(41)

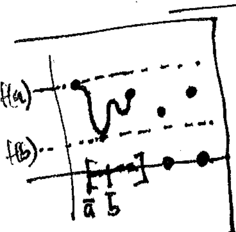
REMARK: This proof is highly non-constructive: even if we specify a concrete sequence $(x_m)_{m=1}^{\infty} \subset C = [-1,+1]$ (in $\mathbb{R}^1$), where $x_m = \sin(10^m)$ (see EXAMPLE 1.6.4),

we have no idea what ~~the~~ a sequence of subboxes $[-1,+1] \supset B_1 \supset B_2 \supset \cdots$ will look like, and how to describe explicitly a convergent subsequence!

10/3/2016

THM 1.6.9 (Extreme Value Thm): For $f: C \longrightarrow \mathbb{R}$, $C \subset \mathbb{R}^n$, continuous with $C$ compact,

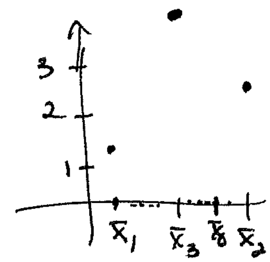
$\exists\, \bar{a}, \bar{b} \in C$ with $f(\bar{a}) \geq f(\bar{x})\ \forall \bar{x} \in C$
$f(\bar{b}) \leq f(\bar{x})$




(i.e. $f$ achieves a minimum $f(\bar{b})$, maximum $f(\bar{a})$ value on $C$)

proof: let's do max; then applying it to $-f(\bar{x})$ gives the min.

First show the values $f(\bar{x})$ are bounded. If not, then $\forall N = 1, 2, \ldots\ \exists\, \bar{x}_N \in C$ with $f(\bar{x}_N) > N$.




Use Bolzano-~~Weierstrass to~~ find a convergent subsequence $(\bar{x}_{N(j)})_{j=1}^{\infty} \subset C$ with $\lim_{j\to\infty} \bar{x}_{N(j)} = \bar{x}_0 \in C$

Continuity implies $\lim_{j\to\infty} f(\bar{x}_{N(j)}) = f(\bar{x}_0)$.

This leads to a contradiction: for $j > f(\bar{x}_0)+1$, one has $f(\bar{x}_{N(j)}) > N(j) \geq j \geq f(\bar{x}_0)+1$,

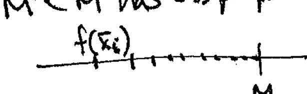
but if we pick $\epsilon$ with $1 > \epsilon > 0$ then $\exists J$ such that $|f(\bar{x}_{N(j)}) - f(\bar{x}_0)| < \epsilon < 1\ \forall j > J$

$\Rightarrow f(\bar{x}_{N(j)}) \leq f(\bar{x}_0) + \epsilon \leq f(\bar{x}_0) + 1$.

When $j > \max\{f(\bar{x}_0)+1, J\}$, these are in conflict.

Once the values of $f(\bar{x})$ are bounded, we know they have a supremum $M$ (sup) in $\mathbb{R}$

least upperbound, i.e. $M \geq f(\bar{x})\ \forall \bar{x} \in C$ but no $M' < M$ has this property.




But then $\exists\, \bar{x}_1, \bar{x}_2, \ldots \in C$ with $\lim_{i\to\infty} f(\bar{x}_i) = M$ (possibly $\bar{x}_1 = \bar{x}_2 = \ldots \in C$ and $f(\bar{x}_1) = M$),

so $\exists$ a convergent subsequence $(\bar{x}_{i(j)})_{j=1}^{\infty}$ with $\lim_{j\to\infty} \bar{x}_{i(j)} = \bar{a} \in C$

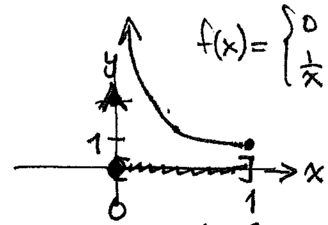
and continuity gives $M = \lim_{j\to\infty} f(\bar{x}_{i(j)}) = f(\bar{a})$. ∎

NON/COUNTER-EXAMPLES:

① Why did we need $f: C \to \mathbb{R}$ continuous?

(EXAMPLE 1.6.10)

$f(x) = \begin{cases} 0 & x=0 \\ \frac{1}{x} & x \in (0,1] \end{cases}$

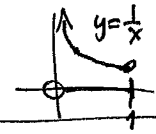



$C=[0,1]$ compact

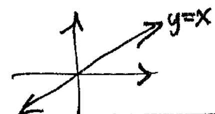
② Why did we need $C$ compact?

EXER 1.6.2 shows every non-compact $C$ gives rise to continuous $f: C \to \mathbb{R}$ which is unbounded!

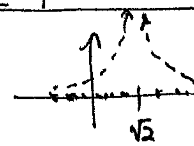
e.g. $C=(0,1] \xrightarrow{f} \mathbb{R}$, $x \longmapsto \frac{1}{x}$



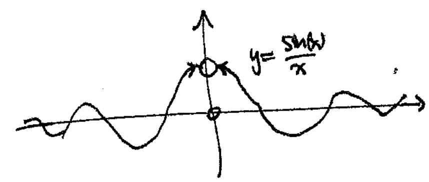

e.g. $C=\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$, $x \longmapsto x$




e.g. $C=\mathbb{Q} \xrightarrow{f} \mathbb{R}$, $x \longmapsto \frac{1}{x-\sqrt{2}}$




e.g. $C=\mathbb{R}\setminus\{0\} \xrightarrow{f} \mathbb{R}$, $x \longmapsto \frac{\sin(x)}{x}$

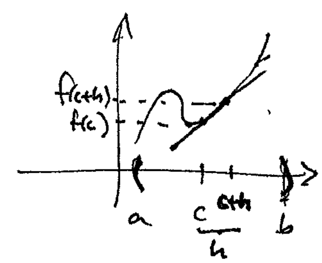



Let's now use this to deduce the Mean Value Thm.

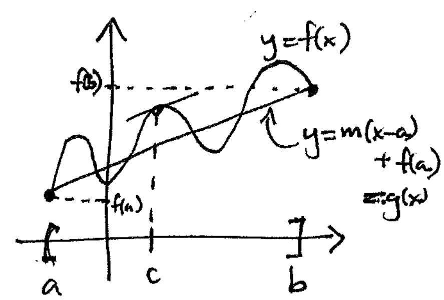
First recall...

DEFIN: For $f:(a,b) \to \mathbb{R}$, one says $f$ is differentiable at $c \in (a,b)$ if $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} =: f'(c)$ exists

Say $f$ is differentiable on $(a,b)$ if it's differentiable at all $c \in (a,b)$




THM: If $f:[a,b] \to \mathbb{R}$ is continuous, and differentiable on $(a,b)$, then $\exists\, c \in (a,b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$




proof: Consider the straight-line function between $\binom{a}{f(a)}, \binom{b}{f(a)}$

$g(x) = f(a) + m(x-a)$ where $m := \frac{f(b)-f(a)}{b-a}$

and its difference from $f(x)$:

$h(x) = f(x) - g(x) = f(x) - m(x-a)$

i.e. $h:[a,b] \to \mathbb{R}$, continuous, differentiable on $(a,b)$.
(Why?) (Why?)

Note $h(a)=h(b)=0$, so either $h(x)=0 \ \forall x\in[a,b]$

(and we're done since $f(x)=g(x)$ is linear and <u>any</u> $c\in(a,b)$ works,)

<u>or</u> $h(x)$ achieves a positive maximum or negative minimum $h(c)$ at some $c\in(a,b)$ (since $[a,b]$ is <u>compact</u>, $h$ <u>continuous</u>).

Assume $f(c)$ is a positive maximum (else consider $-f(x)$ instead).

We claim $h'(c)=0$ (PROP 1.6.11) since

$$h'(c)=\underbrace{\lim_{\epsilon\to 0^+}\frac{\overbrace{h(c+\epsilon)-h(c)}^{\text{negative}}}{\underbrace{\epsilon}_{\text{positive}}}}_{\text{always }\le 0}=\underbrace{\lim_{h\to 0^-}\frac{h(c+\epsilon)-h(c)}{\underbrace{\epsilon}_{\text{negative}}}}_{\text{always }\ge 0}$$

$$\Rightarrow f'(c)\le 0,\ f'(c)\ge 0$$

But $0=h'(c)=f'(c)-m$, i.e. $f'(c)=m=\frac{f(b)-f(a)}{b-a}$ ■

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<u>REMARK</u>: We'll deduce a multi<u>variable</u> MVT from this single-variable one in §1.9.

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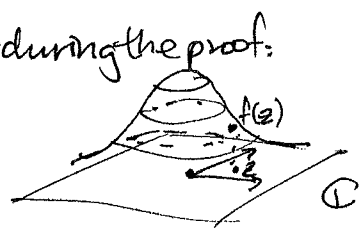
Finally...

THM 1.6.13 (Fundamental Thm of Algebra)

A <u>polynomial</u> $p(z)=z^k+a_{k-1}z^{k-1}+\dots+a_1z^1+a_0$ having $a_i\in\mathbb{C}$ and $k\ge 1$ always has <u>at least one root</u> $z_0\in\mathbb{C}$ with $p(z_0)=0$.

NON-EXAMPLES to ponder during the proof:

① $f(z)=\dfrac{1}{1+|z|^2}$ (not polynomial)




② $f(z)=e^z$ (not polynomial)

③ $f(x)=x^2+1$ has <u>no roots in $\mathbb{R}$</u>, (but has $z_0=\pm i$ as roots in $\mathbb{C}$)