, given any neighborhood $\mathcal{U}_0$ of the origin in Y_0, the set $b + (\mathcal{U}_0 \cap \mathscr{D}(I))$ contains a function $\tilde{b}$ such that the vectors*

$$\begin{aligned}
&(\mu_1(\tilde{b}), \dots, \mu_n(\tilde{b})),\\
&\left(\sqrt{|\mu_1(\tilde{b})|}, \dots, \sqrt{|\mu_n(\tilde{b})|}\right),\\
&\left(\sqrt{|\mu_1(\tilde{b}) - c|}, \dots, \sqrt{|\mu_n(\tilde{b}) - c|}\right)
\end{aligned}$$

are nonresonant up to order k.

Proof. Since the set comprising all vectors in $\mathbb{R}^n$ which are nonresonant up to order k is clearly open and dense in $\mathbb{R}^n$, the conclusion is an immediate consequence of the local surjectivity of the map $\tilde{b} \mapsto (\mu_1(\tilde{b}), \dots, \mu_n(\tilde{b}))$, as stated in Lemma 3.4. □

Proof of the density of $\mathcal{G}_{q,k}$. We prove that $\mathcal{G}_{q,k}$ is dense in Y_m and $\mathcal{G}_{q,k} \cap Y_m^0$ is dense in Y_m^0. Fix any $b \in Y_m$. We need to show that any neighborhood $\mathcal{V}$ of b in Y_m contains a function $\tilde{b} \in \mathcal{G}_{q,k}$. If $b \in Y_m^0$, we want $\tilde{b}$ to be in Y_m^0 as well.

If $J_q(b) = \emptyset$, then b itself belongs to $\mathcal{G}_{q,k}$ and there is nothing else to prove. We proceed assuming that $J_q(b) \neq \emptyset$. As in the proof of the openness of $\mathcal{G}_{q,k}$, the following statements are valid for some $n \in \mathbb{N}$: $J_q(b) = \{1, \dots, n\}$ and there is a neighborhood $\mathcal{V}_0$ of b in Y_0 such that for each $\tilde{b} \in \mathcal{V}_0$ one has $n < \nu(\tilde{b}) + 1$ (so the eigenvalues $\mu_1(\tilde{b}) < \cdots < \mu_n(\tilde{b}) < \beta(\tilde{b})$ are defined) and either $J_q(\tilde{b}) = \{1, \dots, n\}$ or $J_q(\tilde{b}) = \{1, \dots, n-1\}$. Thus, a function $\tilde{b} \in \mathcal{V}_0 \cap \mathcal{V}$ belongs to $\mathcal{G}_{q,k}$ if the vectors $(\mu_1(\tilde{b}), \dots, \mu_n(\tilde{b}))$, $\left(\sqrt{|\mu_j(\tilde{b})|}, \dots, \sqrt{|\mu_n(\tilde{b})|}\right)$ are nonresonant up to order k, and the eigenfunctions $\varphi_j(\tilde{b})$ are algebraically independent up to order k on any interval $I \subset [1/q, q]$ of length $1/(2q)$ (or more).

The rest of the proof consists of two steps. First we prove that at least one function $\tilde{b}$ with the above properties can be found in $\mathcal{V}_0 \cap Y_m$. Then we use an analyticity argument to show that such a function can also be found in the given neighborhood $\mathcal{V}$. This will be achieved by a sequence of perturbations of the function b. The perturbed functions will always be contained in $\mathcal{V}_0 \cap Y_m$ and will be identical to b outside a compact set (so they will be in Y_m^0 if $b \in Y_m^0$).

We note, first of all, that the relation $\mu_n < \beta(b)$ implies that the function $a_0 + b - \mu_n$ is positive somewhere. To prove this, we use the characterization of the bottom of the essential spectrum as given in (3.3). Suppose for a contradiction that $V - \mu_n = a_0 + b - \mu_n \leq 0$ on $(0, \infty)$. Then for each $\psi \in X \cap \mathscr{D}(\mathbb{R}^N)$

$$\begin{aligned}\frac{(\psi, (-\Delta + V - \mu_n)\psi)_{L^2}}{\|\psi\|_{L^2}^2} &= \frac{(\psi, -\Delta\psi)_{L^2}}{\|\psi\|_{L^2}^2} + \frac{(\psi, (V - \mu_n)\psi)_{L^2}}{\|\psi\|_{L^2}^2} \\ &\leq \frac{(\psi, -\Delta\psi)_{L^2}}{\|\psi\|_{L^2}^2}.\end{aligned}$$

Using (3.3) and the fact that the bottom of the essential spectrum for $-\Delta$ (with domain $H^2(\mathbb{R}^N) \cap X$) is 0, we conclude that $\beta(b - \mu_n) \leq 0$. However, by adding a constant to the potential we just shift the whole spectrum by that constant; in particular, $\beta(b - \mu_n) = \beta(b) - \mu_n > 0$, which is a contradiction.

We have thus shown that there is $r_0 > 0$ such that

$$c := a_0(r_0) + b(r_0) > \mu_n(b).$$

Due to the continuity of $\tilde{b} \to \mu_n(\tilde{b})$, we can shrink the neighborhood $\mathcal{V}_0 \subset Y_0$ of b, if necessary, so as to achieve that

$$\mu_n(\tilde{b}) < c \quad (\tilde{b} \in \mathcal{V}_0). \tag{3.21}$$

We choose a *convex* neighborhood $\mathcal{V}_0$ with this property.

We now introduce a first perturbation b_1 of b, modifying b near $r = r_0$ only, such that the perturbed function b_1 satisfies

$$\begin{aligned} a_0(r) + b_1(r) &\equiv c \qquad (r \in (r_0 - \epsilon, r_0 + \epsilon)), \\ b_1(r) &= b(r) \quad (|r - r_0| > 2\epsilon), \end{aligned} \tag{3.22}$$

for some $\epsilon \in (0, r_0/2)$. Adjusting ϵ, as needed, we can clearly choose b_1 such that, in addition, $b_1 \in C^m[0, \infty)$ (the same regularity as assumed of a_0 and b) and $\|b - b_1\|_{Y_0}$ is so small that $b_1 \in \mathcal{V}_0$. (We remark that it may not be possible to have $\|b - b_1\|_{Y_m}$ small for $m > 0$ if (3.22) is to hold.)

By (3.21), (3.22), and Lemma 3.3(i), the functions $\varphi_j^2(b_1)$, $j = 1, \ldots, n$, are linearly independent on the interval $(r_0, r_0 + \epsilon)$. Therefore, applying Corollary 3.5, we find $b_2 \in \mathcal{V}_0 \cap Y_m$ such that

$$b_2(r) = b_1(r) \quad (r \in \mathbb{R} \setminus (r_0, r_0 + \epsilon)), \tag{3.23}$$

and the vectors

$$\begin{aligned}&(\mu_1(b_2), \dots, \mu_n(b_2)),\\&\left(\sqrt{|\mu_1(b_2)|}, \dots, \sqrt{|\mu_n(b_2)|}\right),\\&\left(\sqrt{|\mu_1(b_2) - c|}, \dots, \sqrt{|\mu_n(b_2) - c|}\right)\end{aligned}$$

are nonresonant up to order k. Relations (3.23), (3.22), and Lemma 3.3(ii) subsequently imply that the set $\{\varphi_j(b_2) : j = 1, \dots, n\}$ is algebraically independent up to order k on the interval $(r_0 - \epsilon, r_0)$.

We next use an analytic perturbation to "propagate" these properties of b_2 to the Y_m-neighborhood $\mathcal{V}$ of b. For that aim, we introduce the functions $\tilde{b}_t := b + t(b_2 - b)$, $t \in [0, 1]$. By the convexity of $\mathcal{V}_0$, $\tilde{b}_t \in \mathcal{V}_0$ for all $t \in [0, 1]$. By Lemma 3.1, the eigenvalues $\mu_j(\tilde{b}_t)$, $j = 1, \dots, n$, depend analytically on t and the eigenfunctions $\varphi_j(\tilde{b}_t)$ depend analytically on t as $C[r_0 - \epsilon, r_0]$-valued functions. Consider now the analytic function

$$t \mapsto (\mu_1(\tilde{b}_t), \dots, \mu_n(\tilde{b}_t)) \cdot \alpha \tag{3.24}$$

for any of the finitely many vectors $\alpha \in \mathbb{Z}^n$ with $0 < |\alpha| \leq k$. Since for $t = 1$ we have $\tilde{b}_t = b_2$, the function (3.24) is not identical to zero. Therefore, due to the analyticity, the function is nonzero for all sufficiently small $t > 0$. From this, we conclude that the set $\{\mu_1(\tilde{b}_t), \dots, \mu_n(\tilde{b}_t)\}$ is nonresonant up to order k for all sufficiently small $t > 0$.

By very similar arguments, for all sufficiently small $t > 0$ the set

$$\left\{\sqrt{|\mu_1(\tilde{b}_t)|}, \dots, \sqrt{|\mu_n(\tilde{b}_t)|}\right\}$$

is nonresonant up to order k, and the set $\{\varphi_j(\tilde{b}_t) : j = 1, \dots, n\}$ is algebraically independent up to order k on the interval $(r_0 - \epsilon, r_0)$. For the latter, we use the determinant of the Gram matrix (3.6) (with $I = (r_0 - \epsilon, r_0)$ and $\tilde{b} = \tilde{b}_t$), which is an analytic function of t, different from 0 at $t = 1$.

Choose now $t > 0$ so small that $\tilde{b} := \tilde{b}_t$ has all the above properties and, in addition, $\tilde{b} \in \mathcal{V}$—the originally given neighborhood of b in Y_m. Note also that $\tilde{b} \equiv b$ outside a compact interval (cp. (3.22), (3.23)). Thus $\tilde{b}$ belongs to Y_m^0 if b does.

We see that the function $\tilde{b}$ has the desired properties (p1qk)–(p3qk), except that the algebraic independence in (p3qk) is valid for the specific interval $(r_0 - \epsilon, r_0)$, rather than for all subintervals of $[1/q, q]$ of length at least $1/(2q)$. To remedy the latter, we make one more small perturbation of $\tilde{b}$ on a bounded interval so as to achieve that the perturbed potential is analytic on that interval. Set $\bar{q} := \max\{r_0, q\}$. Using the Stone-Weierstrass theorem, we modify $\tilde{V} := a_0 + \tilde{b}$ slightly in the interval $[0, \bar{q}+1)$—keeping $\tilde{V}$ intact on $[q+1, \infty)$—in such a way that $\tilde{V}$ is analytic in $(0, \bar{q})$ and the new function $\tilde{b}$ (obtained by subtracting a_0 from $\tilde{V}$) still belongs to $\mathcal{V}$ (and to Y^0_m if $b \in Y^0_m$). If the perturbation is small enough, which we will henceforth assume, the above nonresonance and algebraic independence properties are all preserved as well. From the analyticity of $\tilde{V}$ we gain the extra property that the functions $\varphi_j(\tilde{b})$, $j = 1, \ldots, n$, are analytic on $(0, \bar{q})$ (this follows from the eigenfunction equation (2.6) with $\tilde{V}$ in place of V). Therefore, the functions (2.8), with $\psi = (\varphi_1(\tilde{b}), \ldots, \varphi_n(\tilde{b}))$, are analytic on $(0, \bar{q})$. Clearly, for these analytic functions, linear independence on the interval $(r_0 - \epsilon, r_0)$ implies linear independence on any other subinterval of $(0, \bar{q})$. Hence, (p1qk)–(p3qk) are all satisfied for $\tilde{b}$, that is, $\tilde{b} \in \mathcal{G}_{q,k}$, as desired. The proof of the density of $\mathcal{G}_{q,k}$ and $\mathcal{G}_{q,k} \cap Y^0_m$ is complete. □

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