Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^N$: II. Entire solutions

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Abstract. We consider nonautonomous quasilinear parabolic equations satisfying certain symmetry conditions. We prove that each positive bounded solution u on $\mathbb{R}^N \times (-\infty, T)$ decaying to zero at spatial infinity uniformly with respect to time is radially symmetric around some origin in $\mathbb{R}^N$. The origin depends on the solution but is independent of time. We also consider the linearized equation along u and prove that each bounded (positive or not) solution is a linear combination of a radially symmetric solution and (nonsymmetric) spatial derivatives of u. Theorems on reflectional symmetry are also given.

Contents

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1 Introduction and the main results

In this paper we continue our study of symmetry properties of positive solutions of quasilinear parabolic equations

$$u_t = A_{ij}(t,u,\nabla u)u_{x_ix_j} + f(t,u,\nabla u), \quad x \in \mathbb{R}^N,\, t \in (\tau, T) \tag{1.1}$$

(we use the summation convention throughout the paper). The nonlinearities A_{ij} and f satisfy regularity, ellipticity and symmetry conditions formulated below.

In our previous paper [35], we took $\tau = 0$, $T = \infty$ and considered positive solutions of the associated Cauchy problem. Assuming such a solution is global (defined for all $t \geq 0$), bounded, and decays to 0 as $|x| \to \infty$ uniformly with respect to t, we proved that it is asymptotically symmetric about some some center (or about a hyperplane). In this paper we are concerned with solutions defined on $(\tau, T) = (-\infty, T)$, for some $T > 0$. We prove that such solutions are symmetric at each time. We also examine the symmetry properties of solutions of the linearized equation around positive solutions of (1.1). Our results in particular apply to *entire* solutions, by which we mean solutions defined for all $t \in \mathbb{R}$. We remark that suitably extending the solutions and the equation, each solution on $(-\infty, T)$ can be thought of as an entire solution.

1.1 Semilinear equations

To formulate our results in the simplest form, we initially consider semilinear nonautonomous equations

$$u_t = \Delta u + f(t,u), \quad x \in \mathbb{R}^N,\, t \in (-\infty, T). \tag{1.2}$$

Our assumptions on f are as follows.

(S1) $f(t,u)$ is of class C^1 in u uniformly with respect to t, that is, f and f_u are continuous on $(-\infty, T) \times \mathbb{R}$, and for each $M > 0$

$$\lim_{\substack{0 \leq u,v \leq M,\, t<T \\ |u-v| \to 0}} |f_u(t,u) - f_u(t,v)| = 0,$$

(S2) $f(t,0)=0 \quad (t<T)$, and there is a constant $\gamma>0$ such that

$$f_u(t,0)<-\gamma \quad (t<T). \tag{1.3}$$

We consider solutions of (1.2) which are defined on $(-\infty,T)$ and satisfy the uniform decay condition

$$\sup_{-\infty<t<T} u(x,t)\to 0 \text{ as } |x|\to\infty. \tag{1.4}$$

As we show below (see Corollary 2.5), this condition is guaranteed if u is sufficiently small near spatial infinity.

Theorem 1.1. *Assume* (S1), (S2) *and let u be a positive bounded solution of* (1.2) *on* $(-\infty,T)$ *satisfying* (1.4). *Then there exists $\xi\in\mathbb{R}^N$ such that for each $t<T$ and each $x,y\in\mathbb{R}^N$ with $|y-\xi|=|x-\xi|>0$ one has*

$$u(x,t)=u(y,t), \tag{1.5}$$

$$\nabla u(x-\xi,t)\cdot(x-\xi)<0. \tag{1.6}$$

This theorem is an extension of symmetry results on positive solutions of elliptic solutions on the entire space or a ball. For parabolic equations similar results were previously known on bounded domains only. We gave a specific account of these results in [35], here we will be brief. Reflectional and radial symmetry of positive solutions of the Dirichlet problem for elliptic equations was first established in [21] (closely related results and techniques on radial symmetry appeared earlier in [38]). Extensions and generalizations were given in [8, 9, 17, 28, 42], see also the surveys [4, 27, 33] and references therein. Symmetry theorems for positive solutions on the entire space can be found in [6, 22, 29, 30, 39]. See also [6, 5, 7, 11, 37] or the surveys [4, 33] for related results on elliptic equations.

The results for bounded domains are not difficult to extend to periodic solutions of periodic-parabolic equations, see [18]. Symmetry properties of general bounded positive solutions of parabolic equations on bounded domains were first examined in independent works [26] and [1, 2] and later in [3]. Two different, but closely related types of symmetry results can be found in these papers. The first one is the symmetry, radial or reflectional, depending on the domain, of positive entire solutions. The second one is the asymptotic symmetry of bounded positive solutions, or in other words, the symmetry of all functions contained in the ω-limit sets of global positive solutions. We remark that since the bounded domains considered have fixed center (or hyperplane) of symmetry, the former result often implies the latter. Indeed, under suitable regularity assumptions, the functions in the ω-limit set can be viewed as entire solutions of a suitable equation, hence they are symmetric.

For the entire space, similar results for parabolic equations were not available until recently. In autonomous equations, convergence to a ground state, which entails the asymptotic symmetry for the Cauchy problem, is proved in [10, 15, 19], see also [20] for a convergence result for periodic-parabolic equations on $\mathbb{R}$. In the general case, the asymptotic symmetry of positive, suitably bounded solutions is established in our earlier paper [35]. Theorem 1.1 (and the more general Theorem 1.3 below) show the symmetry of entire solutions. Unlike on bounded domains, the relation of these two types of symmetry results is not so close. As the center of symmetry is not fixed and depends on the solution, one has to take into account the possibility that all functions in the ω-limit set of a positive solutions are symmetric, yet they do not share the same center of symmetry (this situation does occur if the strict negativity condition 1.3 is omitted, see [36]).

As we explained in [35], when dealing with parabolic equations on $\mathbb{R}^N$, the usual scenario from the proof of symmetry for elliptic equations or parabolic equations on bounded domains does not lead to desired results. An extra step, in essence a spectral perturbation argument in the evolutionary context, is needed. The method we use here to prove the symmetry theorem follows a similar general scheme as [35] and relies on the same basic ingredients: moving hyperplanes, maximum principles, Harnack-type inequalities and a construction of a subsolution for a perturbation argument.

We remark that the assumption that the decay of $u(x,t)$ as $x \to \infty$ is uniform with respect to t is essential, the symmetry may fail without it. Examples can be found in autonomous equations $u_t = u_{xx} + f(u)$ with $f(0) = f(1) = 0$, $f(u) < 0$ $(u \in (0,1))$ and $f'(0) < 0 < f'(1)$, which is a classical KPP (or Fisher-type) equation. It is well known that under an additional condition the equation has monotone increasing (in x) traveling-front solutions $\Psi_c(x - ct)$ for a continuum of positive speeds c. Combining solutions $\Psi_{c_1}(x - c_1 t)$, $\Psi_{c_2}(-x - c_2 t)$, with $c_1 \neq c_2$, one can construct an entire solution u that decays to zero in space (not uniformly with respect to time), but is not symmetric in x for any t. See [13, 23, 24] for the existence and properties of such solutions.

Let us now formulate, still in the context of semilinear equations, our second main result. It deals with the linearization of (1.2) along a positive solution u:

$$v_t = \Delta v + f_u(t, u(x,t))v, \quad x \in \mathbb{R}^N,\, t \in (-\infty, T). \tag{1.7}$$

Theorem 1.2. *Let u and ξ be as in Theorem 1.1 and let v be a bounded solution of (1.7) on $(-\infty, T)$. Then there exist constants $c_1, \dots, c_N$ and a solution ψ of (1.7) on $(-\infty, T)$ such that $x \mapsto \psi(x - \xi, t)$ is radially symmetric for each $t < T$ and*

$$v \equiv \psi + c_1 \partial_{x_1} u + \dots c_N \partial_{x_N} u. \tag{1.8}$$

If, in addition,

$$v(x,t) \to 0 \text{ as } t \to -\infty \quad (x \in \mathbb{R}^N), \tag{1.9}$$

then $v(x - \xi, t)$ is radially symmetric for each $t < T$.

Let us define the *center-unstable space* of the solution u as the space of all bounded solutions of the linearized equation (1.7) on $(-\infty, T)$, and the *unstable space* as the space of all solutions v of (1.7) such that (1.9) holds. The previous theorem says that the unstable space is spanned by radial solutions, and the center-unstable space is spanned by radial solutions and the spatial derivatives of u (which, of course, are not symmetric). This is a natural extension of the results of [3] on the symmetry of the center-unstable space of positive entire solutions of parabolic equations on bounded domains, which, in its turn, is an extension of similar results for positive steady states of autonomous equations (see [2, 12, 16, 25, 32, 40]) or periodic solutions of periodic-parabolic equations [14]. In case equation (1.2) is autonomous, $f = f(u)$, and the solution u is a positive steady state, Theorem 1.2 restates a well known result on the unstable space of the linearized Schrödinger operator (see [19, 34], for example). Note that the second conclusion of Theorem 1.2 can be proved under a weaker assumption than (1.9), see Theorem 1.6 below.

1.2 Quasilinear equations

We now give precise formulations of our main theorems for quasilinear equations (1.1). We start with results on reflectional symmetry and then state corollaries on the radial symmetry.

We fix a direction e, without loss of generality taken to be $e = (1, 0, \dots, 0)$, and assume that equation (1.1) is invariant under reflections in hyperplanes perpendicular to e. More specifically, for $\lambda \in \mathbb{R}$, let P_λ denote the reflection in the hyperplane $\{x \in \mathbb{R}^N : x_1 = \lambda\}$.

The nonlinearities $A_{ij}, f : (-\infty, T) \times [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$ are assumed to satisfy the following conditions:

(Q1) $A_{ij}(t, u, p)$, $f(t, u, p)$ are of class C^1 in u and $p = (p_1, \dots, p_N)$ uniformly with respect to t. This means that A_{ij}, f are continuous on $(-\infty, T) \times [0, \infty) \times \mathbb{R}^N$ together with their partial derivatives $\partial_u A_{ij}$, $\partial_u f$, $\partial_{p_1} A_{ij}, \dots, \partial_{p_N} A_{ij}$, $\partial_{p_1} f, \dots, \partial_{p_N} f$; and if h stands for any of these partial derivatives, then for each $M > 0$ one has

$$\lim_{\substack{0 \le u, v, |p|, |q| \le M,\, t<T \\ |u-v|+|p-q| \to 0}} |h(t, u, p) - h(t, v, q)| = 0. \tag{1.10}$$

(Q2) $(A_{ij})_{i,j}$ is locally uniformly elliptic in the following sense: for each $M > 0$ there is $\alpha_0^M > 0$ such that

$$\begin{aligned} &A_{ij}(t, u, p)\xi_i\xi_j \ge \alpha_0^M |\xi|^2 \\ &\qquad\qquad (\xi = (\xi_1, \dots \xi_N) \in \mathbb{R}^N, t < T, u \in [0, M], |p| \le M). \end{aligned} \tag{1.11}$$

(Q3) $f(t, 0, 0) = 0$ $(t < T)$ and there is a constant $\gamma > 0$ such that

$$\partial_u f(t, 0, 0) < -\gamma \quad (t < T); \tag{1.12}$$

(Q4) for each $(t, u, p) \in (-\infty, T) \times [0, \infty) \times \mathbb{R}^N$ and $i, j = 1, \dots, N$ one has

$$\begin{gathered} A_{ij}(t, u, P_0 p) = A_{ij}(t, u, p), \quad f(t, u, P_0 p) = f(t, u, p), \\ A_{1j} \equiv A_{j1} \equiv 0 \ \text{ if } j \ne 1. \end{gathered}$$

We now consider positive solutions of (1.1) on $(-\infty, T)$ satisfying the following boundedness and decay conditions:

$$u(x, t), |u_{x_i}(x, t)|, |u_{x_i x_j}(x, t)| < d_0 \quad (x \in \mathbb{R}^N, t < T), \tag{1.13}$$

where d_0 is a positive constant, and

$$\lim_{|x| \to \infty} \sup\{u(x, t), |u_{x_i}(x, t)|, |u_{x_i x_j}(x, t)| : t < T, i, j = 1, \dots, N\} = 0. \tag{1.14}$$

Theorem 1.3. *Assume* (Q1)–(Q4) *and let* u *be a positive solution of* (1.1) *on* $(-\infty, T)$ *satisfying* (1.13) *and* (1.14). *Then there exist* $\lambda \in \mathbb{R}$ *such that for each* $t < T$ *and* x *in the halfspace* $\{x : x_1 > \lambda\}$ *one has*

$$\begin{aligned} u(P_\lambda x, t) &= u(x,t), \\ \partial_{x_1} u(x,t) &< 0. \end{aligned} \tag{1.15}$$

If (1.1) is in the rotationally invariant form

$$u_t = A(t, u, |\nabla u|)\Delta u + f(t, u, |\nabla u|), \tag{1.16}$$

then the following result on the radial symmetry applies.

Corollary 1.4. *Let* (Q1)–(Q3) *hold. Assume that* $A_{ij} \equiv 0$ *for* $i \neq j$, $A_{ii} \equiv A_{jj}$ *for any* i, j, *and*

$$A_{ii}(t, u, p) = A_{ii}(t, u, q), \quad f(t, u, p) = f(t, u, q)$$

whenever $|p| = |q|$. *Let* u *be a positive solution of* (1.1) *on* $(-\infty, T)$ *satisfying* (1.13) *and* (1.14). *Then there exists* $\xi \in \mathbb{R}^N$ *such that for each* $t < T$ *and each* $x, y \in \mathbb{R}^N$ *with* $|y - \xi| = |x - \xi| > 0$ *one has*

$$u(x,t) = u(y,t), \tag{1.17}$$

$$\nabla u(x - \xi, t) \cdot (x - \xi) < 0. \tag{1.18}$$

Proof. The equation is invariant under rotations around the origin. Using this and Theorem 1.3, we obtain that for each direction e there is a hyperplane Γ^e perpendicular to e such that $u(\cdot, t)$ is symmetric with respect to the reflection in Γ^e and has negative derivative in direction e on the halfspace $\{x \cdot e > 0\}$. It follows that the maximum of $u(\cdot, t)$ is achieved at the intersection of the symmetry hyperplanes, which is necessarily a uniquely defined point independent of t. The conclusion of the corollary holds for this point. □

Now assume u is a positive solution of (1.1) on $(-\infty, T)$ as in Theorem 1.3. We examine symmetry properties of bounded solutions of the linearized equation

$$v_t = a_{ij}(x,t) v_{x_i x_j} + b_i(x,t) v_{x_i} + c(x,t) v, \quad x \in \mathbb{R}^N, t \in (-\infty, T) \tag{1.19}$$

where

$$
\begin{aligned}
a_{ij}(x,t) &= A_{ij}(t, u(x,t), \nabla u(x,t)),\\
b_i(x,t) &= f_{p_i}(t, u(x,t), \nabla u(x,t)) + u_{x_k x_\ell}(x,t) A_{k\ell\, p_i}(t, u(x,t), \nabla u(x,t)),\\
c(x,t) &= f_u(t, u(x,t), \nabla u(x,t)) + u_{x_k x_\ell}(x,t) A_{k\ell\, u}(t, u(x,t), \nabla u(x,t)).
\end{aligned}
\tag{1.20}
$$

Theorem 1.5. *Assume* (Q1)–(Q4). *Let u and λ be as in Theorem 1.3 and let v be a bounded solution of* (1.19) *on* $(-\infty, T)$. *Then there exist a constant c_1 and a solution ψ of* (1.19) *on* $(-\infty, T)$ *such that*

$$\psi(P_\lambda x, t) = \psi(x,t) \quad (x \in \mathbb{R}^N, t < T), \tag{1.21}$$

and

$$v \equiv \psi + c_1 \partial_{x_1} u. \tag{1.22}$$

If, in addition,

$$\lim_{t \to -\infty} (v(P_\lambda x, t) - v(x,t)) = 0 \tag{1.23}$$

for some x in $\{x : x_1 > \lambda\}$, then $c_1 = 0$.

We have a similar result concerning the radial symmetry.

Theorem 1.6. *Let the hypotheses of Corollary 1.4 be satisfied and let u and ξ be as in that corollary. Let v be a bounded solution of* (1.19) *on* $(-\infty, T)$. *Then there exist constants $c_1, \dots, c_N$ and a solution ψ of* (1.19) *on* $(-\infty, T)$ *such that $x \mapsto \psi(x - \xi, t)$ is radially symmetric for each $t < T$ and*

$$v \equiv \psi + c_1 \partial_{x_1} u + \cdots + c_N \partial_{x_N} u. \tag{1.24}$$

If, in addition, there is a point x with $x_i > \xi_i$, $i = 1, \dots, N$, such that

$$\lim_{t \to -\infty} (v(P^i_{\xi_i} x, t) - v(x,t)) = 0 \quad (i = 1, \dots, N), \tag{1.25}$$

where $P^i_{\xi_i}$ is the reflection about the hyperplane $\{x : x_i = \xi_i\}$, then $v(x - \xi, t)$ is radially symmetric for each $t < T$.

The paper is organized as follows. In the preliminary Section 2, we give several basic estimates of solutions of linear equations (1.19). The proof of Theorem 1.3 is given in Section 3. Theorems 1.5 and 1.6 are proved in

Section 4. The proofs of Theorems 1.1, 1.2 are simpler than in the quasilinear case and will not be given. The theorems can also be derived from the corresponding results in the quasilinear case upon verifying the stronger boundedness and decay conditions.

We refer the reader to [35, Sect. 4] for a discussion of possible generalizations. The difficulties mentioned there appear when dealing with entire solutions as well.

2 Basic estimates for linear equations

In this section we consider linear problems

$$v_t = a_{ij}(x,t)v_{x_ix_j} + b_i(x,t)v_{x_i} + c(x,t)v, \quad (x,t) \in Q, \tag{2.1}$$

where $Q = \Omega \times (\tau, T)$ is a cylindrical domain in $\mathbb{R}^{N+1}$. Such equations will arise in our analysis in two different ways: as the linearization of (1.1) along a solution u and the equation for the difference of two solutions of (1.1). For the time being we consider a general linear problem with coefficients satisfying the following hypotheses.

(L1) $a_{ij}, b_i, c \in L^\infty(\mathbb{R}^N \times (-\infty, T))$ and there are positive constants β_0 and α_0 such that

$$|a_{ij}(x,t)|, |b_i(x,t)|, |c(x,t)| < \beta_0 \quad (x \in \mathbb{R}^N,\ t < T, i, j = 1, \dots, N), \tag{2.2}$$

$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad (\xi \in \mathbb{R}^N, x \in \mathbb{R}^N, t < T). \tag{2.3}$$

(L2) There are positive constants γ, R such that

$$c(x,t) < -\gamma \quad (|x| \geq R, t < T). \tag{2.4}$$

By a *solution* of (2.1) on Q we always mean a strong solution, that is, a function v in the Sobolev space $W^{2,1}_{N+1,loc}(Q)$ such that the equation is satisfied almost everywhere. We usually consider solutions with the additional property $v \in C(\overline{Q})$. We also use the concept of super and sub-solutions. A *supersolution* of (2.1) on Q is a function in $W^{2,1}_{N+1,loc}(Q)$ which satisfies the following inequality almost everywhere in Q:

$$v_t \geq a_{ij}(x,t)v_{x_ix_j} + b_i(x,t)v_{x_i} + c(x,t)v. \tag{2.5}$$

A *subsolution* is defined analogously.

We denote by $\partial_p Q$ the parabolic boundary of Q:

$$\partial_p = \partial_s Q \cup \partial_b Q,$$

where

$$\partial_s Q = \{(x,t) \in \partial Q : \tau < t < T\},$$
$$\partial_b Q = \begin{cases} \overline{\Omega} \times \{\tau\}, & \text{if } \tau > -\infty, \\ \emptyset, & \text{if } \tau = -\infty, \end{cases}$$

We also use the following notation. For a set $\Omega \subset \mathbb{R}^N$ and functions v and w on Ω, the inequalities $v \geq 0$ and $w > 0$ are always understood in the pointwise sense:

$$v(x) \geq 0, \; w(x) > 0 \quad (x \in \Omega).$$

For a function $z(x)$, we denote by z^+, z^- the positive and negative parts of z, respectively:

$$z^+(x) = (|z(x)| + z(x))/2 \geq 0,$$
$$z^-(x) = (|z(x)| - z(x))/2 \geq 0.$$

If D_0 and D are domains in $\mathbb{R}^N$ the notation $D_0 \subset\subset D$ means that $\overline{D}_0$ is a compact subset of D.

We next recall several basic results on solutions of (2.1). The first one is a variant of the maximum principle. The proof of the maximum principle for strong solutions can be found in [31], for example.

Lemma 2.1. *Assume* (L1). *Let $\tau > -\infty$ and let $v \in C(\overline{Q})$ be a bounded solution of* (2.1) *on Q. Then for each $(x_0, t_0) \in Q$ one has*

$$e^{-mt_0} v^{\pm}(x_0, t_0) \leq \sup_{(x,t) \in \partial_p Q} e^{-mt} v^{\pm}(x,t),$$

where v^+ and v^- stand for the positive and negative parts of v, respectively, and $m = \sup_{(x,t)\in Q} c(x,t)$. The inequality is strict, unless v is constant on $\{(x,t) \in Q : t \leq t_0\}$.

The statement regarding v^+ remains valid if v is a subsolution of (2.1) *and the statement regarding v^- is valid if v is a supersolution.*

The next result is a consequence of the maximum principle and Harnack inequality (see [35] for the proof).

Lemma 2.2. *In addition to* (L1), *assume that the functions a_{ij} are continuous. Let d be a positive constant and let $D \subset\subset D_1$ be bounded domains in $\mathbb{R}^N$ satisfying* $\operatorname{dist}(\overline{D}, \partial D_1) \geq d$. *There exists a constant $\kappa > 0$, depending only on D, d, α_0 and β_0, with the following property. If $\tau > 1$ and $v \in C(\overline{D}_1 \times (\tau - 1, \tau + 1])$ is a solution of* (2.1) *on $Q = D_1 \times (\tau - 1, \tau + 1)$ then*

$$v(x, \tau + 1) \geq \kappa \| v^+(\cdot, \tau + \frac{1}{2}) \|_{L^\infty(D)} - \sup_{\partial_p(D_1 \times (\tau, \tau+1))} e^m v^- \quad (x \in \overline{D}), \tag{2.6}$$

where $m = \sup_{D_1 \times (0,\infty)} c$.

The following lemma facilitates a change of variables in (2.1) which makes the coefficient c negative in a thin slab while not increasing it much elsewhere (see [35] for the proof; more interesting results of this sort can be found in [8]).

Lemma 2.3. *Given positive constants Θ, ε, there exist $\delta > 0$ and a function g on $[0, \infty)$ with the following properties:*

$$\begin{aligned}
&g \in C^1[0, \infty) \cap C^2[0, \delta] \cap C^2[\delta, \infty), \\
&2 \geq g \geq \frac{1}{2}, \\
&g''(\xi) + \Theta(|g'(\xi)| + g(\xi)) \leq 0 \quad (\xi \in (0, \delta)), \\
&g''(\xi) + \Theta|g'(\xi)| - \varepsilon g(\xi) \leq 0 \quad (\xi \in (\delta, \infty)).
\end{aligned}$$

Note that if v is a solution of (2.1) on $Q = \{x : x_1 > \lambda\} \times (-\infty, T)$, g is as in Lemma 2.3 with $\Theta = \beta_0/\alpha_0 + 1$, and $\tilde{g}(\xi) := g(\xi - \lambda)$, then $w(x, t) = v(x, t)/\tilde{g}(x_1)$ is a solution of

$$w_t = a_{ij}(x, t) w_{x_i x_j} + \hat{b}_i(x, t) w_{x_i} + \hat{c}(x, t) w, \quad (x, t) \in Q \tag{2.7}$$

where

$$\begin{aligned}
\hat{b}_i(x, t) &= b_i(x, t) + 2a_{11}(x, t) \frac{\tilde{g}'(x_1)}{\tilde{g}(x_1)}, \\
\hat{c}(x, t) &= \frac{c(x, t)\tilde{g}(x_1) + b_1(x, t)\tilde{g}'(x_1) + a_{11}(x, t)\tilde{g}''(x_1)}{\tilde{g}(x_1)}.
\end{aligned}$$

A straightforward computation shows (see [35]) that

$$\hat{c}(x,t) \le -\alpha_0 \quad (x_1 \in [\lambda, \lambda+\delta)), \tag{2.8}$$

$$\hat{c}(x,t) \le c(x,t) + \alpha_0 \varepsilon \quad (x_1 > \lambda). \tag{2.9}$$

The above remarks remain valid, if v is a subsolution (supersolution), rather than solution, of (2.1) on $\{x : x_1 > \lambda\} \times (-\infty, T)$; w is then a subsolution (supersolution) of (2.7).

In the next lemma we estimate super- and sub-solutions on $Q = \Omega \times (-\infty, T)$.

Lemma 2.4. *Assume* (L1), (L2). *Let* $Q = \Omega \times (-\infty, T)$, *where* Ω *is a domain in* $\mathbb{R}^N$. *The following statements hold.*

(i) If v *is a subsolution of* (2.1) *on* Q *which is bounded above and satisfies* $v \le 0$ *on* $\partial\Omega \times (-\infty, T)$, *then there are positive constants* C *and* ν *such that*

$$v(x,t) \le Ce^{-\nu|x|} \quad ((x,t) \in Q). \tag{2.10}$$

(ii) There exists a constant $\delta > 0$ *depending only on* α_0, β_0 *and* γ *with the following property. If* v *is a supersolution of* (2.1) *on* Q *which is bounded below and satisfies*

$$v(x,t) \ge 0 \quad (x \in \partial\Omega, t < T), \tag{2.11}$$

$$v(x,t) > 0 \quad (x \in D_0, t < T), \tag{2.12}$$

for some domain $D_0 \subset \Omega$ *such that*

$$\Omega \setminus D_0 \subset \{x \in \Omega : \lambda \le x_1 \le \lambda + \delta\} \cup \{x \in \Omega : \sup_{t<T} c(x,t) < -\gamma\}, \tag{2.13}$$

for some $\lambda \in \mathbb{R}$, *then* $v > 0$ *in* Q.

In particular, if v *is a bounded solution of* (2.1) *on* Q *satisfying* (2.12) *and* $v = 0$ *on* $\partial\Omega \times (-\infty, T)$, *then*

$$0 < v(x,t) \le be^{-\nu|x|} \quad ((x,t) \in Q). \tag{2.14}$$

Note that (2.13) holds in particular if

$$\Omega \setminus D_0 \subset \{x \in \Omega : \lambda \le x_1 \le \lambda + \delta\} \cup \{x \in \Omega : |x| > R\}$$

(cf. (L2)).

Proof of Lemma 2.4. We prove (i) using a comparison argument. Let q be an upper bound for v. For a small $\nu > 0$ define

$$\zeta(x,t) := qe^{-\gamma(t-t_0)} + qe^{-\nu(|x|-R)}.$$

Then

$$\zeta(x,t) \geq q \text{ if } |x| = R \text{ or } t = t_0.$$

A simple computation shows that for any $|x| > R$, $t > t_0$,

$$\begin{aligned} &\zeta_t - a_{ij}(x,t)\zeta_{x_i x_j} - b_i(x,t)\zeta_{x_i} - c(x,t)\zeta \\ &\qquad \geq qe^{-\gamma(t-t_0)}(-\gamma - c(x,t)) + qe^{-\nu(|x|-R)}(-c(x,t) + O(\nu)), \end{aligned}$$

as $\nu \to 0$. The last expression is positive (in view of (2.4)) if ν is sufficiently small. Thus ζ is a supersolution of (2.1) on

$$\tilde{Q}_{t_0} = \{(x,t) \in Q : |x| > R, t > t_0\}$$

which dominates v on the parabolic boundary of $\tilde{Q}_{t_0}$. Applying the maximum principle to $\zeta - v$, we obtain

$$v(x,t) \leq qe^{-\gamma(t-t_0)} + qe^{-\nu(|x|-R)} \quad (x \in \Omega, |x| \geq R, t \in (t_0, T)).$$

Taking the limit $t_0 \to -\infty$ yields

$$v(x,t) \leq qe^{-\nu(|x|-R)} \quad (x \in \Omega, |x| > R, t < T). \tag{2.15}$$

In conjunction with the boundedness of v, this estimates implies (2.10).

To prove (ii), set $\epsilon = \alpha_0\gamma/2$ and choose δ and g as in Lemma 2.3 with $\Theta = \beta_0/\alpha_0 + 1$. In view of the remarks following that lemma (see in particular (2.8), (2.9)), $w = v/\tilde{g}$ is a subsolution of equation (2.7) in which

$$\hat{c} \leq -\gamma_0 := \min\{-\alpha_0, -\gamma/2\} \quad ((x,t) \in Q^0_\delta := Q \setminus (D_0 \times (-\infty, T)). \tag{2.16}$$

Since the positivity and decay for w or for v hold equivalently, we may proceed assuming, without loss of generality, that (2.16) holds with $\hat{c}$ replaced by c. Assume v satisfies the conditions in (ii). Let $p < 0$ be a lower bound for v. Then the function $pe^{-\gamma_0(t-t_0)}$ is a subsolution of (2.1) on Q^0_δ and it is smaller than v on $\partial_s Q^0_\delta$ (by (2.11), (2.12)) and on $\Omega \times \{t_0\}$. By comparison,

$$pe^{-\gamma_0(t-t_0)} \leq v(x,t) \quad ((x,t) \in Q^0_\delta, t \geq t_0).$$

Taking the limit $t_0 \to -\infty$ we get $v \geq 0$ in Q^0_δ. By (2.12) and the maximum principle, $v > 0$ in Q. $\square$

We end this section with remarks on decay of positive solutions solutions of (1.1). Although these remarks are not needed below, they show that the decay hypotheses (1.4) or (1.14) can be relaxed somewhat. In fact, the other hypotheses imply that u has to decay exponentially if it is sufficiently small near spatial infinity. Indeed, since $f(t,0,0)=0$, any solution of (1.1) can be viewed as a bounded solution of a linear equation (2.1) with coefficients

$$\begin{aligned} a_{ij}(x,t) &= A_{ij}(t,u(x,t),\nabla u(x,t)), \\ b_i(x,t) &= \int_0^1 f_{p_i}(t,u(x,t),s\nabla u(x,t))\,ds \\ c(x,t) &= \int_0^1 f_u(t,su(x,t)),0)\,ds. \end{aligned} \tag{2.17}$$

If u satisfies (1.13) then hypotheses (Q1), (Q2) imply that (L1) holds. Moreover, (Q3) and (Q1) imply that there is $\varepsilon_0>0$ such that

$$f_u(t,u,0) < -\gamma \quad (t\in\mathbb{R}, u\in[0,\varepsilon_0]). \tag{2.18}$$

It follows that if $0<u(x,t)<\varepsilon_0$ for all t and all x with large $|x|$, then $c(x,t)$ satisfies (L2). Thus, as in Lemma 2.4(i), one shows the following result.

Corollary 2.5. *Assume* (Q1)-(Q3) *and let u be a positive solution of* (1.1) *on $\mathbb{R}^N\times(-\infty,T)$ satisfying* (1.13) *and*

$$\limsup_{|x|\to\infty} u(x,t) \le \epsilon_0$$

uniformly in t. Then for some constants C and $\nu>0$

$$u(x,t)\le Ce^{-\nu|x|} \quad (x\in\mathbb{R}^N, t<T).$$

3 Nonlinear equations: Proof of Theorem 1.3

In the whole section u is a positive solution of (1.1) on $\mathbb{R}^N\times(-\infty,T)$ satisfying (1.13) and (1.14).

We shall use the following notation. For $R,\lambda\in\mathbb{R}$, $\xi\in\mathbb{R}^N$, let

$$\begin{aligned} \mathbb{R}^N_\lambda &:= \{x\in\mathbb{R}^N : x_1>\lambda\}, \\ Q_\lambda &:= \mathbb{R}^N_\lambda\times(-\infty,T), \\ \Gamma_\lambda &:= \partial\mathbb{R}^N_\lambda = \{x\in\mathbb{R}^N : x_1=\lambda\}, \\ B(\xi,R) &:= \{x\in\mathbb{R}^N : |\xi-x|<R\}. \end{aligned} \tag{3.1}$$

As above, let P_λ denote the reflection in the hyperplane Γ_λ. For a function $z(x) = z(x_1, x')$ let z^λ and $V_\lambda z$ be defined by

$$\begin{aligned} z^\lambda(x) &= z(P_\lambda x) = z(2\lambda - x_1, x'), \\ V_\lambda z(x) &= z^\lambda(x) - z(x) \quad (x \in \mathbb{R}^N). \end{aligned} \tag{3.2}$$

Our goal is to prove that for some λ_∞ we have

$$V_{\lambda_\infty} u(x,t) = u(P_{\lambda_\infty} x, t) - u(x,t) = 0 \text{ and } u_{x_1}(x,t) < 0 \quad ((x,t) \in Q_{\lambda_\infty}). \tag{3.3}$$

This gives the symmetry and monotonicity of u, as stated in Theorem 1.3. We use the moving hyperplane method, as outlined the following scheme. Consider the statement

$(\mathrm{T})_\lambda$ $\quad\inf_{t<T} V_\lambda u(x,t) > 0$ for each $x \in \mathbb{R}^N_\lambda$.

We carry out the proof of (3.3) in the following three steps.

STEP 1. $(\mathrm{T})_\lambda$ holds if λ is sufficiently large.

STEP 2. We set

$$\lambda_\infty = \inf\{\mu\text{: } (\mathrm{T})_\lambda \text{ holds for all } \lambda \geq \mu\}, \tag{3.4}$$

and prove that $\lambda_\infty > -\infty$ and $V_{\lambda_\infty} u(\cdot, \hat{t}_n) \to 0$ for some sequence $\hat{t}_n \to -\infty$. Moreover, we prove that $u_{x_1}(x,t) < 0$ in Q_{λ_∞}.

STEP 3. We prove that $V_{\lambda_\infty} u \equiv 0$. Assuming the contrary, we find a contradiction by examining the function $V_\lambda u$ for $\lambda < \lambda_\infty$, $\lambda \approx \lambda_\infty$.

Although the set-up is different from that in [35], several arguments we use here have counterparts in [35]. In some cases, when the arguments are straightforward to adapt, we omit some technical details.

In all these steps we rely on the fact that the function $v = V_\lambda u$ is a solution of a linear problem (2.1). Indeed, by the symmetry assumption (Q4), $u^\lambda(x,t) = u(P_\lambda x, t)$ is a solution of (1.1) for any $\lambda \in \mathbb{R}$, hence $v = V_\lambda u = u^\lambda - u$ satisfies

$$v_t = a_{ij}(x,t) v_{x_i x_j} + b_i^\lambda(x,t) v_{x_i} + c^\lambda(x,t) v, \quad (x,t) \in Q_\lambda, \tag{3.5}$$

with

$$
\begin{aligned}
a_{ij}(x,t) &= A_{ij}(t,u(x,t),\nabla u(x,t)),\\
b_i^\lambda(x,t) &= \int_0^1 f_{p_i}(t,u(x,t),\nabla u(x,t)+s\nabla(u^\lambda(x,t)-u(x,t)))\,ds\\
&\quad + u^\lambda_{x_k x_\ell}(x,t)\int_0^1 A_{k\ell\, p_i}(t,u(x,t),\nabla u(x,t)+s\nabla(u^\lambda(x,t)-u(x,t)))\,ds,\\
c^\lambda(x,t) &= \int_0^1 f_u(t,u(x,t)+s(u^\lambda(x,t)-u(x,t)),\nabla u^\lambda(x,t))\,ds\\
&\quad + u^\lambda_{x_k x_\ell}(x,t)\int_0^1 A_{k\ell\, u}(t,u(x,t)+s(u^\lambda(x,t)-u(x,t)),\nabla u^\lambda(x,t))\,ds.
\end{aligned}
$$

The coefficients have the following properties. By (Q1), (Q2) and (1.13), a_{ij}, b_i^λ and c^λ are continuous, in fact, uniformly continuous on $\mathbb{R}^N \times I$ if I is any compact subinterval of $(-\infty, T)$, and bounded:

$$
|a_{ij}(x,t)|, |b_i^\lambda(x,t)|, |c^\lambda(x,t)| < \beta_0 \quad (x \in \mathbb{R}^N,\ t < T), \tag{3.6}
$$

where β_0 is a constant independent of λ. From the ellipticity of A_{ij}, we get the uniform ellipticity of a_{ij}: there is a constant α_0, such that

$$
a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0|\xi|^2 \quad (\xi \in \mathbb{R}^N, x \in \mathbb{R}^N, t < T). \tag{3.7}
$$

Further observe that by (1.12) and (Q1) we have $c^\lambda(x,t) < -\gamma$ whenever the values of u, u^λ and their first and second spatial derivatives at (x,t) are all sufficiently small. Hence, the decay condition (1.14) implies that there exists $\rho > 0$ such that

$$
c^\lambda(x,t) < -\gamma \quad (t < T, x \in \mathbb{R}^N, |x| \geq \rho, |P_\lambda x| \geq \rho).
$$

We rewrite this condition in the following way

$$
\begin{aligned}
& c^\lambda(x,t) < -\gamma \quad (t < T, x \in \mathbb{R}^N_\lambda \setminus G_\lambda) && (3.8)\\
& \text{with } G_\lambda := B(0,\rho) \cup P_\lambda B(0,\rho). && (3.9)
\end{aligned}
$$

Finally, note that the uniform continuity of the derivatives of A_{ij} and f (hypothesis (Q1)), in conjunction with (1.13), implies

$$
\lim_{|\lambda-\mu|\to 0} \sup_{\substack{x\in\mathbb{R}^N, t<T,\\ i=1,\dots,N}} \big(|b_i^\lambda(x,t) - b_i^\mu(x,t)| + |c^\lambda(x,t) - c^\mu(x,t)|\big) = 0. \tag{3.10}
$$

The following useful sufficient condition for $(\mathrm{T})_\lambda$ is a direct consequence of Lemma 2.4(ii).

Lemma 3.1. *There exists a constant* $\delta_1 > 0$ *independent of* λ *such that* $(\mathrm{T})_\lambda$ *holds provided* $v = V_\lambda u$ *satisfies*

$$v(x,t) > 0 \quad (x \in D_0, t < T), \tag{3.11}$$

for some domain $D_0 \subset \mathbb{R}^N_\lambda$ *such that*

$$D_0 \supset G_\lambda \cap \{x \in \mathbb{R}^N_\lambda : x_1 \geq \lambda + \delta_1\}. \tag{3.12}$$

Next we prove a uniform positivity of $u(\cdot,t)$ on balls.

Lemma 3.2. *Given any ball* $B \subset \mathbb{R}^N$*, there exists a constant* $k(B) > 0$ *such that*

$$u(x,t) \geq k(B) \quad (x \in \overline{B}, t < T). \tag{3.13}$$

Proof. First observe that

$$\liminf_{t \to -\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} > 0. \tag{3.14}$$

Indeed, (Q1) and (Q3) imply that (2.18) holds for some $\varepsilon_0 > 0$. Therefore, by comparison with constants, if $u(\cdot,t) < \varepsilon < \varepsilon_0$ holds for $t = t_0$, then it holds for all $t > t_0$. Thus if (3.14) were not true, u would have to be identical to zero, contrary to the assumption $u > 0$.

Now (3.14) and the continuity of u imply that for some $s > 0$ we have

$$\sup\{u(x,t) : x \in \mathbb{R}^N\} > s \quad (t < T).$$

Then, by (1.14), if the ball B is sufficiently large (which we may assume without loss of generality), we also have

$$\sup\{u(x,t) : x \in B\} > s \quad (t < T).$$

As remarked above, u can be viewed as a solution of a linear equation (2.1) with coefficients (2.17). Applying the Harnack inequality (Lemma 2.2 with with $D = B$ and D_1 equal to a larger ball, noting that $u^- = 0$) we obtain (3.13). □

3.1 Step 1: Large λ

Lemma 3.3. *There exists $\lambda_1 \in \mathbb{R}$ such that* $(\mathrm{T})_\lambda$ *holds for all* $\lambda > \lambda_1$*.*

Proof. If λ is sufficiently large, then

$$\mathbb{R}^N_\lambda \cap G_\lambda = P_\lambda B(0, \rho). \tag{3.15}$$

Moreover, since

$$\inf\{|x| : x \in P_\lambda B(0, \rho)\} \to \infty \text{ as } \lambda \to \infty,$$

the decay assumption (1.14) implies that for λ sufficiently large, say for $\lambda > \lambda_1$, (3.15) holds together with

$$u(y, t) < \frac{k(B(0, \rho))}{2} \quad (y \in P_\lambda B(0, \rho), t < T)$$

where $k(B(0, \rho))$ is as in Lemma 3.2. Consequently, for $\lambda > \lambda_1$ we have

$$u(x, t) - u(P_\lambda x, t) > \frac{k(B(0, \rho))}{2} \quad (x \in B(0, \rho), t < T),$$

or, equivalently,

$$V_\lambda u(x, t) = u(P_\lambda x, t) - u(x, t) > \frac{k(B(0, \rho))}{2} \quad (x \in P_\lambda B(0, \rho), t < T).$$

By Lemma 3.1 (with $D_0 = P_\lambda B(0, \rho)$), this and (3.15) imply that $(\mathrm{T})_\lambda$ holds for $\lambda > \lambda_1$. □

3.2 Step 2: $\lambda = \lambda_\infty$

Let λ_1 be as in Lemma 3.3 and λ_∞ as in (3.4).

Lemma 3.4. *The following statements hold:*

(i) $-\infty < \lambda_\infty \leq \lambda_1$.

(ii) $V_{\lambda_\infty} u(x, t) \geq 0 \quad ((x, t) \in Q_{\lambda_\infty})$.

(iii) There is a sequence $\hat{t}_n \to -\infty$ *such that*

$$\|V_{\lambda_\infty} u(\cdot, \hat{t}_n)\|_{L^\infty(\mathbb{R}^N_{\lambda_\infty})} \to 0. \tag{3.16}$$

Proof. (i) Analogously to Step 1, one proves that $V_\lambda < 0$ on Q_λ if λ is sufficiently large negative. Clearly, $(\mathrm{T})_\lambda$ does not hold for such λ which proves $\lambda_\infty > -\infty$. The relation $\lambda_\infty \le \lambda_1$ is trivial.

(ii) This statement is obvious since $V_\lambda u(x,t) \to V_{\lambda_\infty} u(x,t)$ as $\lambda \searrow \lambda_\infty$.

(iii) Assume the statement is not true. Then by (ii) and the maximum principle, $v = V_{\lambda_\infty} u > 0$ in Q_{λ_∞} and there is $s > 0$ such that

$$\sup\{v(x,t) : x \in \mathbb{R}^N_{\lambda_\infty}\} > s \quad (t < T). \tag{3.17}$$

By (1.14), $v(x,t) < s/2$ for any t if $|x|$ is sufficiently large. Also, since $v(x,t) = 0$ for $x_1 = \lambda$ and ∇v is bounded, $v(x,t) < s/2$ if x_1 is close to λ. It follows that for some $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ (3.17) can be strengthened to

$$\sup\{v(x,t) : x \in D_0\} > s \quad (t < T).$$

Consequently, by Harnack inequality, for any domain $D \subset\subset \mathbb{R}^N_{\lambda_\infty}$ with $D_0 \subset D$ there is $s(D) > 0$ such that

$$v(x,t) > s(D) \quad (x \in \overline{D}, t < T). \tag{3.18}$$

Choose D (bounded and) so large that for each $\lambda \le \lambda_\infty$ sufficiently close to λ_∞ we have

$$D \supset G_\lambda \cap \{x : x_1 \ge \lambda + \delta_1\}, \tag{3.19}$$

where δ_1 is as in Lemma 3.1. Since ∇u is bounded, (3.18) implies that for each $\lambda \le \lambda_\infty$ sufficiently close to λ_∞ we have

$$V_\lambda u(x,t) > \frac{s(D)}{2} \quad (x \in \overline{D}, t < T). \tag{3.20}$$

This, together with (3.19) and Lemma 3.1, imply that $(\mathrm{T})_\lambda$ holds for each $\lambda \le \lambda_\infty$ sufficiently close to λ_∞, contradicting the definition of λ_∞. The contradiction proves (iii). □

The next lemma completes Step 2.

Lemma 3.5. *The following statements hold.*

(i) $u_{x_1} < 0$ *in* Q_{λ_∞}*. Moreover, for each domain* $D \subset\subset \mathbb{R}^N_{\lambda_\infty}$ *there is a constant* $m_1(D) > 0$ *such that*

$$-u_{x_1}(x,t) \ge m_1(D) \quad (x \in \overline{D}, t < T) \tag{3.21}$$

(ii) For each $\lambda > \lambda_\infty$ and each domain $D \subset\subset \mathbb{R}^N_\lambda$ there is a constant $m_2(D, \lambda) > 0$ such that

$$V_\lambda u(x,t) \geq m_2(D,\lambda) \quad (x \in \overline{D}, t < T) \tag{3.22}$$

Proof. For each $\lambda > \lambda_\infty$, $v = V_\lambda u$ is a positive solution of (3.5) on Q_λ. Using $(\mathrm{T})_\lambda$ and applying Harnack inequality to v we obtain (3.22) (recall that $D \subset\subset \mathbb{R}^N_{\lambda_\infty}$ entails the boundedness of D). Further, since v vanishes on $\Gamma_\lambda \times (-\infty, T)$, the Hopf boundary lemma gives

$$-2u_{x_1}\Big|_{x_1=\lambda} = v_{x_1}\Big|_{x_1=\lambda} > 0,$$

which shows that $u_{x_1} < 0$ in Q_{λ_∞}.

To prove (3.21), first note that u_{x_1} is a (strong) solution of (1.19) with coefficients (1.20). The equation is obtained by formal differentiation of (1.1). To justify it, one views (1.1) as a linear nonhomogeneous equation with coefficients a_{ij} (as in (1.20)) and the inhomogeneity $g(x,t) = f(t, u(x,t), \nabla u(x,t))$. Since a_{ij} and g are of class C^1 in x, the differentiation of (1.1) can be justified in a standard way using local L^p-estimates for the difference quotient $(u(x_1+h, x', t) - u(x_1, x', t))/h$ ($x = (x_1, x')$, $h > 0$) and passing to the limit.

Since $u_{x_1} < 0$ on Q_{λ_∞}, (3.21) follows from Harnack inequality, provided, we show that

$$\inf_{t<T} \|u_{x_1}(\cdot, t)\|_{L^\infty(B(x^0,r))} > 0$$

for a ball $B(x^0, r) \subset\subset D$. Suppose

$$\|u_{x_1}(\cdot, \tau_n)\|_{L^\infty(B(x^0,r))} \to 0 \tag{3.23}$$

for some sequence τ_n. Choose $\lambda > \lambda_\infty$ so that Γ_λ contains x_0. Then (3.23) readily implies

$$\|V_\lambda u(\cdot, \tau_n)\|_{L^\infty(B(x^0,r))} \to 0$$

contradicting (3.22). This contradiction completes the proof. $\square$

3.3 Step 3: $\lambda < \lambda_\infty$, $\lambda \approx \lambda_\infty$

We prove that $V_{\lambda_\infty} u \equiv 0$. This will complete Step 3 and the proof of Theorem 1.3.

We first use the method of moving hyperplanes starting with λ near $-\infty$. Proceeding analogously as in the steps above, we obtain the following result.

Lemma 3.6. *There exists* $\lambda_\infty^- \in (-\infty, \lambda_\infty]$ *with the following properties.*

(i) $V_{\lambda_\infty^-} u(x,t) \le 0 \quad ((x,t) \in Q_{\lambda_\infty^-})$.

(ii) $\sup_{t<T} V_\lambda u(x,t) < 0 \quad (x \in \mathbb{R}^N_{\lambda_\infty^-}, \lambda < \lambda_\infty^-)$.

(iii) *There is a sequence* $t_n \to -\infty$ *such that*

$$\|V_{\lambda_\infty^-} u(\cdot, t_n)\|_{L^\infty(\mathbb{R}_{\lambda_\infty^-})} \to 0. \tag{3.24}$$

Identity $V_{\lambda_\infty} u \equiv 0$ now clearly follows from the claim

$$\lambda_\infty^- = \lambda_\infty. \tag{3.25}$$

In the proof of this claim, the following lemma is crucial. It plays a similar role as Lemma 3.8 in [35], where also the intuitive meaning of the lemma is explained.

Lemma 3.7. *Given any domain* $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ *and any* $\theta > 0$*, there exist* $\lambda_2 < \lambda_\infty$*, domain* D *and a function* $\varphi : \overline{D} \times (-\infty, T) \to \mathbb{R}$ *with the following properties:*

(i) $D_0 \subset\subset D \subset\subset \mathbb{R}^N_{\lambda_\infty}$,

(ii) φ *is* C^2 *in* x *and* C^1 *in* t *on* $\overline{D} \times (-\infty, T)$,

(iii) $\varphi > 0$ *in* $D_0 \times (-\infty, T)$,

(iv) $\varphi < 0$ *on* $\partial D \times (-\infty, T)$,

(v) *one has*

$$\frac{\|\varphi^+(\cdot,t)\|_{L^\infty(D)}}{\|\varphi^+(\cdot,s)\|_{L^\infty(D)}} \ge Ce^{-\theta(t-s)} \quad (T > t \ge s \ge -\infty), \tag{3.26}$$

for some constant $C > 0$ *independent of* t *and* s,

(vi) *for each* $\lambda \in [\lambda_2, \lambda_\infty]$, φ *is a (strict) subsolution of* (3.5) *on* $D \times (-\infty, T)$:

$$\varphi_t < a_{ij}(x,t)\varphi_{x_i x_j} + b_i^\lambda(x,t)\varphi_{x_i} + c^\lambda(x,t)\varphi, \quad x \in D,\ t < T.$$

One constructs a subsolution φ using the function $v := V_\mu u$ for $\mu > \lambda_\infty$, $\mu \approx \lambda_\infty$. Specifically, it can be shown that for suitable λ_2 and domain D the statements are satisfied with

$$\varphi(x,t) = e^{-\theta t} v^\alpha(x,t) + s\left(-e^{-\theta t}(x_1-\mu)^\beta\right), \tag{3.27}$$

where $\alpha > 1 > \beta$ and $s > 0$ are some constants. The proof uses the same estimates as the proof of Lemma 3.8 in [35] and we refer the reader to that paper for details.

We now prove (3.25) by contradiction. Assume $\lambda_\infty^- < \lambda_\infty$.

Fix $\delta > 0$ as in Lemma 2.3 with

$$\Theta = \beta_0/\alpha_0 + 1, \quad \varepsilon = \frac{\gamma}{2\alpha_0}. \tag{3.28}$$

Choose a domain $D_0 \subset\subset \mathbb{R}^N_{\lambda_\infty}$ such that the following inclusion holds for $\lambda = \lambda_\infty$:

$$G_\lambda \cap \{x \in \mathbb{R}^N : x_1 \geq \lambda + \delta\} \subset\subset D_0. \tag{3.29}$$

Clearly, this is still valid if $\lambda \leq \lambda_\infty$ is close enough to λ_∞, say if $\lambda \in (\lambda_3, \lambda_\infty]$, for some $\lambda_3 < \lambda_\infty$.

Let $\lambda_2 < \lambda_\infty$ and D be as in Lemma 3.7 with

$$\theta := \min\{\gamma/2, \alpha_0\}. \tag{3.30}$$

Fix any λ satisfying $\max\{\lambda_3, \lambda_2\} < \lambda < \lambda_\infty$, and set $v = V_\lambda u$.

Let $t_n, \hat{t}_n \to -\infty$ be as in Lemmas 3.6 and 3.4, respectively. We claim that for some constant $q > 0$ the following inequalities hold if n is sufficiently large:

$$\begin{aligned} v(x,\hat{t}_n) &< -q \quad (x \in \overline{D}), \\ v(x,t_n) &> q \quad\;\;\, (x \in \overline{D}). \end{aligned} \tag{3.31}$$

Indeed, we have for any $x \in \overline{D}$

$$\begin{aligned} v(x,\hat{t}_n) = [u(2\lambda - x_1, x', \hat{t}_n) &- u(2\lambda_\infty - 2\lambda + x_1, x', \hat{t}_n)] \\ &+ [u(2\lambda_\infty - 2\lambda + x_1, x', \hat{t}_n) - u(x_1, x', \hat{t}_n)]. \end{aligned}$$

Since $\lambda < \lambda_\infty$, the expression in the second pair of brackets is bounded above by a negative constant, by Lemma 3.5(i), whereas the expression in the first brackets converges to zero uniformly with respect to x, by the defining property of $\hat{t}_n$ in Lemma 3.4. This proves the first inequality in (3.31), the other one is analogous.

It follows from (3.31) that for each large n there exists $T_n > t_n$ such that

$$v(x,t) > 0 \quad ((x \in \overline{D},\, t \in [t_n, T_n)), \tag{3.32a}$$

$$v(\cdot, T_n) \text{ vanishes somewhere on } \partial D. \tag{3.32b}$$

In the rest of the proof we use almost identical arguments as in the proof of Lemma 3.7 in [35]. We only sketch them. One shows that (3.32) have the following consequences (with θ as in (3.30)):

(C1) $T_n - t_n > 2$ for all n large enough,

(C2) $\sup_{t\in[t_n,T_n]} e^{\theta(t-t_n)}\|v^-(\cdot,t)\|_{L^\infty(\mathbb{R}^N_\lambda)} \to 0$ as $n \to \infty$,

(C3) there is a constant $C_0 > 0$ such that

$$\inf_{t\in[t_n,T_n]} e^{\theta(t-t_n)}\|v^+(\cdot,t)\|_{L^\infty(D)} \geq C_0 \text{ for all } n \text{ large enough.}$$

Let us give brief remarks on how these properties are proved (see [35] for details). (C1) is obtained by a continuity argument using the facts that $v(\cdot, t_n) > q > 0$ in D (see 3.31) and

$$\|v^-(\cdot, t_n)\|_{L^\infty(\mathbb{R}^N_\lambda)} \to 0 \quad \text{as } n \to \infty \tag{3.33}$$

(this follows from statements (ii) and (iii) of Lemma 3.6). (C2) follows from the fact that $v(x,t)$ is positive for $(x,t) \in D_0 \times [t_n, T_n)$ and outside this domain the equation can be modified, as in (2.7), so that the coefficient $\hat{c}$ has a negative upper bound. One can thus estimate v^- by the maximum principle using (3.33) again. Finally, one uses the subsolution φ of Lemma 3.7 to verify (C3).

Let us show that (C1)-(C3) lead to a contradiction (which proves (3.25)). Choose any bounded domain $D_1 \subset \mathbb{R}^N_\lambda$ such that $D \subset\subset D_1$. Using Lemma 2.2 with $\tau = T_n - 1$, we find constants κ and m independent of n such that

$$v(x, T_n) \geq \kappa \|v^+(\cdot, T_n - \frac{1}{2})\|_{L^\infty(D)} - e^m \|v^-\|_{L^\infty(\mathbb{R}^N_\lambda \times (T_n - 1, T_n))} \quad (x \in \overline{D}).$$

By (C1) - (C3), this inequality implies that for each $x \in \overline{D}$

$$\begin{aligned} v(x, T_n) &\geq e^{-\theta(T_n - t_n)} (C_0 e^{\theta/2} \kappa - e^m e^{\theta(T_n - t_n)} \|v^-\|_{L^\infty(\mathbb{R}^N_\lambda \times (T_n - 1, T_n))}) \\ &> e^{-\theta(T_n - t_n)} C_0 e^{\theta/2} \kappa/2 > 0 \end{aligned}$$

if n is sufficiently large. This is a contradiction to (3.32b), which completes Step 3.

4 Linearizations: Proofs of Theorems 1.5, 1.6

Here we prove the results concerning the linearization along a positive solution.

Proof of Theorem 1.5. Assume the hypotheses of the theorem are satisfied. To prove the first part of the conclusion, consider the linearized equation (1.19) with coefficients (1.20). Similarly as for equation (3.5), there are positive constants α_0, β_0 and R such that the coefficients satisfy (2.2),(2.3) and (2.4). Let v be a bounded solution of (1.19). What we have to prove is equivalently formulated as saying that the odd part of v, $v_o(x, t) := (v(x, t) - v(P_\lambda x, t))/2$, is a scalar multiple of $w := -u_{x_1}$. Observe that v_o solves (1.19), by the symmetry of u, and so does w (cf. the proof of Lemma 3.5). Thus, without loss of generality, we may proceed assuming that v itself is odd around $x_1 = \lambda$ (and we have to prove that is it is a multiple of w). Choose a domain $D_0 \subset\subset \mathbb{R}^N_\lambda$ such that

$$B(0, R) \cap \mathbb{R}^N_\lambda \setminus D_0 \subset \{x \in \mathbb{R}^N_\lambda : \lambda < x_1 < \lambda + \delta\} \tag{4.1}$$

where δ so small that the conclusion of Lemma 2.3 holds with $\Theta = \beta_0/\alpha_0 + 1$ and $\varepsilon = \gamma/(2\alpha_0)$ and the conclusion of Lemma 2.4(ii) holds with $Q_\lambda = \mathbb{R}^N_\lambda \times (-\infty, T)$. The latter guarantees, since $w = v = 0$ for $x_1 = \lambda$, that if ℓ and k are constants and $\ell w - kv$ is positive in $D_0 \times (-\infty, T)$ then it is positive in Q_λ.

Replacing v by $-v$, we may further assume that v is positive somewhere, hence also somewhere in $D_0 \times (-\infty, T)$. Since v is bounded and w is bounded below by a positive constant on $D_0 \times (-\infty, T)$ (see Lemma 3.5(i)), for each small constant $k_1 > 0$ we have $w - k_1 v > 0$ on $D_0 \times (-\infty, T)$ (hence on Q_λ). Let k be the supremum of all k_1 for which the inequality holds. Then $w - kv \geq 0$ in Q_λ and, by the maximum principle, either $w - kv \equiv 0$ or

$w - kv > 0$. The proof will be complete if we rule out the latter. Suppose it holds. We distinguish two cases:

a) $\|w(\cdot, t_n) - kv(\cdot, t_n)\|_{L^\infty(D_0)} \to 0$ for some sequence $t_n \to -\infty$,

b) $\|w(\cdot, t) - kv(\cdot, t)\|_{L^\infty(D_0)} \geq M \quad (t < T)$, for some $M > 0$.

In the case b), Harnack inequality implies $w - kv > M_1$ in $\overline{D}_0 \times (-\infty, T)$ for some constant $M_1 > 0$. Then, since v is bounded, we find $\epsilon > 0$ so that $w - (k + \epsilon)v > 0$ in $\overline{D}_0 \times (-\infty, T)$ contradicting the definition of k.

It remains to find a contradiction in the case a). It will consist in proving that $kv - w \geq 0$. Observe that for any $\ell > k$, the convergence in a) and the lower bound on w give

$$\ell v(\cdot, t_n) - w(\cdot, t_n) = (\ell - k)v(\cdot, t_n) + kv(\cdot, t_n) - w(\cdot, t_n) \approx \frac{\ell - k}{k} w(\cdot, t_n) > 0$$

in $\overline{D}_0$ if n is sufficiently large. If $\ell v - w > 0$ in $D_0 \times (-\infty, T)$ (hence in Q_λ) for each $\ell > k$, then taking the limit we get the contradiction immediately: $kv - w \geq 0$. In the opposite case, there are $\ell > k$ and a sequence $T_n \to -\infty$, $T_n > t_n$, such that

$$\ell v(x, t) - w(x, t) > 0 \quad (x \in \overline{D}_0, t \in [t_n, T_n)), \tag{4.2}$$

$$\ell v(x, T_n) - w(x, T_n) = 0 \quad \text{for some } x \in \partial D_0. \tag{4.3}$$

We can show that this leads to a contradiction, similarly as in Step 3 of the proof of Theorem 1.3. By our choice of D_0, we can repeat the arguments following (3.32a), (3.32b) (replacing everywhere v by $\ell v - w$), provided we construct a subsolution $\tilde{\varphi}$ of (1.19) which has similar properties as φ in Lemma 3.7. This is done in Lemma 4.1 below. Once that lemma is proved, one derives a contradiction to (4.3) as in the end of the proof of Theorem 1.3. We omit the details.

It remains to prove the second part of the conclusion of Theorem 1.5. By (1.22) and the symmetry of u and ψ we have

$$-2c_1 u_{x_1}(x, t) = c_1(u_{x_1}(P_\lambda x, t) - u_{x_1}(x, t)) = v(P_\lambda x, t) - v(x, t) \to 0$$

as $t \to -\infty$. By Lemma 3.5(i), this is possible only if $c_1 = 0$. $\square$

Lemma 4.1. *Let D_0 be as above. For any $\theta > 0$, there exist domain D and a function $\tilde{\varphi} : \mathbb{R}^N_\lambda \times (-\infty, T) \to \mathbb{R}$ with the following properties:*

(i) $D_0 \subset\subset D \subset\subset \mathbb{R}^N_\lambda$,

(ii) $\tilde{\varphi} \in W^{2,1}_{N+1,loc}(\mathbb{R}^N_\lambda \times (-\infty, T))$,

(iii) $\tilde{\varphi} > 0$ *in* $D_0 \times (-\infty, T)$,

(iv) $\tilde{\varphi} < 0$ *on* $\partial D \times (-\infty, T)$,

(v) one has

$$\frac{\|\tilde{\varphi}^+(\cdot, t)\|_{L^\infty(D)}}{\|\tilde{\varphi}^+(\cdot, s)\|_{L^\infty(D)}} \geq Ce^{-\theta(t-s)} \quad (T > t \geq s \geq -\infty),$$

for some constant $C > 0$ *independent of* t *and* s,

(vi) $\tilde{\varphi}$ *is a subsolution of* (1.19) *on* $D \times (-\infty, T)$:

$$\tilde{\varphi}_t - (a_{ij}(x,t)\tilde{\varphi}_{x_i x_j} + b_i(x,t)\tilde{\varphi}_{x_i} + c(x,t)\tilde{\varphi}) < 0, \tag{4.4}$$

for almost all $(x, t) \in D \times (-\infty, T)$.

Proof. Let θ be given. Set $\tilde{\varphi}(x, t) = e^{-\theta t}(w(x, t) - \epsilon)$, where $w = -u_{x_1}$, as above, and $\epsilon > 0$ is so small that

$$\epsilon < \inf_{x \in D_0, t < T} w(x, t) \quad \text{(cf. Lemma 3.5)}, \tag{4.5}$$

$$\epsilon\beta_0 < \theta. \tag{4.6}$$

We have $w(x, t) \to 0$ as $|x| \to \infty$ uniformly in t. Also $w(x, t) \to 0$ as $x_1 \searrow \lambda$ uniformly in t and $(x_2, \ldots, x_N)$, since $w_{x_1} = u_{x_1 x_1}$ is bounded. This implies that there is a domain D satisfying (i) such that (iv) holds. The regularity requirement (ii) is satisfied, for w is a strong solution of (1.19) (cf. proof of Lemma 3.5), and (4.5) implies (iii). Boundedness of v and (4.5) readily imply (v). Finally, the left-hand side of (4.4) is almost everywhere equal to $e^{-\theta t}(-\theta + \epsilon c(x, t))$, which is negative by (4.6). Thus $\tilde{\varphi}$ has all the stated properties. $\square$

Proof of Theorem 1.6. Using a translation, we may assume that $\xi = 0$. Observe that under the assumptions of the theorem, equation (1.19) takes the form

$$v_t = \tilde{a}(r, t)\Delta v + \tilde{b}(r, t)v_r + \tilde{c}(r, t)v, \tag{4.7}$$

where $r = |x|$ and $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ are some continuous bounded functions. The rotational invariance and Theorem 1.5 yield the following: given any solution v of (4.7) and any direction $e \in \mathbb{R}^N \setminus \{0\}$, there is a constant s such that $v(\cdot, t) - se \cdot \nabla u(\cdot, t)$ is symmetric about the hyperplane $\{x : x \cdot e = 0\}$ (for each t).

We first find c_1 so that $\psi_1 = v - c_1 u_{x_1}$ is even in x_1. Applying the above to the solution ψ_1, we next find c_2 such that $\psi_2 = \psi_1 - c_2 u_{x_2}$ is even in x_2. Clearly, ψ_2 is also even in x_1, as ψ_1 and u_{x_2} are. Continuing this way, we find constants $c_1, \ldots, c_N$ such that

$$\psi := v - c_1 \partial_{x_1} u - \cdots - c_N \partial_{x_N} u$$

is even in all the variables $x_1, \ldots, x_N$. We prove that ψ is radially symmetric.

Let S be the unit sphere in $\mathbb{R}^N$ centered at the origin. We introduce an orthonormal basis h_j, $j = 0, \ldots, \infty$ of the space $L^2(S)$ consisting of spherical harmonics (eigenfunctions of the spherical Laplacian). Recall (see [41]) that each spherical harmonic is the restriction to S of a homogeneous harmonic polynomial on $\mathbb{R}^N$. In particular, spherical harmonics of degree one are the restrictions of the coordinate functions $x_1, \ldots, x_N$ and their linear combinations. We assume that in the orthonormal basis, h_0 is a constant and h_j is a multiple of x_j, $j = 1, \ldots, N$. Any continuous function $z(x) = z(r\omega)$, $\omega \in S$, on $\mathbb{R}^N$ can be written as an infinite Fourier series of spherical harmonics with (uniquely determined) r-dependent coefficients:

$$\int_S z(r\omega) h_j(\omega)\, d\sigma_\omega.$$

It is clear that if a function is even in all variables $x_1, \ldots, x_N$, then the coefficients of the first order spherical harmonics in this series must vanish for each r. This in particular applies to the solution $\psi(x, t)$:

$$\int_S \psi(r\omega, t) h_j(\omega)\, d\sigma_\omega = 0 \quad (r > 0, t < T, j = 1, \ldots, N).$$

Now, if $j \in \{1, \ldots, N\}$, e is any unit vector in $\mathbb{R}^N$, and P^e is the reflection about the hyperplane $\{x : x \cdot e = 0\}$, then $h_j P^e$ is still a spherical harmonic of degree one. Thus, by a change of variable, the above identities remain valid if $\psi(\cdot, t)$ is replaced by $\psi(P^e \cdot, t)$. Given any unit vector e, let s be such that $\psi(\cdot, t) - se \cdot \nabla u(\cdot, t)$ is symmetric about the hyperplane $\{x : x \cdot e = 0\}$. Using this symmetry and the radial symmetry of u we obtain

$$\psi(P^e x, t) - \psi(x, t) = 2se \cdot \nabla u(x, t) \quad (x \in \mathbb{R}^N, t < T).$$

As mentioned above, in the Fourier series for the left-hand side the coefficients of the spherical harmonics of degree one vanish. On the other hand, by the radial symmetry, the function on the right-hand side has the form $sK(r)e \cdot x$ and its Fourier series contains only the spherical harmonics of degree one. Hence necessarily $s = 0$. This proves that $\psi(x,t) = \psi(P^e x, t)$ for any direction e and therefore ψ is radially symmetric in x, as claimed.

If the additional condition (1.25) is satisfied, then, by Theorem 1.5, in the above construction we have $c_1 = \cdots = c_N = 0$, hence $v = \psi$ is radially symmetric. $\square$

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