(20)

9/21/2016 **Determinants & cross products** - will help us compute areas, volumes, and normal vectors pretty easily algebraically

DEF'N: For a 1×1 ~~matrix~~ matrix $A = [a]$, $\det A := a$

2×2 matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $\det A := a_1 b_2 - a_2 b_1$

$= a_1 \det[b_2] - a_2 \det[b_1]$

3×3 matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \bar{a} & \bar{b} & \bar{c} \\ | & | & | \end{bmatrix}$

$$\det A := a_1 \det\begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - a_2 \det\begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} + a_3 \det\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} +\det\begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} \\ -\det\begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} \\ +\det\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \end{bmatrix}$$

$$= \underline{a} \cdot (\underbrace{\bar{b} \times \bar{c}}_{\text{the cross product of } \bar{b}, \bar{c} \text{, defined as this}})$$

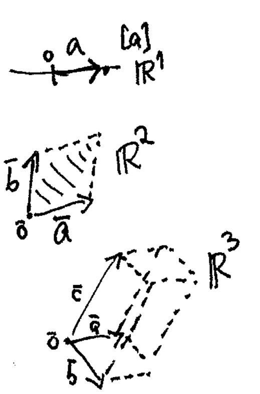
REMARK: For $n\times n$ matrices, this recursive definition, expanding along 1st column, will still be the definition of $\det A$

THEOREM: • For $n = 1,2,3$ an $n\times n$ matrix A has $|\det A|$ (the absolute value)

1.4.14
1.4.19
1.4.20

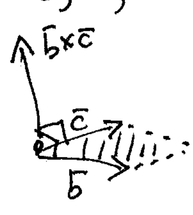
equal to the n-dimensional volume (= length for $n=1$, area for $n=2$) of the parallelepiped/box

spanned by the columns of A, with sign $\pm$ determined by a right-hand rule (described in §1.4).



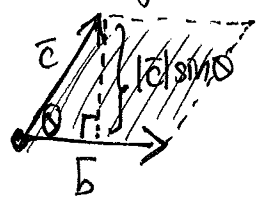
• Also, $\bar{b}\times\bar{c}$ in $\mathbb{R}^3$ has length $|\bar{b}\times\bar{c}|$ = area of parallelogram spanned by $\bar{b}, \bar{c}$

with direction orthogonal to $\bar{b}, \bar{c}$, and ~~oriented~~ $\bar{b}, \bar{c}, \bar{b}\times\bar{c}$ oriented via a right-hand rule.



Proof: (up to sign statements!)

Let's first check $|\bar{b}\times\bar{c}| \overset{?}{=}$ area of parallelogram $= \underbrace{|\bar{b}|}_{\text{base}} \cdot \underbrace{|\bar{c}|\sin\theta}_{\text{height}}$



or equivalently, let's check

(easier to work with; gets rid of $\sqrt{(\sim)}$)

$$|\bar{b}\times\bar{c}|^2 \overset{?}{=} |\bar{b}|^2|\bar{c}|^2\sin^2\theta$$

$$|\bar{b}\times\bar{c}|^2 = (b_2c_3-b_3c_2)^2+(b_1c_3-b_3c_1)^2+(b_1c_2-b_2c_1)^2$$

$$= \begin{pmatrix} b_1^2c_2^2 + b_1^2c_3^2 \\ +b_2^2c_1^2 \quad +b_2^2c_3^2 \\ +b_3^2c_1^2+b_3^2c_2^2 \end{pmatrix} - 2(b_1b_2c_1c_2+b_1b_3c_1c_3+b_2b_3c_2c_3)$$

$$|\bar{b}|^2|\bar{c}|^2\sin^2\theta = |\bar{b}|^2|\bar{c}|^2(1-\cos^2\theta) = |\bar{b}|^2|\bar{c}|^2 - \underbrace{|\bar{b}|^2|\bar{c}|^2\cos^2\theta}_{(\bar{b}\cdot\bar{c})^2}$$

$$= (b_1^2+b_2^2+b_3^2)(c_1^2+c_2^2+c_3^2) - (b_1c_1+b_2c_2+b_3c_3)^2$$

$$= \begin{pmatrix} \cancel{b_1^2c_1^2} + b_1^2c_2^2 + b_1^2c_3^2 \\ +b_2^2c_1^2 + \cancel{b_2^2c_2^2} + b_2^2c_3^2 \\ +b_3^2c_1^2 + b_3^2c_2^2 + \cancel{b_3^2c_3^2} \end{pmatrix} - \left(\cancel{b_1^2c_1^2} + \cancel{b_2^2c_2^2} + \cancel{b_3^2c_3^2}\right) - 2(b_1b_2c_1c_2+b_1b_3c_1c_3+b_2b_3c_2c_3)$$

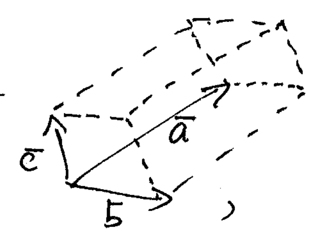
Can also check directly that $\bar{b}\times\bar{c} \perp \bar{b},\bar{c}$ ($\perp$: orthogonal to)

via dot products: $\bar{b}\cdot(\bar{b}\times\bar{c}) = b_1(b_2c_3-b_3c_2) - b_2(b_1c_3-b_3c_1) + b_3(b_1c_2-b_2c_1)$

$= \underbrace{b_1b_2c_3 - b_1b_3c_2 - b_1b_2c_3 + b_2b_3c_1 + b_1b_3c_2 - b_2b_3c_1}_{=0}$

and similarly for $\bar{c}\cdot(\bar{b}\times\bar{c}) = 0$.

To show $\det\begin{bmatrix} | & | & | \\ \bar{a} & \bar{b} & \bar{c} \\ | & | & | \end{bmatrix} = \bar{a}\cdot(\bar{b}\times\bar{c})$ is the volume of



first note that $\bar{b}\times\bar{c} = \bar{0} \iff \bar{b},\bar{c}$ are collinear

by what we've already shown (Why?)

But in that collinear case, both $\bar{a}\cdot(\bar{b}\times\bar{c}) = \det A$ and the volume above are 0,

so we'd be done.

(22) So we can assume $\bar{b}, \bar{c}$ are not collinear, and $\bar{b}\times\bar{c} \neq \bar{0}$.

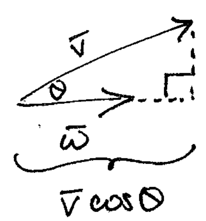
Note that the volume can be computed as base · height taking the $\bar{b}, \bar{c}$ parallelogram as the base, and the <u>length of the projection of $\bar{a}$ onto $\bar{b}\times\bar{c}$</u> as the height.

Note we can compute that projection length, via dot products:

<u>COR</u> 1.4.4: If $\bar{w} \neq \bar{0}$, then $\bar{v}\cdot\bar{w} = |\bar{w}| \cdot |\bar{v}| \cos\theta$

$= |\bar{w}|$ (length of projection of $\bar{v}$ onto line spanned by $\bar{w}$)

<u>proof:</u>



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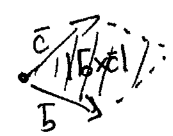
Hence the height above is $\dfrac{\bar{a}\cdot(\bar{b}\times\bar{c})}{|\bar{b}\times\bar{c}|}$

and the volume above is base · height

$$|\bar{b}\times\bar{c}| \cdot \frac{\bar{a}\cdot(\bar{b}\times\bar{c})}{|\bar{b}\times\bar{c}|} = \bar{a}\cdot(\bar{b}\times\bar{c})$$

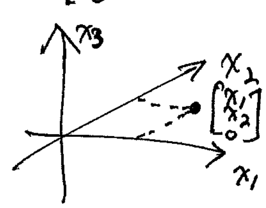
$$= \det\begin{bmatrix} | & | & | \\ \bar{a} & \bar{b} & \bar{c} \\ | & | & | \end{bmatrix}$$ ■



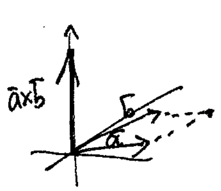
This proves all the $\mathbb{R}^3$ assertions, but then the $\mathbb{R}^2$ assertions about $\det\begin{bmatrix} | & | \\ \bar{a} & \bar{b} \\ | & | \end{bmatrix} = a_1b_2 - a_2b_1$ are special cases of the $\mathbb{R}^3$ assertions, if we embed $\mathbb{R}^2$ in $\mathbb{R}^3$ as the x_1, x_2-plane:

$\det \underset{2\times 2}{A}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$



e.g. $$\left|\det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right| = \left| \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} \right|$$



$$= \left| \begin{bmatrix} \det\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} \\ \det\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \\ \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \end{bmatrix} \right| = \left| \begin{bmatrix} 0 \\ 0 \\ \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \end{bmatrix} \right| = \left| \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \bar{e}_3 \right|$$ ■

(23) What about the $\pm$ sign rules involving right-hands?? (see end of §1.4)

It may be better to think about the more general statement for $n\times n$ matrices A and the $\mathrm{sign}(\det(A)) = \pm 1$, which might even prove later (after §4.8?)

DEFN. Say that two $n\times n$ matrices $A = \begin{bmatrix} | & & | \\ \bar{a}_1 & \cdots & \bar{a}_n \\ | & & | \end{bmatrix}$, $B = \begin{bmatrix} | & & | \\ \bar{b}_1 & \cdots & \bar{b}_n \\ | & & | \end{bmatrix}$ with $\det A, \det B \neq 0$

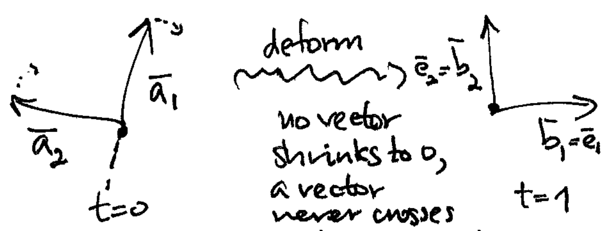
can be (invertibly) deformed into each other if you can slowly change $\bar{a}_1$ into $\bar{b}_1$, $\vdots$, $\bar{a}_n$ into $\bar{b}_n$ keeping the ~~matrix~~ determinant $\neq 0$ along the way

More formally: $\exists$ a parametrized matrix $A(t) = \begin{bmatrix} | & & | \\ \bar{a}_1(t) & \cdots & \bar{a}_n(t) \\ | & & | \end{bmatrix}$

having $A(0) = A$
$A(1) = B$
$\det A(t) \neq 0 \ \forall t$

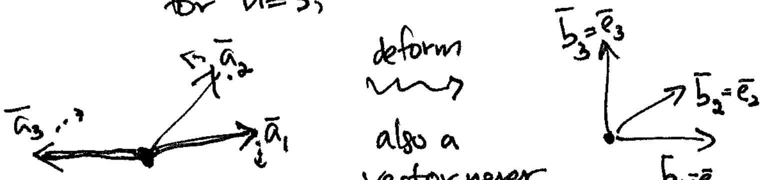
This entire page was only alluded to in lecture; we decided to skip it as detailed discussion

e.g. for $n=2$,



no vector shrinks to 0, a vector never crosses the line spanned by the other

for $n=3$,



also a vector never crosses the plane spanned by another.

FACT: One can always deform A $n\times n$ into either $\begin{bmatrix} | & | & & | \\ \bar{e}_1 & \bar{e}_2 & \cdots & \bar{e}_n \\ | & | & & | \end{bmatrix} = I_n$

or $\begin{bmatrix} | & | & | & & | \\ \bar{e}_2 & \bar{e}_1 & \bar{e}_3 & \cdots & \bar{e}_n \\ | & | & | & & | \end{bmatrix}$ (swapped 1st two columns)

THM: $\det(A) = \begin{cases} +1 & \text{if } A \text{ can be deformed to } \begin{bmatrix} | & | & & | \\ \bar{e}_1 & \bar{e}_2 & \cdots & \bar{e}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_n \\ -1 & \text{if } \text{---}\ " \ \text{---} \ \begin{bmatrix} | & | & | & & | \\ \bar{e}_2 & \bar{e}_1 & \bar{e}_3 & \cdots & \bar{e}_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & 0 \\ & 0 & 1 \ddots 1 \end{bmatrix} \end{cases}$

9/23/2016 >