Addendum to the paper

P. Poláčik, Threshold solutions and sharp transitions for nonautonomous parabolic equations on $\mathbb{R}^N$. Arch. Rational Mech. Anal. 199 (2011), 69-97.

The phase plane analysis, as referred to in the proof of Lemma 3.5 of the above paper, applies only if $\theta \in (\beta^*, \gamma)$, where $\beta^* \in (\beta, \gamma)$ is determined by $\int_0^{\beta*} g^I(u)\,du = 0$ (the notation used here is introduced in formulas (2.1)-(2.5) of the paper). Accordingly, this proves Lemma 3.5 for $\theta \in (\beta^*, \gamma)$ only (thanks to Yihong Du and Bendong Lou for pointing this out to me). For $\theta \in (\beta, \beta^*]$ an additional argument is needed, as indicated below.

First modify the sentence

"By elementary phase plane analysis (cp. [2, Proposition 4.3]), such a solution exists for $c = 0$, and hence for $c \approx 0$, if condition (2.3) is satisfied (which we assume in the bistable case)."

at the end of the first paragraph of the proof as follows

"By elementary phase plane analysis (cp. [2, Proposition 4.3]), such a solution exists for $c = 0$, and hence for $c \approx 0$, if condition (2.3) is satisfied (which we assume in the bistable case) and if $\theta \in (\beta^*, \gamma)$, where $\beta^* \in (\beta, \gamma)$ is determined by $\int_0^{\beta*} g^I(u)\,du = 0$."

Then add the following text right after this sentence.

We have thus proved the conclusion of Lemma 3.5 for each $\theta \in (\beta^*, \gamma)$. It remains to prove the conclusion for $\theta \in (\beta, \beta*]$. Fix any such θ. Choose some $\theta_0 \in (\beta^*, \gamma)$ and let $R_0 := R(f, \theta_0)$ be the corresponding radius in the conclusion of Lemma 3.5. It is sufficient to prove that if $R > 0$ is sufficiently large, then the solution u^R of (3.3) satisfies the following condition.

$(*)$ There is $t_0 > 0$ such that $u^R(x, t_0) > \theta_0$ for each $x \in \mathbb{R}^N$ with $|x| \leq R_0$.

Indeed, the conclusion then follows immediately by comparison of $u^R(x, t - t_0)$ and the solution of (3.3) with $\theta = \theta_0$, $R = R_0$.

Let us first consider the solution $\bar{u}$ of (1.1) with initial condition identical to θ. Clearly, $\bar{u}$ coincides with the solutions of the ODE $\bar{u}_t = f(\bar{u})$ with $\bar{u}(0) = \theta$, hence there is $t_0 > 0$ such that $\bar{u}(t_0) > \theta_0$. We now claim that $u^R(\cdot, t_0) \to \bar{u}(t_0)$, as $R \to \infty$, uniformly on the ball $\{x : |x| \leq R_0\}$; in particular, $(*)$ holds for all large enough R.

The claim is a consequence of a continuity of the solutions of (1.1) with respect to the initial conditions. More specifically, consider the space $\mathcal{B}_\gamma$ of all continuous functions on $\mathbb{R}^N$ taking values in $[0, \gamma]$. We equip $\mathcal{B}_\gamma$ with the metric given by the weighted sup-norm

$$\|v\|_\rho \equiv \sup_{x \in \mathbb{R}} \rho(x)|v(x)|,$$

where $\rho(x) := 1/(1 + |x|^2)$. Then, given any $t > 0$ and any two solutions u, $\tilde{u}$ of (1.1) with $u(\cdot, 0), \tilde{u}(\cdot, 0) \in \mathcal{B}_\gamma$, one has

$$(**) \qquad \|u(\cdot, 0) - \tilde{u}(\cdot, t)\|_\rho \leq L(t)\|u(\cdot, 0) - \tilde{u}(\cdot, 0)\|_\rho,$$

where $L(t)$ is a constant depending on t, but not on the solutions. This continuity result, which clearly implies the claim, is proved easily by considering the linear parabolic equation satisfied by $w(x, t) := \rho(x)(u(x, t) - \tilde{u}(x, t))$. The equation has bounded coefficients, hence $(**)$ follows by standard estimates. For $N = 1$ the details can be found in [14, Lemma 6.2] and a similar computation applies in any dimension.