(64)

Chapter 2 Solving equations (i.e. back to more linear algebra!)

How would you solve the system

$$\begin{aligned} x+y+z+w &= 1 \\ 2x+2y+6z+8w &= 0 \\ 4x+4y+8z+10w &= 1 \end{aligned} \quad ?$$

In matrix form, it's $A\bar{x} = \bar{b}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 \\ 4 & 4 & 8 & 10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\bar{b}}$$

Adding / scaling equations is easiest to perform/mimic on the augmented matrix $[A|\bar{b}] =$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 4 & 4 & 8 & 10 & 1 \end{array}\right]$$

by performing (invertible) row operations: (DEFN 2.1.1)

1. scale a row by some $c \in \mathbb{R} - \{0\}$
2. add a multiple of a row to another
3. exchange rows

One systematic way to do this brings $[A|\bar{b}]$ into what is called (row-reduced) echelon form (DEFN 2.1.4):

schematically:

$$\begin{bmatrix} 0 & 0 \cdots 0 & 1 & * & * & * & * & 0 & * & * & * & 0 & * & * & * & * & * \\ 0 & 0 \cdots 0 & \cdots & & & & 0 & 1 & * & * & * & 0 & * & * & * & * & * \\ 0 & 0 \cdots 0 & \cdots & & & & 0 & \cdots & & & 0 & 1 & * & * & * & * & * \\ 0 & 0 \cdots 0 & \cdots & & & & 0 & \cdots & & & 0 & \cdots & & & & & 0 \\ 0 & 0 \cdots 0 & \cdots & & & & 0 & \cdots & & & 0 & \cdots & & & & & 0 \end{bmatrix}$$

10/24/2016 >

rules:
1. every row has leftmost nonzero entry scaled to 1 (called a pivotal 1)
2. pivot 1's go left-to-right as row index increases
3. pivot 1's are the only nonzero entry in their columns
4. zero rows can only go at the bottom

(65) How to do this: Gaussian elimination. Starting at the topmost nonzero row, do this...

EXAMPLE:

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 4 & 4 & 8 & 10 & 1 \end{array}\right] \xrightarrow[\text{in same column}]{\text{use pivot 1 to clear entries below}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 6 & -2 \\ 0 & 0 & 4 & 6 & -3 \end{array}\right] \xrightarrow[\text{next pivot 1}]{\text{scale row 2 to create}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 4 & 6 & -3 \end{array}\right]$$

$\downarrow$ use next pivot 1 to clear below/above

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] \xrightarrow[\text{next pivot 1}]{\text{Scale row 3 to create}} [A'|b'] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(might seem like overkill!)

in echelon form

$$= \left(\left[\begin{array}{cccc|c} 1 & * & 0 & * & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]\right)$$

$A'\bar{x} = \bar{b}'$ corresponds to the equivalent, and easily solved system

$$\left.\begin{aligned} x + y + \quad w &= 3/2 \\ z + 3/2\, w &= -1/2 \\ \boxed{0 = 1} \end{aligned}\right\}$$ no solutions!

EXAMPLE

If we had changed the right side slightly to

$A\bar{x} = \bar{b}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 6 & 8 & 0 \\ 4 & 4 & 8 & 10 & 2 \end{array}\right] \longmapsto \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 6 & -2 \\ 0 & 0 & 4 & 6 & -2 \end{array}\right] \xrightarrow{\text{similar steps as above}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [A'|b']$$ in echelon form

$$= \left(\left[\begin{array}{cccc|c} 1 & 1 & 0 & -\frac{1}{2} & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\right)$$

then $[A'|b']$ also corresponds to an easily solved (equivalent) system

$$\left.\begin{aligned} x + y + -\tfrac{1}{2} w &= \tfrac{3}{2} \\ z + \tfrac{3}{2} w &= -\tfrac{1}{2} \\ 0 &= 0 \quad \text{ignore} \end{aligned}\right\}$$

infinitely many solutions, parametrizable by picking the non-pivotal variables y, w arbitrarily, and then pivotal variables x, z are forced:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -y + \frac{1}{2}w + \frac{3}{2} \\ y \\ -\frac{3}{2}w - \frac{1}{2} \\ w \end{bmatrix}$$

$$= y \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} +\frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Let's summarize this...

THM: (2.1.2, 2.1.7 part 1, 2.2.1) Given a system $A\bar{x} = \bar{b}$ with $m \times n$ matrix A and $\bar{b} \in \mathbb{R}^m$, one can form the augmented matrix $[A|\bar{b}]$ and bring it into echelon form $[\tilde{A}|\tilde{b}]$ without changing the solutions: $A\bar{x} = \bar{b} \Leftrightarrow \tilde{A}\bar{x} = \tilde{b}$ $\forall \bar{x} \in \mathbb{R}^n$.

pivotal column locations determine

# of solutions	
0	Furthermore, 1. there are no solutions $\bar{x}$ $\Leftrightarrow$ $[\tilde{A}\|\tilde{b}]$ has $\tilde{b}$ as a pivotal column
1	2. if $\tilde{b}$ is not pivotal and every column of $\tilde{A}$ is pivotal, there is a unique solution $\bar{x} \in \mathbb{R}^n$
∞	3. if $\tilde{b}$ is not pivotal, and at least one column of $\tilde{A}$ is not pivotal, there are infinitely many solutions $\bar{x} \in \mathbb{R}^n$, parametrized by arbitrarily specifying the variables x_j from non-pivotal columns j of $\tilde{A}$ (and x_j from pivotal columns are then determined)

Example for 1.:

$$\text{e.g. } [\tilde{A}|\tilde{b}] = \left[\begin{array}{ccccccc|c} 1 & ** & 0 & ** & 0 & ** & ** & 0 \\ & & 1 & ** & 0 & ** & ** & 0 \\ & & & & 1 & ** & ** & 0 \\ 0 & 0 & \cdots & & & & 0 & 1 \end{array}\right]$$

Example for 2.:

$$\text{e.g. } [\tilde{A}|\tilde{b}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example for 3.:

$$\text{e.g. } [\tilde{A}|\tilde{b}] = \begin{array}{c} \begin{array}{ccccc} x_1 & \underline{x_2} & x_3 & \underline{x_4} & \underline{x_5} \end{array} \\ \left[\begin{array}{ccccc|c} 1 & * & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

e.g. $x_3 = -*x_4 - *x_5 + *$

$x_1 = -*x_2 - *x_4 - *x_5 + *$

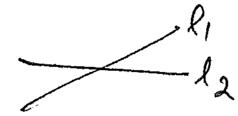
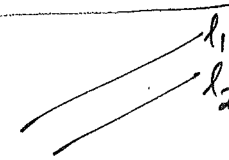
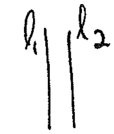
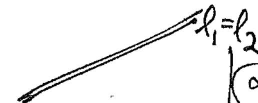
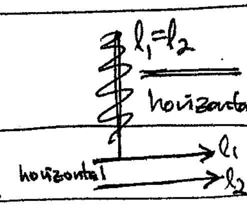
proof: Not much to say, really! ∎

(67)

EXAMPLES of the geometry

for $\boxed{\begin{matrix} n=2 \\ m=2 \end{matrix}}$ $\quad A\bar{x}=\bar{b}$ $\quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\rightsquigarrow [A|\bar{b}] \rightsquigarrow \underbrace{[\tilde{A}|\tilde{b}]}$ echelon form

Solving is intersecting lines in $\mathbb{R}^2$
$$\begin{cases} l_1 = \{a_{11}x + a_{12}y = b_1\} \\ l_2 = \{a_{21}x + a_{22}y = b_2\} \end{cases}$$

$[\tilde{A}\|\tilde{b}]$ =		# of sol'ns
$\left[\begin{array}{cc\|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right]$		1
$\left[\begin{array}{cc\|c} 1 & * & 0 \\ 0 & 0 & 1 \end{array}\right]$		0
$\left[\begin{array}{cc\|c} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$		0
$\left[\begin{array}{cc\|c} 1 & * & * \\ 0 & 0 & 0 \end{array}\right]$		∞
$\left[\begin{array}{cc\|c} 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right]$		∞
$\left[\begin{array}{cc\|c} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$		0

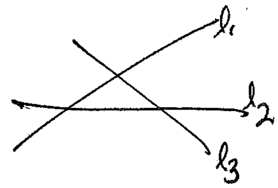
Q: What would $[\tilde{A}|\tilde{b}] = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ mean?

What about $[\tilde{A}|\tilde{b}] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$?

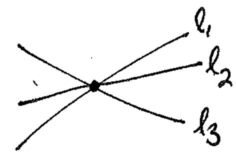
10/26/2016 >

(some) EXAMPLES for $n=2$, $m=3$

$[\tilde{A}|\tilde{b}] =$

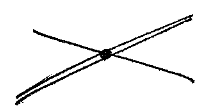
$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$ 

0

$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right]$ 

1

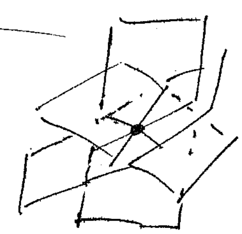
or

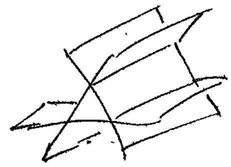


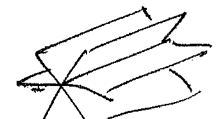
(some) EXAMPLES for $n=3$, $m=3$

Solving is intersecting planes in $\mathbb{R}^3$

$[\tilde{A}|\tilde{b}] =$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$  1

$\left[\begin{array}{ccc|c} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$  0

$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$  ∞