Math 3592H Honors Math I
Final exam, Friday December 16, 2016

**Name:**

**Instructions:**
3 hours, closed book, no electronic devices, but a standard 8.5 by 11 page of notes (front and back) is allowed.
There are 8 problems, worth a total of 100 points.

1. (12 points; 6 points each part)
Consider the linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \\ -1 & -2 & -3 \end{bmatrix}.$$

Write down a basis for ...

(a) the image of $A$.

$$A \overset{\text{row-reduction}}{\rightsquigarrow} \tilde{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the 1st column is the only pivot column, the corresponding column $\begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}$ gives a basis for $\mathrm{img}(A)$.

(b) the kernel (nullspace) of $A$.

$$\ker(A) = \ker(\tilde{A}) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : [1\ 2\ 3] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [0] \right\}$$

$$= \left\{ \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\} = \left\{ y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} : y, z \in \mathbb{R} \right\}$$

$$\Rightarrow \ker(A) \text{ has basis } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2. (12 points) Use Newton's method to approximately solve the system

$$\begin{aligned} x^3 + y^3 &= xy \\ x^4 + y^4 &= x + y \end{aligned}$$

starting with $\bar{\mathbf{a}}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, and finding the next approximation $\bar{\mathbf{a}}_1$.
(Make sure to clarify your procedure for the sake of partial credit.)

Consider $\mathbb{R}^2 \xrightarrow{\bar{f}} \mathbb{R}^2$

$$\bar{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^3+y^3-xy \\ x^4+y^4-(x+y) \end{pmatrix}$$

and want to solve $\bar{f}\begin{pmatrix} x \\ y \end{pmatrix} = \bar{0}$ approximately,

so solve for $\bar{y} = \bar{0}$ in $\bar{y} - \bar{f}(\bar{a}_0) = D\bar{f}(\bar{a}_0)(\bar{x} - \bar{a}_0)$ instead,

i.e. $-\bar{f}(\bar{a}_0) = D\bar{f}(\bar{a}_0)(\bar{a}_1 - \bar{a}_0)$ gives us next approximation $\bar{a}_1$.

$$\Rightarrow \bar{a}_1 = \bar{a}_0 - [D\bar{f}(\bar{a}_0)]^{-1} \bar{f}(\bar{a}_0)$$

$\bar{a}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\bar{f}(\bar{a}_0) = \begin{pmatrix} 0^3 + (-1)^3 - 0\cdot(-1) \\ 0^4 + (-1)^4 - (0-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$D\bar{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 3x^2-y & 3y^2-x \\ 4x^3-1 & 4y^3-1 \end{bmatrix}$, $D\bar{f}(\bar{a}_0) = \begin{bmatrix} 1 & 3 \\ -1 & -5 \end{bmatrix}$

$$[D\bar{f}(\bar{a}_0)]^{-1} = \frac{1}{-5+3}\begin{bmatrix} -5 & -3 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 & 3 \\ -1 & -1 \end{bmatrix}$$

$$\begin{aligned} \bar{a}_1 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{bmatrix} 5 & 3 \\ -1 & -1 \end{bmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix} \end{aligned}$$

3. (12 points total; 6 points each part)

Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{x^3+y^2+z}$.

(a) Compute the Jacobian matrix $Jf(\overline{\mathbf{x}})$ at a general point $\overline{\mathbf{x}}$.

$$Jf(\bar{x}) = \left[ \frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z} \right]$$

$$= \left[ 3x^2 e^{x^3+y^2+z} \quad 2y e^{x^3+y^2+z} \quad e^{x^3+y^2+z} \right]$$

(b) For which unit vector $\overline{\mathbf{u}}$ in $\mathbb{R}^3$ will the directional derivative of $\overline{\mathbf{f}}$ at $\overline{\mathbf{x}} = \overline{\mathbf{0}}$ in the direction $\overline{\mathbf{u}}$ be largest? Explain.

$$Jf(\bar{0}) = \left[ 3 0 \cdot e^{0^3+0^2+0} \quad 2 \cdot 0 \cdot e^{0^3+0^2+0} \quad e^{0^3+0^2+0} \right]$$

$$= [0 \quad 0 \quad 1]$$

$\Rightarrow$ directional derivative at $\bar{x} = \bar{0}$ in direction $\bar{u}$

is $Jf(\bar{0})(\bar{u}) = [0 \ 0 \ 1] \cdot \bar{u}$

$= |[0 \ 0 \ 1]| \, |\bar{u}| \cos\theta$ where $\theta$ = angle between $[0 \ 0 \ 1]$ and $\bar{u}$

$= \cos\theta$

which is maximized when $\bar{u} = [0 \ 0 \ 1]$.

4. (12 points) Describe, with explanation, the set of all points in $\mathbb{R}^2$ where this function is differentiable:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \frac{2xy}{x^2+5y^2} & \text{if } \overline{\mathrm{x}} \neq \overline{0}, \\ 0 & \text{if } \overline{\mathrm{x}} = \overline{0} \end{cases}$$

Since $f$ is a rational function $f\binom{x}{y} = \dfrac{p\binom{x}{y}}{q\binom{x}{y}}$,

it is differentiable at every $\binom{x}{y}$ where its denominator polynomial $q\binom{x}{y} \neq 0$, at least. In this case,

$q\binom{x}{y} = x^2 + 5y^2 \neq 0 \quad \forall \binom{x}{y} \neq \binom{0}{0}$.

So the only question is whether $f$ happens to also be differentiable at $\overline{x} = \overline{0}$, but it is not even continuous there, since when we approach $\binom{0}{0}$ along lines $y = cx$, the limits are not all the same:

$$\lim_{\substack{\binom{x}{y} \to \binom{0}{0} \\ \text{with } y = cx}} f\binom{x}{y} = \lim_{t \to 0} f\binom{t}{ct} = \lim_{t \to 0} \frac{2 \cdot t \cdot ct}{t^2 + 5(ct)^2} = \frac{\cancel{t^2} \cdot 2c}{\cancel{t^2}(1+5c^2)}$$

$$= \frac{2c}{1+5c^2}$$

depends upon c

Hence $f$ is differentiable exactly on $\mathbb{R}^2 - \left\{\binom{0}{0}\right\}$.

5. (13 points total)
Recall the *trace* of a matrix $X$ in $\mathrm{Mat}(n,n)$ is $\mathrm{Tr}(X) := \sum_{i=1}^{n} x_{i,i}$.
Prove or disprove.

(a) (7 points) The function $f : \mathrm{Mat}(n,n) \to \mathbb{R}$ given by $f(X) = \mathrm{Tr}(X)$ is differentiable at every $X = A$ in $\mathrm{Mat}(n,n)$, with

$$Df(A)(H) = \mathrm{Tr}(H)$$

for all $H$ in $\mathrm{Mat}(n,n)$.

<u>True</u>. $f(X) = \mathrm{Tr}(X)$ is <u>linear</u> in $X$ : $\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$
$\mathrm{Tr}(cA) = c\,\mathrm{Tr}(A)$.

Hence $Df(A)(H) = f(H) = \mathrm{Tr}(H) \quad \forall A \in \mathrm{Mat}(n,n)$

(b) (6 points)The function $\overline{\mathbf{g}} : \mathrm{Mat}(n,n) \to \mathrm{Mat}(n,n)$ defined by

$$\overline{\mathbf{g}}(X) = \mathrm{Tr}(X)^5 \cdot X^2$$

is differentiable everywhere on $\mathrm{Mat}(n,n)$.

<u>True</u>. $\bar{g}(X) = f(X)\bar{h}(X)$ where $\mathrm{Mat}(n,n) \xrightarrow{f} \mathbb{R}$
$X \longmapsto \mathrm{Tr}(X)^5$

is <u>differentiable everywhere</u>, because it is a polynomial in the entries of $X$, or the composite of $X \mapsto \mathrm{Tr}(X)$ with $y = x^5$, etc...

and $\mathrm{Mat}(n,n) \xrightarrow{\bar{h}} \mathrm{Mat}(n,n)$
$X \longmapsto X^2$

is <u>differentiable everywhere</u> (as seen on HW), e.g. because its entries are polynomials in the entries of $X$, or because $Df(A)(H) = AH + HA$

(Then apply one of our <u>product rules</u> for differentiation to $\bar{g} = f\bar{h}$)

... or just argue directly that the entries of $\bar{g}(X)$ are all polynomials in the entries of $X$, so $\bar{g}$ is differentiable everywhere

6. (13 points total) Let $x$ in $\mathbb{R}$ be a constant, and $A(x) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$.

(a) (3 points) Compute $A(x)^2, A(x)^3$.

$$A(x)^2 = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2x & 1 \end{bmatrix}$$

$$A(x)^3 = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3x & 1 \end{bmatrix}$$

(b) (3 points) Give a formula for $A(x)^n$ as a function of $n = 1, 2, \ldots$, with proof.

$A(x)^n = \begin{bmatrix} 1 & 0 \\ nx & 1 \end{bmatrix}$ by induction on $n$,

since we checked the base cases $n = 1, 2, 3$ above,

and in the inductive step,

$$A(x)^n = A(x) \cdot A(x)^{n-1}$$

$$\underset{\text{by induction}}{=} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (n-1)x & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ nx & 1 \end{bmatrix}$$

(c) (3 points) Compute explicitly the entries of the $2 \times 2$ matrix

$$e^{A(x)} = I + A(x) + \frac{A(x)^2}{2!} + \frac{A(x)^3}{3!} + \frac{A(x)^4}{4!} + \cdots$$

leaving no summations in your answer.

$$e^{A(x)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} + \frac{\begin{bmatrix} 1 & 0 \\ 2x & 1 \end{bmatrix}}{2!} + \frac{\begin{bmatrix} 1 & 0 \\ 3x & 1 \end{bmatrix}}{3!} + \cdots$$

$$= \begin{bmatrix} 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots & 0 \\ 0 + x + \frac{2x}{2!} + \frac{3x}{3!} + \cdots & 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^1 & 0 \\ x\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) & e^1 \end{bmatrix} = \begin{bmatrix} e & 0 \\ xe & e \end{bmatrix}$$

(d) (4 points) For $\overline{\mathbf{f}} : \mathbb{R} \to \mathrm{Mat}(2,2)$ given by $\overline{\mathbf{f}}(x) = e^{A(x)}$, consider the linear map $D\overline{\mathbf{f}}(\pi) : \mathbb{R} \to \mathrm{Mat}(2,2)$, its derivative at $x = \pi = 3.14159....$

Write down the $2 \times 2$ matrix $D\overline{\mathbf{f}}(\pi)(h)$, that is, the linear map $D\overline{\mathbf{f}}(\pi)$ evaluted on $h$ in $\mathbb{R}$.

Thinking of $\overline{f} : \mathbb{R} \to \mathrm{Mat}(2,2) \cong \mathbb{R}^4$

as $\quad x \longmapsto f(x) = e^{A(x)}$

$$= \begin{bmatrix} e & 0 \\ xe & e \end{bmatrix} = \begin{bmatrix} e \\ 0 \\ xe \\ e \end{bmatrix}$$

then $D\overline{f}(a) : \mathbb{R} \longrightarrow \mathbb{R}^4$

is given ~~by the matrix~~

$$\begin{bmatrix} d/dx(e) \\ d/dx(0) \\ d/dx(xe) \\ d/dx(e) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ e & 0 \end{bmatrix}$$

independent of $a$,

and hence

$$D\overline{f}(\pi)(h) = \begin{bmatrix} 0 \\ 0 \\ e \\ 0 \end{bmatrix} [h] = \begin{bmatrix} 0 \\ 0 \\ eh \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ eh & 0 \end{bmatrix}$$

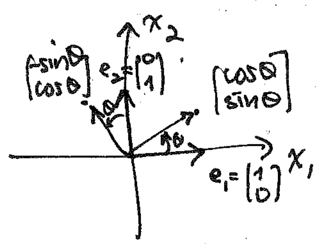
7. (13 points total) For each $\theta$ in $[0, 2\pi)$, consider the linear transformation $A_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates about the $x_3$-axis and whose restriction to the $(x_1, x_2)$-plane rotates by an angle $\theta$ counterclockwise.

(a) (4 points) Write down the matrix $A_\theta$ representing this map with respect to the standard basis $\overline{e}_1, \overline{e}_2, \overline{e}_3$.

$$A_\theta(\overline{e}_3) = \overline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_\theta(\overline{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$A_\theta(\overline{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$




$$\Rightarrow \quad \mathbb{R}^3 \xrightarrow{A_\theta} \mathbb{R}^3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (5 points) Prove there are exactly two angles $\theta$ in $[0, 2\pi)$ for which $A_\theta$ is diagonalizable with eigenvalues in $\mathbb{R}$ and eigenvectors in $\mathbb{R}^3$.

$$\chi_A(t) = \det(t\cdot I_3 - A) = \det\begin{bmatrix} t-\cos\theta & \sin\theta & 0 \\ -\sin\theta & t-\cos\theta & 0 \\ 0 & 0 & t-1 \end{bmatrix}$$

$$= (t-1)\left((t-\cos\theta)^2+\sin^2\theta\right)$$

$$= (t-1)(t^2-2\cos\theta+\cos^2\theta+\sin^2\theta)$$

$$= (t-1)\underbrace{(t^2-2\cos\theta+1)}_{\text{roots given by quadratic formula}}$$

$$t = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta-4}}{2}$$

$$t = \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$t = \cos\theta \pm i\sin\theta$$

Since these eigenvalues lie in $\mathbb{R} \iff \theta = 0, \pi$ and since $A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A_\pi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonalizable, these are the only 2 such angles.

(c) (4 points) For which angles $\theta$ in $[0, 2\pi)$ is $A_\theta$ diagonalizable if we allow eigenvalues in $\mathbb{C}$ and eigenvectors in $\mathbb{C}^3$?

Since $\chi_A(t) = (t-1)(t^2-2\cos\theta+1)$

$$= (t-1)(t-(\cos\theta+i\sin\theta))(t-(\cos\theta-i\sin\theta))$$

has distinct roots whenever $\theta \in [0,2\pi) - \{0,\pi\}$, A will be diagonalizable over $\mathbb{C}^3$ in those cases. But it is already diagonalizable over $\mathbb{R}^3$ (and hence also over $\mathbb{C}^3$) when $\theta \in \{0,\pi\}$, so it is diagonalizable for all $\theta \in [0,\pi)$ over $\mathbb{C}^3$.

8. (13 points total) Consider an $m \times n$ matrix $A$ and $n \times m$ matrix $B$, so the product $AB$ is well-defined and square $m \times m$. Recall that the *rank* of a matrix is the dimension of its image, considered as a linear transformation.

(a) (4 points) Prove that $\text{rank}(AB) \leq \text{rank}(A)$.

$\text{img}(AB) \subseteq \text{im}(A)$ since $AB\vec{x} = A(B\vec{x}) = A\vec{y}$ where $\vec{y} = B\vec{x}$.

Hence $\underbrace{\dim(\text{img}(AB))}_{= \text{rank}(AB)} \leq \underbrace{\dim(\text{img}(A))}_{= \text{rank}(A)}$

(b) (4 points) Prove that $\text{rank}(AB) \leq \text{rank}(B)$.

$\text{rank}(AB) = \text{rank}((AB)^T) = \text{rank}(B^T A^T) \underset{\substack{\uparrow \\ \text{by part (a)}}}{\leq} \text{rank}(B^T) = \text{rank}(B)$

(c) (5 points) Prove that if $AB$ is invertible then $A$ is surjective, $B$ is injective, and $m \leq n$.

If $\mathbb{R}^m \xrightarrow{AB} \mathbb{R}^m$ is invertible, then

$A$ is surjective ($\mathbb{R}^n \to \mathbb{R}^m$) since $\underbrace{\text{img}(AB)}_{= \mathbb{R}^m} \subseteq \text{img}(A) \Rightarrow \text{img}(A) = \mathbb{R}^m$.

Also $B$ is injective since $\ker(B) \subseteq \underbrace{\ker(AB)}_{= \{\vec{0}\}}$ (if $B\vec{x} = \vec{0}$ then $AB\vec{x} = A\vec{0} = \vec{0}$)

To see $m \leq n$, note ~~[illegible]~~ $m = \dim(\text{img}(A)) = \text{rank}(A) \leq \underbrace{\#\text{columns of } A}_{= n}$.