(76) Combining spanning & lin. independence gives an important concept.

DEF'N: Given a subspace $E \subset \mathbb{R}^n$, a <u>basis</u> for E is a subset $\{\bar{v}_1, \dots, \bar{v}_k\} \subset E$ that <u>spans</u> E and is <u>lin. indep.</u>

Equivalently, $\{\bar{v}_1, \dots, \bar{v}_k\}$ is a basis for E $\Leftrightarrow$ every $v \in E$ has a $!$ expression $v = c_1\bar{v}_1 + \dots + c_k\bar{v}_k$.

<u>EXAMPLES</u>: ① For an $m \times n$ matrix A, the solutions $\bar{x} \in \mathbb{R}^n$ to $A\bar{x} = 0$ always form a subspace of $\mathbb{R}^n$, and we can use row-reduction to find a basis.

$A\bar{x} = \bar{0} \Rightarrow A(c\bar{x}) = cA\bar{x} = \bar{0}$

$A\bar{x}_i = \bar{0} \Rightarrow A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \bar{0} + \bar{0} = \bar{0}$

e.g. $A = [1 \; -1 \; -1]$

$$E := \{\bar{x} \in \mathbb{R}^3 : A\bar{x} = 0\} = \left\{ \text{sol'ns to } [1 \; -1 \; -1]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\Rightarrow E$ has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} =: \bar{v}_1, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =: \bar{v}_2 \right\}$ (Why?)
- spans E?
- lin. indep.?

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② Every basis for $\mathbb{R}^n$ has exactly n elements (e.g. $\{\bar{e}_1, \dots, \bar{e}_n\}$)

<u>PROP</u>: For a subspace $E \subset \mathbb{R}^n$, and $\{\bar{v}_1, \dots, \bar{v}_k\} \subset E$, TFAE

(a) $\{\bar{v}_i\}_{i=1,\dots,k}$ are a basis for E

(b) they are a <u>minimal</u> <u>spanning set</u> for E, i.e. removing any $\bar{v}_i$ no longer spans E

(c) they are a <u>maximal</u> <u>lin. indep. set</u> in E, i.e. adding any $\bar{v} \in E$ ruins their lin. independence.

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(77) proof: (a) $\Rightarrow$ (b): A basis spans, and if removing $\bar{v}_i$ it still spans, then $\bar{v}_i \in \text{span}\{\bar{v}_j\}_{j \neq i}$

$$\Rightarrow \bar{v}_i = \sum_{j \neq i} c_j \bar{v}_j$$

$$\Rightarrow 1 \cdot \bar{v}_i - \sum_{j \neq i} c_j \bar{v}_j = \bar{0}$$ gives a nontrivial dependence among $\{\bar{v}_i\}$.

(a) $\Rightarrow$ (c): A basis is lin. indep., and if ~~adding~~ you add any $\bar{v} \in E$ to it, then the expression $\bar{v} = \sum_{j=1}^{k} c_j \bar{v}_j$ leads to a nontrivial dependence among $\{\bar{v}_j\}_{j=1,\dots,k} \cup \{\bar{v}\}$

(b) $\Rightarrow$ (a): A minimal spanning set $\{\bar{v}_i\}_{i=1,\dots,k}$ is also lin. indep., else a dependence $c_1\bar{v}_1 + \dots + c_k v_k = \bar{0}$ with $c_k \neq 0$ (WLOG) leads to a smaller spanning set $\bar{v}_1, \dots, \bar{v}_{k-1}$, as $\bar{v}_k = -\left(\frac{c_1}{c_k}\bar{v}_1 + \dots + \frac{c_{k-1}}{c_k}\bar{v}_{k-1}\right) \in \text{span}(\bar{v}_1, \dots, \bar{v}_{k-1})$.

(c) $\Rightarrow$ (a): A maximal indep set $\{\bar{v}_i\}_{i=1,\dots,k}$ is also spanning, since if $\bar{v} \in E$ has $\bar{v} \notin \text{span}\{\bar{v}_1, \dots, \bar{v}_k\}$ one cannot have a nontrivial dependence among $\bar{v}_1, \dots, \bar{v}_k, \bar{v}$ as it would need a nonzero coefficient on $\bar{v}$, show $\bar{v} \in \text{span}\{\bar{v}_1, \dots, \bar{v}_k\}$. ∎

COR (PROP-DEFN 2.4.19)
Every subspace $E \subset \mathbb{R}^n$ has a basis, and any 2 such bases contain the same number of vectors, called the dimension $\dim(E)$ of E.

EXAMPLE: $E := \left\{ \text{sol'ns to } \begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\}$ has $\dim E = 2$ since it had basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

proof: To find a basis for E, either $E = \{\bar{0}\}$ and the empty set is a basis (Why?), or one can start with some $\bar{v}_1 \in E \setminus \{\bar{0}\}$

and either $E = \text{span}(\bar{v}_1)$

or $\exists$ some $\bar{v}_2 \in E \setminus \text{span}(\bar{v}_1)$ (so $(\bar{v}_1, \bar{v}_2)$ are lin. indep. - Why?)

and either $E = \text{span}(\bar{v}_1, \bar{v}_2)$

or $\exists$ some $\bar{v}_3 \in E \setminus \text{span}(\bar{v}_1, \bar{v}_2)$ (so $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ are lin. indep. - Why?)

etc.

This must stop with $E = \text{span}(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k)$ for some $k \leq n$ (since $E \subset \mathbb{R}^n$)

and since $\{\bar{v}_i\}_{i=1,\ldots,k}$ are lin. indep., they're a basis for E.

Given 2 bases $(\bar{v}_1, \ldots, \bar{v}_m)$ for E
$(\bar{w}_1, \ldots, \bar{w}_p)$

uniquely express $\bar{w}_j = \sum_{i=1}^{m} a_{ij} \bar{v}_i$ $\qquad \overset{m \times p}{A} = (a_{ij})$

and $\bar{v}_i = \sum_{l=1}^{p} b_{li} \bar{w}_l$ $\qquad \overset{p \times m}{B} = (b_{li})$

Then one has $\bar{w}_j = \sum_{i=1}^{m} a_{ij} \bar{v}_i = \sum_{i=1}^{m} a_{ij} \sum_{l=1}^{p} b_{li} \bar{w}_l$

$$0 \cdot \bar{w}_1 + \ldots + 0\bar{w}_{j-1} + 1 \cdot \bar{w}_j + 0 \cdot \bar{w}_{j+1} + \ldots + 0\bar{w}_p \qquad = \sum_{l=1}^{p} \underbrace{\left(\sum_{i=1}^{m} b_{li} a_{ij} \right)}_{(BA)_{l,j}} \bar{w}_l$$

coefficients on $\bar{w}_j$ must be identical, since $\{\bar{w}_j\}$ are lin. indep.

$$\Rightarrow (BA)_{l,j} = \begin{cases} 1 & \text{if } j = l \\ 0 & \text{otherwise} \end{cases}$$

i.e. $\boxed{BA = I_p}$

Similarly using $\bar{v}_i = \sum_{l=1}^{p} b_{li} \sum_{k=1}^{m} a_{kl} \bar{v}_k$, conclude $\boxed{AB = I_m}$. Hence $A = B^{-1}$ and $m = p$. ∎

Some choices of bases for a subspace are more convenient than others...

DEF'N (sort of 2.4.16)

A set of vectors $\{\bar{v}_1, \dots, \bar{v}_k\}$ is orthogonal if $\bar{v}_i \cdot \bar{v}_j = 0 \ \forall i \neq j$

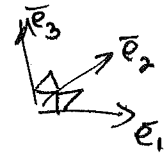
and it is orthonormal if additionally $|\bar{v}_i| = 1 \ \forall i = 1, \dots, k$

i.e. $\underbrace{\bar{v}_i \cdot \bar{v}_i}_{\text{"}|\bar{v}_i|^2\text{"}} = 1$

EXAMPLES:

① $\{\bar{e}_1, \dots, \bar{e}_n\}$ are orthonormal in $\mathbb{R}^n$ since $\bar{e}_i \cdot \bar{e}_j = 0 \ \forall i \neq j$

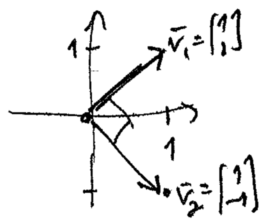
$|\bar{e}_i| = 1 \ \forall i$



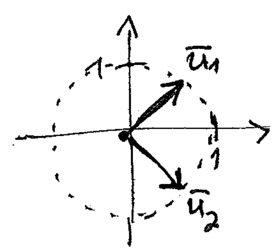
② $\left\{ \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ are orthogonal, but not orthonormal,

however $\left\{ \bar{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \bar{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

are orthonormal



normalize them



~~NON-EXAMPLES~~

PROP (sort of 2.4.17) (i) Orthogonal sets of non-zero vectors $\{\bar{v}_1, \dots, \bar{v}_k\}$ are always lin. independent, hence a basis for their span.

(ii) Orthonormal sets of vectors $\{\bar{v}_1, \dots, \bar{v}_k\}$ have the convenient property that any $\bar{v} \in \mathrm{span}(\bar{v}_1, \dots, \bar{v}_k)$

has ~~its~~ ! expansion $\bar{v} = \sum_{i=1}^{k} c_i \bar{v}_i$ given by the easily computed coefficients

$c_i = \bar{v} \cdot \bar{v}_i, \ i = 1, \dots, k$

proof: For (i), if $\{\bar{v}_i\}_{i=1,\dots,k}$ are orthogonal and one has a dependence $c_1 \bar{v}_1 + \dots + c_k \bar{v}_k = \bar{0}$, then

for each $j = 1, \dots, k$ take dot product with $\bar{v}_j$ to get

$$c_1 \underbrace{\bar{v}_1 \cdot \bar{v}_j}_{0} + \dots + c_j \bar{v}_j \cdot \bar{v}_j + \dots + c_k \underbrace{\bar{v}_k \cdot \bar{v}_j}_{0} = \bar{0} \cdot \bar{v}_j = 0$$

$$\Rightarrow c_j \underbrace{|\bar{v}_j|^2}_{\neq 0 \text{ since } \bar{v}_j \neq \bar{0}} = 0 \Rightarrow c_j = 0.$$

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For (ii), if $\{\bar{v}_i\}_{i=1,\dots,k}$ are orthonormal,

given $\bar{v} = c_1\bar{v}_1 + \dots + c_k\bar{v}_k$

again for $j=1,\dots,k$ take dot product with $\bar{v}_j$ to get

$$\bar{v}\bullet\bar{v}_j = c_1\underbrace{\bar{v}_1\bullet\bar{v}_j}_{0} + \dots + c_j\underbrace{\bar{v}_j\bullet\bar{v}_j}_{1} + \dots + c_k\underbrace{\bar{v}_k\bullet\bar{v}_j}_{0}$$

$$\Rightarrow \bar{v}\bullet\bar{v}_j = c_j \qquad \blacksquare$$

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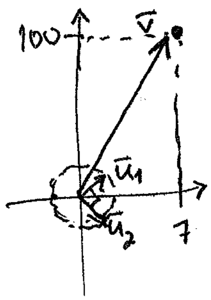
EXAMPLE: $\left\{ \bar{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \bar{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$

and hence $\bar{v} = \begin{bmatrix}7\\100\end{bmatrix} = c_1\bullet\bar{u}_1 + c_2\bullet\bar{u}_2$ where $c_1 = \bar{v}\bullet\bar{u}_1 = \begin{bmatrix}7\\100\end{bmatrix}\bullet\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$= \frac{107}{\sqrt{2}}$

$c_2 = \bar{v}\bullet\bar{u}_2 = \begin{bmatrix}7\\100\end{bmatrix}\bullet\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$= \frac{-93}{\sqrt{2}}$



NON-EXAMPLE: $E = \left\{ \text{sol'ns to } [1\ -1\ -1]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = [0] \text{ in } \mathbb{R}^3 \right\}$

had basis $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$, but it is neither orthogonal nor orthonormal.