Math 3593H Honors Math II
Midterm exam 1, Thursday February 16, 2017

Instructions:
50 minutes, closed book and notes, no electronic devices.
There are four problems, worth 25 points each.

1.(i) (10 points) Show the solution set in $\mathbb{R}^3$ to this system is a manifold:

$$x^2 + y^2 = z,$$
$$x + y + z = 4.$$

The solution set $M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2+y^2-z \\ x+y+z-4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

and $\left[DF\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{bmatrix} 2x & 2y & -1 \\ 1 & 1 & 1 \end{bmatrix}$ $\underline{\text{has (full) rank 2}}$ unless $\left\{ \begin{matrix} 2x+1=0 \\ 2y+1=0 \end{matrix} \right\}$,

$\xrightarrow{\text{row-reduce}} \begin{bmatrix} 2x+1 & 2y+1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

i.e. unless $x = -\frac{1}{2} = y$.
But then $z = x^2+y^2 = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$
would mean $x+y+z = -\frac{1}{2}-\frac{1}{2}+\frac{1}{2} \neq 4$,
i.e. there are $\underline{\text{no such points on } M}$.

(ii) (5 points) What is its dimension as a manifold?

Since $\mathbb{R}^3 \xrightarrow{F} \mathbb{R}^2$ has $DF(\vec{x})$ of full rank $\forall \vec{x} \in M$, its dimension is $3-2=1$

(iii) (10 points) Find equations that cut out its tangent space $T_{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} M$.

$$T_{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} M = \ker\left[DF\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right] = \ker\begin{bmatrix} 2\cdot 1 & 2\cdot 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \ker\begin{bmatrix} 2 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \underbrace{\begin{matrix} 2x+2y-z=0 \\ x+y+z=0 \end{matrix}}_{\text{these are such equations}} \right\}$$

2. (i) (10 points) Compute the 4^{th} degree Taylor polynomial $P^4_{f,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}$ at the origin in $\mathbb{R}^2$, for $f\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = e^{x^2-3y^2+y^3}$.

$$e^{x^2-3y^2+y^3} = 1 + \frac{(x^2-3y^2+y^3)^1}{1!} + \frac{(x^2-3y^2+y^3)^2}{2!} + \mathcal{O}\left(\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|^4\right)$$

$$= \underbrace{1 + x^2-3y^2+y^3 + \frac{x^4-6x^2y^2+9y^4}{2}}_{\text{this is } P^4_{f,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)} + \mathcal{O}\left(\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|^4\right)$$

(ii) (5 points) Prove that f has a critical point at the origin in $\mathbb{R}^2$.

The above calculation shows $P^1_{f,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = 1 + 0\cdot x + 0\cdot y + \mathcal{O}\left(\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|^1\right)$,

so $\left.\frac{\partial f}{\partial x}\right|_{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)} = \left.\frac{\partial f}{\partial y}\right|_{\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)} = 0$.

(iii) (10 points) Classify this critical point as either a local maximum, a local minimum, a saddle, or something indeterminate.

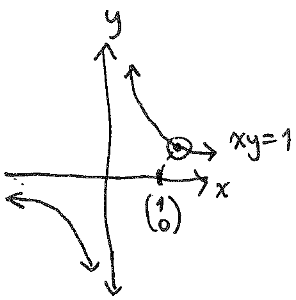
The above calculation shows $P^2_{f,\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = 1 + \underbrace{x^2-3y^2}_{\substack{\text{call this}\\ Q\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)}} + \mathcal{O}\left(\left\|\begin{bmatrix}x\\y\end{bmatrix}\right\|^2\right)$

Since $Q\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) = x^2-3y^2$ has signature $(1,1)$, this critical point is a saddle.

3. Write down a system of m equations in m unknowns, for some value of m, whose solution would let you compute the point(s) in $\mathbb{R}^2$ on the hyperbola $xy = 1$ closest to the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Don't bother with solving the system, but do explain what you would do with its solution to find the closest point(s).

Want to minimize $f\begin{pmatrix}x\\y\end{pmatrix} = (x-1)^2 + (y-0)^2 \leftarrow$ squared distance to $\begin{pmatrix}1\\0\end{pmatrix}$

$$= (x-1)^2 + y^2$$



subject to the constraint $xy = 1$

i.e. $F\begin{pmatrix}x\\y\end{pmatrix} = xy - 1 = 0$

so Lagrange multipliers says to calculate critical points in $\begin{pmatrix}x\\y\\\lambda\end{pmatrix}$ for the Lagrangian $\mathcal{L}\begin{pmatrix}x\\y\\\lambda\end{pmatrix} = f\begin{pmatrix}x\\y\end{pmatrix} - \lambda F\begin{pmatrix}x\\y\end{pmatrix}$

$$= (x-1)^2 + y^2 - \lambda(xy-1)$$

Critical points for $\mathcal{L}$ solve

$$\begin{cases} 0 = \dfrac{\partial \mathcal{L}}{\partial x} = 2(x-1) - \lambda y \\[2ex] 0 = \dfrac{\partial \mathcal{L}}{\partial y} = 2y - \lambda x \\[2ex] 0 = \dfrac{\partial \mathcal{L}}{\partial x} = -(xy-1) \end{cases}$$

Once one finds all solutions $\begin{pmatrix}x_0\\y_0\\\lambda_0\end{pmatrix}$ to the above system, one could plug in to find their values $f\begin{pmatrix}x_0\\y_0\end{pmatrix}$, to see which one is smallest.

4. Find the signature of the quadratic form $\mathbb{R}^4 \xrightarrow{Q} \mathbb{R}$ in the four variables w, x, y, z defined by

$$Q\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \det \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

$$Q\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = xw - yz$$

$$= \frac{1}{4}\left[(x+w)^2 - (x-w)^2\right] - \frac{1}{4}\left[(y+z)^2 - (y-z)^2\right]$$

$$= + \left(\frac{x+w}{2}\right)^2 - \left(\frac{x-w}{2}\right)^2 - \left(\frac{y+z}{2}\right)^2 + \left(\frac{y-z}{2}\right)^2$$

$\Rightarrow$ signature $(2,2)$

— OR —

$$Q\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = [w\ x\ y\ z]\left[\begin{array}{cc|cc} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{array}\right]\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$= \frac{1}{2} \bar{x}^T \underbrace{\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]}_{\text{call this } A} \bar{x}$$

and $\varphi_A(t) = \det(tI_4 - A) = \det \left[\begin{array}{cc|cc} t & -1 & 0 & 0 \\ -1 & t & 0 & 0 \\ \hline 0 & 0 & t & 1 \\ 0 & 0 & 1 & t \end{array}\right]$

$$= (t^2-1)(t^2-1)$$

$$= (t+1)^2(t-1)^2$$

so A has eigenvalues $(+1,+1,-1,-1)$

and $Q(\bar{x})$ has signature $(2,2)$