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EXAMPLE: $\bar{f}: \mathbb{R}^2 - \{\binom{x}{y} : x=0\} \longrightarrow \mathbb{R}^3$

$$\binom{x}{y} \longmapsto \begin{pmatrix} \sin(x^2y) \\ e^{x+y} \\ \frac{1}{x} \end{pmatrix}$$

has directional derivative at $\binom{x}{y} = \binom{\pi/2}{0}$ in direction $\bar{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

given by
$$\left[J\bar{f}\binom{\pi/2}{0}\right]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} \pi & \pi^2/4 \\ e^{\pi/2} & e^{\pi/2} \\ \frac{4}{\pi^2} & 0 \end{bmatrix}}_{3\times 2, \text{ as we'd expect for } L:\mathbb{R}^2\to\mathbb{R}^3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \pi + \frac{\pi^2}{2} \\ 2e^{\pi/2} \\ 4/\pi^2 \end{bmatrix}$$

Worth noting, as in 1-variable case...

PROP 1.7.11: $\bar{f}: \underset{\mathbb{R}^n}{U} \to \mathbb{R}^m$ differentiable at $\bar{a} \in U$ $\Rightarrow$ $\bar{f}$ continuous at $\bar{a}$ also.

proof:
$$\bar{0} = \lim_{\bar{h}\to\bar{0}} \frac{(\bar{f}(\bar{a}+\bar{h}) - \bar{f}(\bar{a})) - L(\bar{h})}{|\bar{h}|}$$

$\lim_{\bar{h}\to\bar{0}} |\bar{h}| = 0$

$$\bar{0} = \lim_{\bar{h}\to\bar{0}} \left(\left(\bar{f}(\bar{a}+\bar{h}) - \bar{f}(\bar{a})\right) - L(\bar{h})\right)$$

$\lim_{\bar{h}\to\bar{0}} L(\bar{h}) = L(\bar{0}) = \bar{0}$ since L is linear so continuous

$$\bar{0} = \lim_{\bar{h}\to\bar{0}} \left(\bar{f}(\bar{a}+\bar{h}) - \bar{f}(\bar{a})\right)$$

i.e. $\lim_{\bar{h}\to\bar{0}} \bar{f}(\bar{a}+\bar{h}) = \bar{f}(\bar{a})$ ∎

Sometimes one can get away with avoiding a Jacobian calculation...

EXAMPLE:
(1.7.18) Consider the squaring map $S: \mathrm{Mat}(n,n) \longrightarrow \mathrm{Mat}(n,n) = \mathbb{R}^{n^2}$

$:= \left\{ n\times n \text{ matrices } A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \right\} = \mathbb{R}^{n^2}$

$$A \longmapsto S(A) = A^2$$

What is its derivative $\underset{\text{near } A}{DS(A)}: \mathrm{Mat}(n,n) \longrightarrow \mathrm{Mat}(n,n)$?

Near A, say for $H \in \mathrm{Mat}(n,n)$ with $\underbrace{|H|}_{\text{norm of } H}$ small,

we might <u>guess</u> from calculating

$$\begin{aligned} S(A+H) - S(A) &= (A+H)^2 - A^2 \\ &= \cancel{A^2} + AH + HA + H^2 - \cancel{A^2} \\ &= \underbrace{AH+HA}_{\substack{\text{a linear} \\ \text{function of } H \\ \text{i.e. } H \mapsto AH+HA}} + \underbrace{H^2}_{\substack{\text{shrinks faster than } |H| \\ \text{as } |H| \to 0}} \end{aligned}$$

that $DS(A) : \mathrm{Mat}(n,n) \to \mathrm{Mat}(n,n)$
$$H \longmapsto AH + HA$$

and this is <u>correct</u> since $H \mapsto AH+HA$ is <u>linear</u> in H (check this)

and $$\lim_{H \to \bar{0}} \frac{S(A+H) - S(A) - (AH+HA)}{|H|}$$

$$= \lim_{H \to \bar{0}} \frac{H^2}{|H|} = \bar{0}$$ since $$\left| \frac{H^2}{|H|} \right| \leq \frac{|H|^2}{|H|} = |H| \to 0 \text{ as } H \to \bar{0}.$$

A similar one...

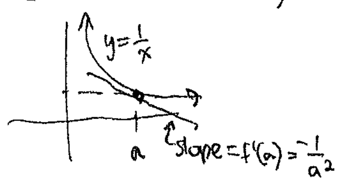
<u>PROP 1.7.19</u>. The function $f : \mathcal{U} = \left\{ \substack{\text{invertible} \\ n \times n \text{ matrices}} \right\} \longrightarrow \mathrm{Mat}(n,n)$

$\mathcal{U} \subset \mathrm{Mat}(n,n) = \mathbb{R}^{n^2}$ $\quad X \longmapsto f(X) = X^{-1}$

is differentiable at $X = A$ (if A^{-1} exists)

and $Df(A) : \mathrm{Mat}(n,n) \longrightarrow \mathrm{Mat}(n,n)$
$$H \longmapsto -A^{-1} H A^{-1}$$

(Compare with $n=1$ case, where $f(x) = \frac{1}{x}$ has



$f'(a) = \frac{-1}{a^2}$)

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<u>proof:</u> Near $X=A$, we have

$$f(A+H) = (A+H)^{-1}$$

imagine $H \to \bar{O}$ i.e. $|H|$ is small or $|H| \to 0$

$$= (A(I+A^{-1}H))^{-1}$$

$$= (I+A^{-1}H)^{-1}A^{-1}$$

$$= (I-B)^{-1}A^{-1} \quad \text{where } B := -A^{-1}H$$

$$= (I+B+B^2+B^3+\dots)A^{-1}$$

so $f(A+H) - f(A) = (I+B+B^2+B^3+\dots)A^{-1} - A^{-1}$

$$= (B+B^2+B^3+\dots)A^{-1}$$

$$= -A^{-1}HA^{-1} + (B^2+B^3+\dots)A^{-1}$$

and $f(A+H) - f(A) - (-A^{-1}HA^{-1}) = (B^2+B^3+\dots)A^{-1}$

Then
$$\frac{|f(A+H)-f(A)-(-A^{-1}HA^{-1})|}{|H|} = \frac{|(B^2+B^3+\dots)A^{-1}|}{|H|}$$

$$\leq \frac{(|B|^2+|B|^3+|B|^4+\dots)|A^{-1}|}{|H|}$$

$$= \frac{|B|^2\,|A^{-1}|}{|H|(1-|B|)} \leq \frac{|A^{-1}|\,|H|^2\,|A^{-1}|}{|H|(1-|A^{-1}||H|)}$$

$$= \frac{|A^{-1}|^2\,|H|}{1-|A^{-1}||H|} \longrightarrow 0 \text{ as } |H| \to 0.$$

Therefore $\displaystyle\lim_{H\to\bar{O}} \frac{f(A+H)-f(A)-(-A^{-1}HA^{-1})}{|H|} = \bar{O}$, as desired. ∎

§1.8 Derivative rules

THM 1.8.1:

1. $\bar{f}: \underset{\mathbb{R}^n}{\overset{\text{open}}{U \cap}} \to \mathbb{R}^m$ constant $f(\bar{x}) = \bar{c}$ $\Rightarrow$ $\bar{f}$ diff'ble with $D\bar{f} = \bar{\mathbf{0}}$

2. —— " —————— linear $f(\bar{x}) = A\bar{x}$ $\Rightarrow$ $\bar{f}$ diff'ble with $D\bar{f} = \bar{f}$
i.e. $J\bar{f}(\bar{a}) = A$ $\forall \bar{a} \in \mathbb{R}^m$

3. $\bar{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ diff'ble at $\bar{x} = \bar{a}$ $\Leftrightarrow$ each f_j is

4. $D(\bar{f} + \bar{g})(\bar{a}) = D\bar{f}(\bar{a}) + D\bar{g}(\bar{a})$

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5. $f: U \to \mathbb{R}^1$
$\bar{g}: U \to \mathbb{R}^m$ both diff'ble at $\bar{x} = \bar{a}$
$\Rightarrow f \cdot \bar{g}: U \to \mathbb{R}^m$ is also, with $[D(f \cdot \bar{g})(\bar{a})]\bar{v}$
$$f\cdot\bar{g} = \begin{pmatrix} fg_1 \\ \vdots \\ fg_m \end{pmatrix}$$
$$= \underbrace{f(\bar{a})}_{\in \mathbb{R}} \cdot \underbrace{[D\bar{g}(\bar{a})]\bar{v}}_{\in \mathbb{R}^m} + \underbrace{[Df(\bar{a})]\bar{v}}_{\in \mathbb{R}} \cdot \underbrace{\bar{g}(\bar{a})}_{\in \mathbb{R}^m}$$

6. If $f(\bar{a}) \neq 0$ in 5, then
$$\left[D\left(\frac{\bar{g}}{f}\right)(\bar{a})\right]\bar{v} \left(= \frac{D\bar{g}(\bar{a})\bar{v}}{f(\bar{a})} - \frac{[Df(\bar{a})]\bar{v} \cdot g(\bar{a})}{f(\bar{a})^2}\right)$$
$$= \frac{f(\bar{a}) \cdot D\bar{g}(\bar{a})\bar{v} - [Df(\bar{a})]\bar{v} \cdot g(\bar{a})}{f(\bar{a})^2}$$

7. $\bar{f}, \bar{g}: U \to \mathbb{R}^m$ both diff'ble at $\bar{x} = \bar{a}$
$\Rightarrow \bar{f} \bullet \bar{g}: U \to \mathbb{R}^1$ is also, with $[D(\bar{f} \bullet \bar{g})(\bar{a})]\bar{v} =$
$$\underbrace{\underbrace{[D\bar{f}(\bar{a})]\bar{v}}_{\in \mathbb{R}^m} \bullet \underbrace{\bar{g}(\bar{a})}_{\in \mathbb{R}^m}}_{\in \mathbb{R}} + \underbrace{\underbrace{\bar{f}(\bar{a})}_{\in \mathbb{R}^m} \cdot \underbrace{[D\bar{g}(\bar{a})]\bar{v}}_{\in \mathbb{R}^m}}_{\in \mathbb{R}}$$

Note that 1, 2, 3, 4 immediately imply ...

COR 1.8.2: Polynomials $f: \mathbb{R}^n \to \mathbb{R}$ are diff'ble everywhere,
and rational functions $\frac{f}{g}: ~~\mathbb{R}~~ \to \mathbb{R}$ are diff'ble on $U = \{\bar{x} \in \mathbb{R}^n : g(\bar{x}) \neq 0\}$.