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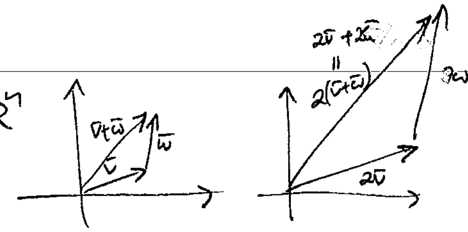
EXAMPLES:

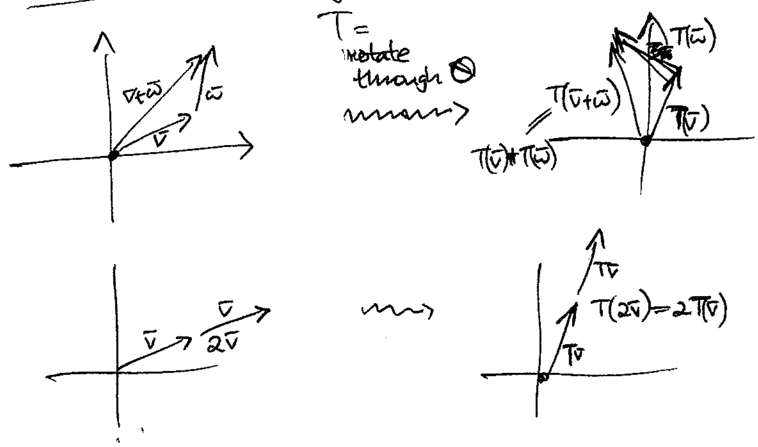
① Scalings $T(\bar{x}) = c\cdot\bar{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_n \end{bmatrix}$, $T: \mathbb{R}^n \to \mathbb{R}^n$

$$T(\bar{v}+\bar{w}) = c(\bar{v}+\bar{w}) = c\bar{v} + c\bar{w}$$

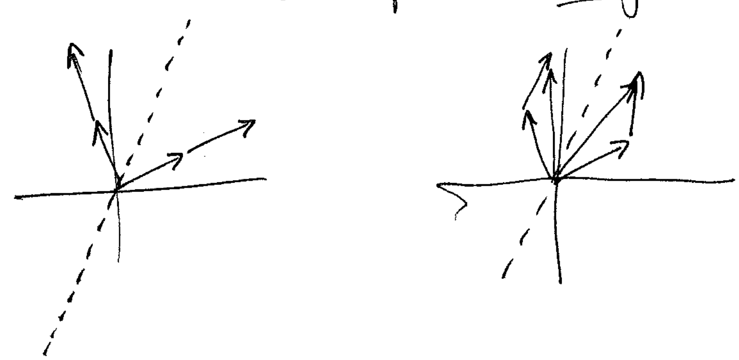


called rigid motions that fix $\bar{0}$

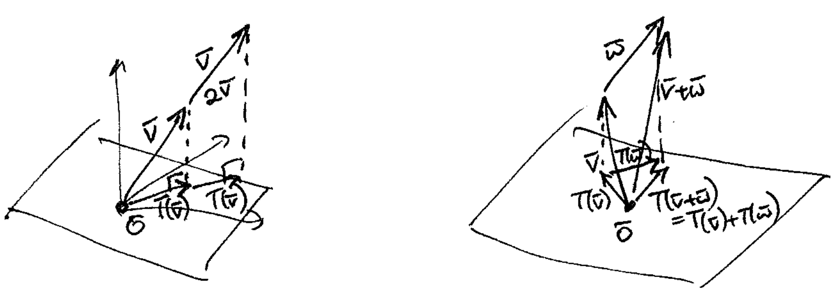
② Rotations through some angle θ about $\bar{0}$ in $\mathbb{R}^2$



③ Reflections ~~in~~ in a line through $\bar{0}$ in $\mathbb{R}^2$
or a plane through $\bar{0}$ in $\mathbb{R}^3$



④ Projection orthogonally (perpendicularly) onto a line through $\bar{0}$ in $\mathbb{R}^2$ or a plane through $\bar{0}$ in $\mathbb{R}^3$



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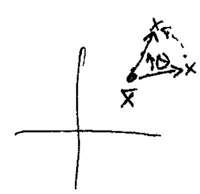
NON-EXAMPLES:



⑤ Rotations not about $\bar{0}$, about some $\bar{x} \neq \bar{0}$, in $\mathbb{R}^2$

⑥ Reflections in lines not through $\bar{0}$ in $\mathbb{R}^2$

⑦ Projections orthogonally onto lines not through $\bar{0}$

⑩ $f(x) = mx + b$ with $b \neq 0$

⑧ $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, since $(x+y)^2 \neq x^2 + y^2$

⑨ — " — $f(x) = \sin(x)$, since $\sin(x+y) \neq \sin(x) + \sin(y)$

KEY POINT:

Matrix multiplication will give us all linear transformations (!),
(multiplying a matrix by a vector) once we learn how to do it...

DEF'N: An $m \times n$ matrix A is an array of real numbers

m rows, n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: [a_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ is a 2×3 matrix

$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ (so $a_{21} = 6$, for example)

DEF'N: Its transpose A^T is an $n \times m$ matrix obtained by "flipping" A across the diagonal: $A = \begin{bmatrix} & A & \end{bmatrix}$

$A^T = \begin{bmatrix} & A^T & \end{bmatrix}$

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$

that is, $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$ (or A^T has (i,j)-entry $a^T_{ij} = a_{ji}$)

A column vector $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is an example of an $n \times 1$ matrix

e.g. $\bar{v} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

A row vector $\bar{v}^T = [v_1 \ v_2 \cdots v_n]$ is an example of $1 \times n$ matrix

e.g. $\bar{v}^T = [2 \ 0 \ 3]$

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One can think of an $m \times n$ matrix A as ~~[scribbled out]~~

– a sequence of n column vectors $\bar{c}_i \in \mathbb{R}^m$ $\quad A = \begin{bmatrix} | & | & & | \\ \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \\ | & | & & | \end{bmatrix}$

e.g. $A = m{=}2\left\{\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}}_{n=3}\right.$ has $\bar{c}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \bar{c}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \bar{c}_3 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

– a stack of m row vectors $\bar{r}_i^T \in \mathbb{R}^n$

$$A = \begin{bmatrix} — & \bar{r}_1^T & — \\ — & \bar{r}_2^T & — \\ & \vdots & \\ — & \bar{r}_m^T & — \end{bmatrix}$$

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ has $\bar{r}_1^T = [1\ 2\ 3]$, $\bar{r}_2^T = [6\ 5\ 4]$

and one can form the dot product of 2 vectors of the same length

$\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \bar{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ have $\bar{v} \bullet \bar{w} := v_1w_1 + v_2w_2 + \ldots + v_nw_n \in \mathbb{R}$

dot product

e.g. $\bar{v} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}, \bar{w} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ have $\bar{v} \cdot \bar{w} = 6 \cdot 2 + 0 \cdot 5 + 4 \cdot 3 = 24$

which is a special case of matrix multiplication

for a row vector $\bar{v}^T$ (a $1 \times n$ matrix) times a column vector $\bar{w}$ (an $n \times 1$ matrix) of the same length n:

Get used to the different, but equivalent ways to say it!

$$\boxed{[\bar{v}^T][\bar{w}] = [v_1\ v_2 \cdots v_n]\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} := \bar{v} \bullet \bar{w} = v_1w_1 + \ldots + v_nw_n = \sum_{i=1}^{n} v_i w_i}$$

e.g. $[6\ 5\ 4]\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 6 \cdot 2 + 5 \cdot 0 + 4 \cdot 3 = 24$

Then for an $m \times n$ matrix A and $n \times p$ matrix B,

their matrix product $\underset{m \times p}{C} = \underset{m \times n}{A}\underset{n \times p}{B}$ is an $m \times p$ matrix with $\boxed{c_{ij} := \sum_{k=1}^{n} a_{ik} b_{kj}}$

$= \underset{\text{row } i \text{ of } A}{\bar{r}_i} \bullet \underset{\text{column } j \text{ of } B}{\bar{c}_j}$

$= [\bar{r}_i^T][\bar{c}_j]$

i.e. $AB = m\left\{\underbrace{\begin{bmatrix} — & \bar{r}_1^T & — \\ — & \bar{r}_2^T & — \\ & \vdots & \\ — & \bar{r}_m^T & — \end{bmatrix}}_{n}\right. n\left\{\underbrace{\begin{bmatrix} | & | & & | \\ \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_p \\ | & | & & | \end{bmatrix}}_{p}\right. = m\left\{\underbrace{\begin{bmatrix} \bar{r}_1^T\bar{c}_1 & \bar{r}_1^T\bar{c}_2 & \cdots & \bar{r}_1^T\bar{c}_p \\ \bar{r}_2^T\bar{c}_1 & \bar{r}_2^T\bar{c}_2 & \cdots & \bar{r}_2^T\bar{c}_p \\ \vdots & & & \\ \bar{r}_m^T\bar{c}_1 & & & \bar{r}_m^T\bar{c}_p \end{bmatrix}}_{p}\right. = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & & & \\ \vdots & & & \\ c_{m1} & & \cdots & c_{mp} \end{bmatrix} = C$

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<u>EXAMPLES</u> ①
$$\begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 2\cdot 0 + 3\cdot 3 \\ 6\cdot 2 + 5\cdot 0 + 4\cdot 3 \end{bmatrix} = \begin{bmatrix} 11 \\ 24 \end{bmatrix}$$
$$\underset{2\times 3}{A} \quad \underset{3\times 1}{B} = \underset{2\times 1}{C}$$

$$\begin{bmatrix} - \bar{r}_1^T - \\ - \bar{r}_2^T - \end{bmatrix}\begin{bmatrix} | \\ \bar{c}_1 \\ | \end{bmatrix} = \begin{bmatrix} \bar{r}_1^T \bar{c}_1 \\ \bar{r}_2^T \bar{c}_2 \end{bmatrix} = \begin{bmatrix} \bar{r}_1 \cdot \bar{c}_1 \\ \bar{r}_2 \cdot \bar{c}_2 \end{bmatrix}$$

②
$$\begin{array}{c} \\ \bar{r}_2^T \rightarrow \\ \\ \end{array}\begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}\begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} & & & \bar{c}_3 \downarrow \\ \end{array}$$
$$\begin{bmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 7 & 26 \\ 3 & 0 & 7 & 24 \\ 1 & 0 & 7 & 22 \end{bmatrix}$$
$$\underset{3\times 2}{A} \quad \underset{2\times 4}{B} = \underset{3\times 4}{C}$$

$\bar{r}_2^T \bar{c}_3 =$

$\bar{r}_2 \cdot \bar{c}_3 =$

$\begin{bmatrix} 2 \\ 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} =$

$2\cdot 2 + 5\cdot 4 = 24$

Matrix multiplication has lots of good properties that are not hard to verify, such as its interactions with

<u>scaling</u> matrices ~~componentwise~~ (entrywise) $\quad cA := \begin{bmatrix} ca_{11} & ca_{12} & \cdots \\ ca_{21} & \ddots & \\ \vdots & & \end{bmatrix}$

<u>adding</u> matrices (entrywise) $\quad A+B := \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \\ a_{21}+b_{21} & \ddots & \\ \vdots & & \end{bmatrix}$

like
$$\left.\begin{aligned} (cA)\cdot B &= A(cB) = cAB \\ (A_1+A_2)B &= A_1B + A_2B \\ A(B_1+B_2) &= AB_1 + AB_2 \end{aligned}\right\} \text{all pretty easy}$$

associativity: (PROP 1.2.9 in book) $\quad (AB)C = A(BC)$ for $\overset{n\times m}{A}, \overset{m\times p}{B}, \overset{p\times q}{C}$

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<u>proof:</u> Let's calculate (i,j) entry on both sides for $i=1,\dots,n$, $j=1,\dots,q$:

$$\big[(AB)C\big]_{i,j} = \sum_{l=1}^{p} (AB)_{il} c_{lj} = \sum_{l=1}^{p}\Big(\sum_{k=1}^{m} a_{ik} b_{kl}\Big) c_{lj} = \sum_{l=1}^{p}\sum_{k=1}^{m} a_{ik} b_{kl} c_{lj}$$

$$\big[A(BC)\big]_{i,j} = \sum_{k=1}^{m} a_{ik} (BC)_{kj} = \sum_{k=1}^{m} a_{ik}\Big(\sum_{l=1}^{p} b_{kp} c_{pj}\Big) = \sum_{k=1}^{m}\sum_{l=1}^{p} a_{ik} b_{kl} c_{lj}$$

same!