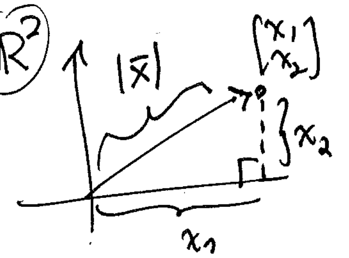
(17)

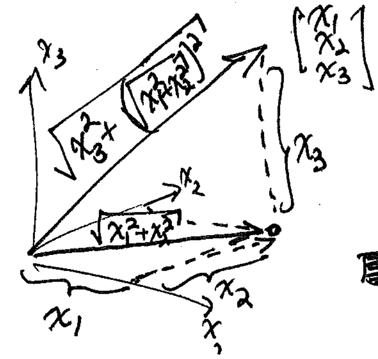
COR: In $\mathbb{R}^1$ and $\mathbb{R}^2$, $\mathbb{R}^3$, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ has length squared $= \bar{x} \bullet \bar{x} =: |\bar{x}|^2$
$= \sum_{i=1}^{n} x_i^2$

(and as definition in $\mathbb{R}^n$)

proof: $\mathbb{R}^2$



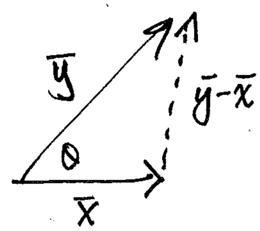
$\mathbb{R}^3$



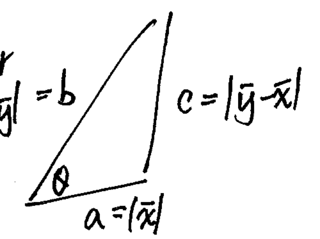
Thus $|\bar{x}| :=$ length of $\bar{x} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\bar{x} \cdot \bar{x}}$

COR (PROP 1.43) More generally, ~~for~~ any $\bar{x}, \bar{y} \in \mathbb{R}^n$, with $n = 1, 2, 3$, their dot product $\bar{x} \bullet \bar{y} = |\bar{x}| \cdot |\bar{y}| \cos\theta$ $\left(:= \sum_{i=1}^{n} x_i y_i\right)$

if θ is the angle between them:



proof: Consider



and law of cosines:

$c^2 = a^2 + b^2 - 2ab\cos\theta$

$\Downarrow$

$\underbrace{(\bar{y}-\bar{x}) \bullet (\bar{y}-\bar{x})}$ $= \cancel{\bar{x} \cdot \bar{x}} + \cancel{\bar{y} \cdot \bar{y}} - 2|\bar{x}||\bar{y}|\cos\theta$

distributivity of dot product (= matrix multiplication)

$= \cancel{\bar{y} \cdot \bar{y}} - \bar{x} \cdot \bar{y} - \bar{y} \cdot \bar{x} + \cancel{\bar{x} \cdot \bar{x}}$

$\Rightarrow \quad -2\bar{x} \cdot \bar{y} = -2|\bar{x}||\bar{y}|\cos\theta$

$\bar{x} \cdot \bar{y} = |\bar{x}||\bar{y}|\cos\theta$ ■

9/19/2016 >

So $\bar{x} \cdot \bar{y} =$ (in $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$)

- $0 \Leftrightarrow \bar{x} \perp \bar{y}$ perpendicular/orthogonal ($\cos\theta = 0$, $\theta = \pi/2$)
- $>0 \Leftrightarrow \bar{x}, \bar{y}$ acute (so $\cos\theta > 0$)
- $<0 \Leftrightarrow \bar{x}, \bar{y}$ obtuse (so $\cos\theta < 0$)

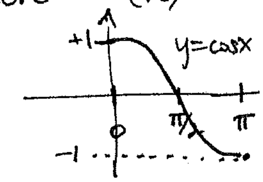
(18) In higher dimensions, this is the way we will define the angle θ between $\bar{x}, \bar{y}$:

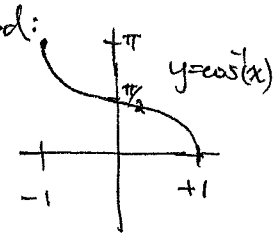
$$\text{i.e.}\quad \theta := \cos^{-1}\left(\frac{\bar{x}\cdot\bar{y}}{|\bar{x}||\bar{y}|}\right) \in [0,\pi]$$

However, we can't make sense of this definition until we know that

$$-1 \le \frac{\bar{x}\cdot\bar{y}}{|\bar{x}||\bar{y}|} \le +1$$, where $\cos^{-1}(x)$ is defined:

$$\text{i.e.}\quad \frac{|\bar{x}\cdot\bar{y}|}{|\bar{x}||\bar{y}|} \le +1$$





That is, we need to know...

THM 1.4.5 (Cauchy-Schwarz-Bunyakovsky) inequality

$$|\bar{x}\cdot\bar{y}| \le |\bar{x}|\,|\bar{y}| \quad \forall\, \bar{x},\bar{y}\in\mathbb{R}^n$$

absolute value of a real number — length of a vector!

with equality if and only if $\bar{x}, \bar{y}$ are scalar multiples of each other.

proof: If $\bar{y}=\bar{0}$ then it's trivially true (with equality): $|\bar{x}\cdot\bar{0}| \overset{?}{\le} |\bar{x}||\bar{0}| \geq 0$ ✓, where $|\bar{x}\cdot\bar{0}| = |0| = 0$ and $|\bar{0}| = 0$.

If $\bar{y}\ne\bar{0}$, here's a cute proof from Wikipedia (different from the book's cute proof!):

For any vector $\bar{v}$, $|\bar{v}|^2 = \sum_{i=1}^{n} v_i^2 \ge 0$ with equality $|\bar{v}|^2=0 \Leftrightarrow \bar{v}=\bar{0}$.

So consider $0 \le |\bar{x} - \lambda\bar{y}|^2 = (\bar{x}-\lambda\bar{y})\cdot(\bar{x}-\lambda\bar{y})$

where $\lambda := \dfrac{\bar{x}\cdot\bar{y}}{|\bar{y}|^2}$ ← not zero, since $\bar{y}\ne\bar{0}$

$$= \bar{x}\cdot\bar{x} - \lambda\bar{y}\cdot\bar{x} - \lambda\bar{x}\cdot\bar{y} + \lambda^2\bar{y}\cdot\bar{y}$$

$$= |\bar{x}|^2 - 2\lambda\,\bar{x}\cdot\bar{y} + \lambda^2|\bar{y}|^2$$

def'n of λ

$$= |\bar{x}|^2 - 2\frac{(\bar{x}\cdot\bar{y})^2}{|\bar{y}|^2} + \frac{(\bar{x}\cdot\bar{y})^2}{|\bar{y}|^2}$$

$$= |\bar{x}|^2 - \frac{(\bar{x}\cdot\bar{y})^2}{|\bar{y}|^2}$$

The idea in the cute proof: $\bar{x}-\lambda\bar{y}$ is this vector

$\dfrac{\bar{x}\cdot\bar{y}}{|\bar{y}|^2}\bar{y}$

It has length $\bar{0}$ $\Leftrightarrow$ $\bar{x}$ is a multiple of $\bar{y}$

Therefore $\dfrac{(\bar{x}\cdot\bar{y})^2}{|\bar{y}|^2} \le |\bar{x}|^2$

i.e. $(\bar{x}\cdot\bar{y})^2 \le |\bar{x}|^2|\bar{y}|^2$

$\Rightarrow |\bar{x}\cdot\bar{y}| \le |\bar{x}||\bar{y}|$

Also, equality here traces back to forcing equality in $0 \le |\bar{x}-\lambda\bar{y}|^2$

i.e. $0 = |\bar{x}-\lambda\bar{y}|^2$, so $\bar{0} = \bar{x}-\lambda\bar{y}$

i.e. $\lambda\bar{y} = \bar{x}$ and $\bar{x}$ is a multiple of $\bar{y}$ ∎

COROLLARY (THM 1.4.9) "Triangle inequality": $|\bar{x}+\bar{y}| \le |\bar{x}|+|\bar{y}|$

proof: $|\bar{x}+\bar{y}|^2 = |\bar{x}|^2 + 2\bar{x}\cdot\bar{y} + |\bar{y}|^2 \le |\bar{x}|^2 + 2|\bar{x}||\bar{y}| + |\bar{y}|^2 = (|\bar{x}|+|\bar{y}|)^2$ ∎

(19)

It's useful to also measure how large a matrix is, as it will bound how much it stretches vectors.

DEF'N: $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} - & \bar{r}_1^T & - \\ & \vdots & \\ - & \bar{r}_m^T & - \end{bmatrix} = \begin{bmatrix} | & & | \\ \bar{c}_1 & \cdots & \bar{c}_n \\ | & & | \end{bmatrix}$

has squared norm or length $|A|^2 := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 = |\bar{r}_1|^2 + \ldots + |\bar{r}_m|^2 = |\bar{c}_1|^2 + \ldots + |\bar{c}_n|^2$

Here's the stretching bound...

PROP 1.4.11 Given matrices $\underset{m\times n}{A}$, $\underset{n\times p}{B}$, then $|AB| \leq |A||B|$

In particular, if $\bar{v} \in \mathbb{R}^n$ (i.e. the $p=1$ case), then $\underbrace{|A\bar{v}|}_{\text{usual length}} \leq \underbrace{|A|}_{\text{norm of } A} \underbrace{|\bar{v}|}_{\text{usual length}}$

proof: To show $|AB| \leq |A||B|$, it's equivalent to show

$$|AB|^2 \overset{?}{\leq} |A|^2 |B|^2$$

$$|AB|^2 = \left| \overbrace{\begin{bmatrix} - & \bar{r}_1^T & - \\ & \vdots & \\ - & \bar{r}_m^T & - \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} | & & | \\ \bar{c}_1 & \cdots & \bar{c}_p \\ | & & | \end{bmatrix}}^{B} \right| = \Bigg| \underbrace{\begin{bmatrix} \bar{r}_1 \cdot \bar{c}_1 & \cdots & \bar{r}_1 \cdot \bar{c}_p \\ \vdots & & \\ \bar{r}_m \cdot \bar{c}_1 & \cdots & \bar{r}_m \cdot \bar{c}_p \end{bmatrix}}_{AB} \Bigg|$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{p} \underbrace{(\bar{r}_i \cdot \bar{c}_j)^2}_{\leq |\bar{r}_i|^2 |\bar{c}_j|^2} \leq \sum_{i=1}^{m} \sum_{j=1}^{p} |\bar{r}_i|^2 |\bar{c}_j|^2$$

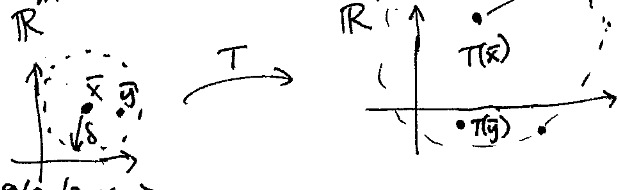
$$|A|^2|B|^2 = \left| \begin{bmatrix} - & \bar{r}_1^T & - \\ & \vdots & \\ - & \bar{r}_m^T & - \end{bmatrix} \right| \left| \begin{bmatrix} | & & | \\ \bar{c}_1 & \cdots & \bar{c}_p \\ | & & | \end{bmatrix} \right| = (|\bar{r}_1|^2 + \ldots + |\bar{r}_m|^2)(|\bar{c}_1|^2 + \ldots + |\bar{c}_p|^2) = \sum_{i=1}^{m} \sum_{j=1}^{p} |\bar{r}_i|^2 |\bar{c}_j|^2$$

∎

COR 1.4.12: A linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous,

meaning $\forall \bar{x} \in \mathbb{R}^m$ and $\forall \epsilon > 0$ $\exists$ some $\delta > 0$ such that

$$|\bar{y} - \bar{x}| < \delta \Rightarrow |T(\bar{y}) - T(\bar{x})| < \epsilon.$$



Proof: Taken the $\epsilon > 0$ (supplied by the Devil), we can pick $\delta = \frac{\epsilon}{|A|}$ where $A = [T]$, since then

$|\bar{y} - \bar{x}| < \delta = \frac{\epsilon}{|A|} \Rightarrow |T(\bar{y}) - T(\bar{x})| = |A\bar{y} - A\bar{x}| = |A(\bar{y} - \bar{x})|$
$\leq |A| \cdot |\bar{y} - \bar{x}| < |A| \cdot \frac{\epsilon}{|A|} = \epsilon$ ∎

9/21/2016 >