DEF'N 2.6.10–2.6.14: As before, we define

- Lin. independence of $\{\bar{v}_i\}_{i=1,\dots,k}$ in V, i.e. $\sum_{i=1}^{k} c_i\bar{v}_i = \bar{0} \Rightarrow c_1=\dots=c_k=0$
- $\mathrm{span}(\bar{v}_1,\dots,\bar{v}_k) := \{c_1\bar{v}_1+\dots+c_k\bar{v}_k : c_i \in \mathbb{R}\}$
- $\{\bar{v}_i\}_{i=1,\dots,k}$ are a basis for V if they are lin. indep. & $\mathrm{span}\{\bar{v}_i\}_{i=1,\dots,k} = V$,

~~or~~ equivalently, every $\bar{v} \in V$ can be written uniquely as $\bar{v} = \sum_{i=1}^{k} x_i \bar{v}_i$

(in which case, you call $\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$ the coordinates of $\bar{v}$ with respect to the (ordered) basis $(\bar{v}_1,\dots,\bar{v}_k)$)

11/11/2016 Once you have picked (ordered) bases $(\bar{v}_1,\dots,\bar{v}_n)$ for V
$(\bar{w}_1,\dots,\bar{w}_m)$ for W

one can express any linear transformation $T: V \to W$ uniquely in these bases via an $m\times n$ matrix $A = (a_{ij})$

where $T(\bar{v}_j) = \sum_{i=1}^{m} a_{ij}\bar{w}_i$

that is, $A = \begin{bmatrix} | & | & & | \\ T(\bar{v}_1) & T(\bar{v}_2) & \cdots & T(\bar{v}_n) \\ | & | & & | \end{bmatrix}$ where $T(\bar{v}_j)$ is expressed in coordinates with respect to $(\bar{w}_1,\dots,\bar{w}_m)$

As before, composition of linear maps corr. to multiplying matrices

EXAMPLES:

① $P_d = \{a_0 + a_1x + \dots + a_dx^d\}$ (polynomials) has basis $(1, x, x^2, \dots, x^d)$

P_{d-1} has basis $(1, x, x^2, \dots, x^{d-1})$

The linear transformation $P_d \xrightarrow{d/dx} P_{d-1}$ expressed with respect to these choices of bases has matrix

$$A = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ \vdots \\ x^{d-1} \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & x & x^2 & x^3 \cdots & x^{d-1} & x^d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & d-1 \;\; 0 & d \end{bmatrix} \end{array}$$

e.g. $d=3$

$$A = \begin{array}{c} \\ 1 \\ x \\ x^2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \end{array}$$

② The ~~integration~~ map

$$P_{d-1} \xrightarrow{\int_0^x} P_d$$
$$p(x) \longmapsto \int_0^x p(t)\,dt$$

is also linear (Why?)

and with respect to the same chosen of bases has matrix

$$B = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ \vdots \\ x^{d-1} \\ x^d \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & \cdots & x^{d-1} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & \frac{1}{2} & 0 & & 0 \\ 0 & 0 & \frac{1}{3} & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 \cdots & \frac{1}{d-1} & 0 \\ 0 & 0 & 0 \cdots & 0 & \frac{1}{d} \end{bmatrix} \end{array}$$

e.g. $d=3$
$$B = \begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 0 & \frac{1}{2} & 0 \\ x^3 & 0 & 0 & \frac{1}{3} \end{array}$$

③ Since the composite map

$$P_{d-1} \xrightarrow{\int_0^x} P_d \xrightarrow{d/dx} P_{d-1}$$
$$p(x) \longmapsto \int_0^x p(t)\,dt \longmapsto \frac{d}{dx}\int_0^x p(t)\,dt = p(x)$$

is the identity map $1_{P_{d-1}}$, one should have $AB = I_d$

$$\left(\text{i.e. } \frac{d}{dx} \circ \int_0^x = 1_{P_{d-1}}\right)$$

and this is what happens, e.g. for $d=3$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{\substack{A \\ d/dx}} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}_{\substack{B \\ \int_0^x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

A has B as a right-inverse; $d/dx : P_d \to P_{d-1}$ is surjective (but not injective; who is $\ker(d/dx)$?)

B has A as a left-inverse; $\int_0^x : P_{d-1} \to P_d$ is injective. (but not surjective; who is not in $\mathrm{img}(\int_0^x)$?)

④ Similarly one could do the composite in the other order:

$$P_d \xrightarrow[A]{d/dx} P_{d-1} \xrightarrow[B]{\int_0^x} P_d$$

The composite $\int_0^x \circ \frac{d}{dx} : P_d \to P_d$ has matrix (w.r.t. to the ordered basis $(1, x, x^2, \ldots, x^d)$ for P_d)

$$BA = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{array}{c c} & \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array} & \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

e.g. take $d=4$

$$\left(\int_0^x \circ \frac{d}{dx}\right)(x^m) = \begin{cases} x^m & \text{if } m > 0 \\ 0 & \text{if } m = 0 \end{cases}$$

DEF'N 2.6.5: + 2.6.14 An isomorphism of vector spaces V, W is a bijective linear transformation $T: V \to W$
(injective, surjective)
$\ker T = \{\bar{0}\}$, $\operatorname{img}(T) = W$

Picking an ordered basis $(\bar{v}_1, \ldots, \bar{v}_n)$ for V leads to the isomorphism

$$\mathbb{R}^n \xrightarrow{\Phi_{\{\bar{v}\}}} V$$
$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto x_1\bar{v}_1 + \ldots + x_n\bar{v}_n$$

which the book calls the "concrete to abstract" function.

$\Phi^{-1}_{\{\bar{v}\}}(\bar{w})$ gives the coordinates $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ of $\bar{w}$ with respect to the ordered basis $(\bar{v}_1, \ldots, \bar{v}_n)$

i.e. $\bar{w} = x_1\bar{v}_1 + \ldots + x_n\bar{v}_n$

What is interesting is how one changes the coordinates in V from one choice of an ordered basis $(\bar{v}_1, \ldots, \bar{v}_n)$ to another one $(\bar{w}_1, \ldots, \bar{w}_m)$

First, let's check that we should have $m = n$ here.

THM 2.6.22: & DEFIN 2.6.23 If $\{\bar v_1, \ldots, \bar v_n\}$, $\{\bar w_1, \ldots, \bar w_m\}$ are 2 bases for V, then $m = n$; $n =: \dim(V)$ dimension of V

Proof: We'd like to say that the composite bijection

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\Phi_{\{\bar v\}}", "\text{a bijection}"'] \arrow[rr, bend right, "\Phi^{-1}_{\{\bar w\}} \circ \Phi_{\{\bar v\}}"'] & V \arrow[r, "\Phi^{-1}_{\{\bar w\}}", "\text{a bijection}"'] & \mathbb{R}^m \end{tikzcd}$$

is a lin. transformation $\mathbb{R}^n \xrightarrow{\Phi} \mathbb{R}^m$ which is bijective, hence $m = n$.

However, 1st we should check that $\Phi^{-1}_{\{\bar w\}}$ is linear,

by noting LEMMA: If $T: V \to W$ is a bijective, linear map then $T^{-1}: W \to V$ is also linear.

proof (same as for PROP 1.3.14):

Want to check $T^{-1}(a\bar v_1 + b\bar v_2) = aT^{-1}(\bar v_1) + bT^{-1}(\bar v_2)$,

but since T is bijective, this is true if and only if

$$TT^{-1}(a\bar v_1 + b\bar v_2) \overset{?}{=} T(aT^{-1}(\bar v_1) + bT^{-1}(\bar v_2))$$

$$\begin{array}{ccc} TT^{-1}(a\bar v_1 + b\bar v_2) & \overset{?}{=} & T(aT^{-1}(\bar v_1) + bT^{-1}(\bar v_2)) \\ \| & & \| \leftarrow T \text{ is linear} \\ a\bar v_1 + b\bar v_2 & & aTT^{-1}(\bar v_1) + bTT^{-1}(\bar v_2) \\ & \checkmark & \| \\ & & a\bar v_1 + b\bar v_2 \end{array}$$

∎

Thus $\Phi^{-1}_{\{\bar w\}}$ is linear,

so the composite $\Phi^{-1}_{\{\bar w\}} \circ \Phi_{\{\bar v\}}: \mathbb{R}^n \to \mathbb{R}^m$ is linear, bijective, forcing $m = n$ ∎

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In fact, the matrix that represents

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "\Phi^{-1}_{\{\bar w\}} \circ \Phi_{\{\bar v\}} =: \Phi_{\{\bar v \to \bar w\}} \text{ in book}"] \arrow[dr, "\Phi_{\{\bar v\}}"'] & & \mathbb{R}^n \\ & V \arrow[ur, "\Phi^{-1}_{\{\bar w\}}"'] & \end{tikzcd}$$

is the matrix that converts the coordinates w.r.t. $\{\bar v_i\}$ for a vector in V to its coordinates w.r.t. $\{\bar w_j\}$:

$$\begin{tikzcd} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \arrow[dr, mapsto] & \mathbb{R}^n \arrow[rr, "\Phi^{-1}_{\{\bar w\}} \circ \Phi_{\{\bar v\}} = \Phi_{\{\bar v \to \bar w\}}"] \arrow[d, "\Phi_{\{\bar v\}}"] & & \mathbb{R}^n \arrow[dll, "\Phi_{\{\bar w\}}"'] & \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \arrow[dlll, mapsto] \\ & x_1\bar v_1 + \ldots + x_n\bar v_n = y_1\bar w_1 + \ldots + y_n\bar w_n & & & \end{tikzcd}$$

i.e. $\Phi_{\{\bar v \to \bar w\}}$ sends $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$