DEF'N 2.9: The <u>rank</u> of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is $\dim(\mathrm{img}(T)) =: \mathrm{rank}(T)$

The <u>nullity</u> is $\dim(\ker(T)) =: \mathrm{nullity}(T)$

<u>THM</u> 2.5.8 (rank-nullity formula)

For linear maps $T: \mathbb{R}^n \to \mathbb{R}^m$,

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = n.$$

<u>proof</u>: If $\underset{m \times n}{A} = [T]$, then

$$\mathrm{rank}(T) = \mathrm{rank}(A) = \#\text{ pivotal columns}$$
$$\mathrm{nullity}(T) = \mathrm{nullity}(A) = \#\text{ non-pivotal columns}$$

TOTAL: # columns $= n$ ∎

A remarkable consequence...

<u>COR 2.5.10</u>: For linear maps $T: \mathbb{R}^n \to \mathbb{R}^n$, (must be same!)

$T(\bar{x}) = \bar{b}$ has a sol'n for every $\bar{b} \in \mathbb{R}^n$ $\iff$ $T(\bar{x}) = \bar{0}$ has only the trivial sol'n $\bar{x} = \bar{0}$

<u>proof</u>:

$$\begin{array}{ccc} \Updownarrow & & \Updownarrow \\ T \text{ surjective (onto } \mathbb{R}^n) & & T \text{ injective} \\ \Updownarrow & & \Updownarrow \\ \mathrm{rank}(T) = n & \xleftrightarrow{\text{THM 2.5.8}} & \mathrm{nullity}(T) = 0 \end{array}$$ ∎

RMK: Compare with FACT: If A, B are finite sets of <u>same cardinality</u> n, then $f: A \to B$ is injective $\iff$ surjective

<u>RMK</u>: See p. 201 for some equivalent ways to phrase $A: \mathbb{R}^n \to \mathbb{R}^m$ is surjective or is injective, when $m \neq n$.

Another curious fact relates A and A^T.

$\mathrm{img}(A) = \mathrm{span}\{\text{columns of } A\}$ has $\dim(\mathrm{img}(A)) = \mathrm{rank}(A)$

$= \{x_1\bar{a}_1 + \ldots + x_n\bar{a}_n : x_i \in \mathbb{R}\}$ = the <u>column space</u> of A

so $\mathrm{img}(A^T) = \mathrm{span}\{\text{rows of } A\}$ = the <u>row space</u> of A.

PROP 2.5.11 ("row rank = column rank")

For any matrix A, $\operatorname{rank}(A) = \operatorname{rank}(A^T)$,

i.e. column space and row space have same dimension.

proof: Invertible row operations don't ~~affect the row space~~ (Why?),

so if $A \underset{\text{row-reduce}}{\rightsquigarrow} \tilde{A}$ then $\operatorname{rank}(A^T) = \operatorname{rank}(\tilde{A}^T)$
$$\operatorname{rank}(\tilde{A}^T) = \dim(\text{row space of } \tilde{A})$$

But the rows containing a pivotal 1 in $\tilde{A}$ give a basis for the row space, since the other rows are zero, and restricting to the pivotal columns they look like $I_r = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

e.g.
$$\tilde{A} = \begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivotal columns

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

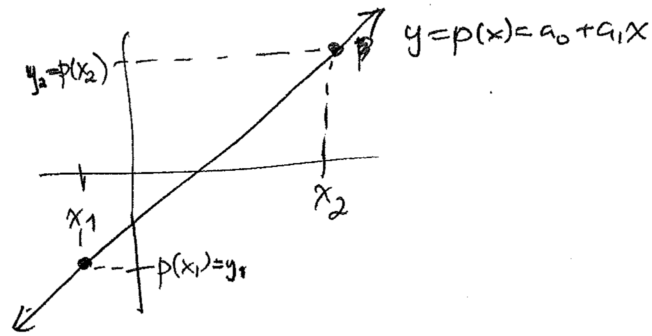
Hence $\dim(\operatorname{img} \tilde{A}^T) = \#\text{pivotal ones in } \tilde{A} = \operatorname{rank}(A) = \dim(\operatorname{img} A)$ ∎

An application of COR 2.5.10 lets us answer this question:

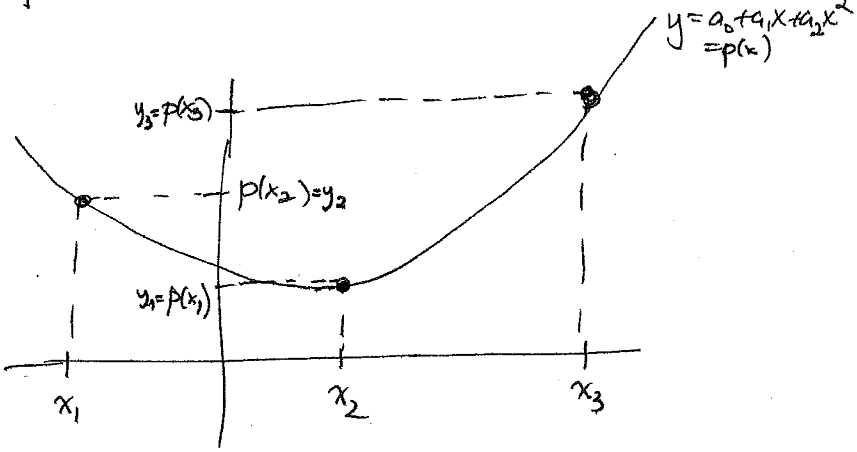
Can we find a polynomial of degree d, say $p(x) = a_0 + a_1 x + \dots + a_d x^d$, that has $d+1$ specified values $y_1 = p(x_1), \dots, y_{d+1} = p(x_{d+1})$ at $d+1$ specified $x_1, \dots, x_{d+1}$?

This is called interpolating the points $(x_1, \underset{y_1}{p(x_1)}), \dots, (x_k, \underset{y_{d+1}}{p(x_{d+1})})$ with the polynomial $p(x)$.

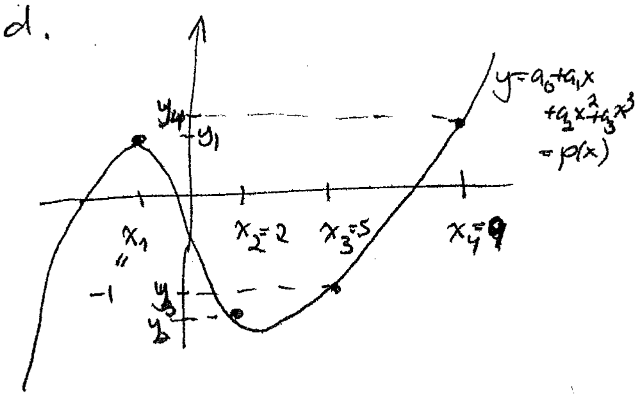
EXAMPLES: ① $d=1$ Yes:



② $d=2$: Seems plausible:



THM: Yes, it can always be done, for any d.



Proof: If we want to find a polynomial

$p(x)=a_0+a_1x+\ldots+a_dx^d$ that

interpolates $(x_1,y_1),\ldots,(x_{d+1},y_{d+1})$,

we'd like to solve for $a_0,a_1,\ldots,a_d$ as unknowns in a system of equations

$$y_1=p(x_1)=a_0+a_1x_1+a_2x_1^2+\ldots+a_dx_1^d$$

$$\vdots$$

$$y_{d+1}=p(x_{d+1})=a_0+a_1x_{d+1}+a_2x_{d+1}^2+\ldots+a_dx_{d+1}^d$$

$$\Updownarrow$$

$$\begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_{d+1}\end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^d \\ 1 & x_2 & x_2^2 & \cdots & x_2^d \\ \vdots & & & & \\ 1 & x_{d+1} & x_{d+1}^2 & \cdots & x_{d+1}^d \end{bmatrix}}_{\text{called a Vandermonde matrix for } x_1,x_2,\ldots,x_{d+1}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_{d+1}\end{bmatrix}$$

e.g. $d=3$

$x_1=-1$
$x_2=2$
$x_3=5$
$x_4=9$

$$\underbrace{\begin{bmatrix} 1 & -1 & (-1)^2 & (-1)^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 5 & 5^2 & 5^3 \\ 1 & 9 & 9^2 & 9^3 \end{bmatrix}}_{\text{Vandermonde matrix for } (-1,2,5,9)} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$x_1 \; x_2 \; x_3 \; x_4$

We'd like to know that this $(d+1)\times(d+1)$ matrix is invertible, since then a solution always exists.

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