Math 3593H Honors Math II
Quiz 2, Thursday March 2, 2017

Instructions:
20 minutes, closed book, no electronic devices,
but an 8.5×11 page of notes is OK.
There are two problems, worth a total of 20 points.

1. (8 points) On the surface in $\mathbb{R}^3$ which is the graph of

$$z = 8 + x^2 + 2xy + y^2 + 6x^3 - y^5$$

compute the Gauss curvature at the point $\begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}$.

Since $z = f\begin{pmatrix} x \\ y \end{pmatrix} = 8 + x^2 + 2xy + y^2 + O(\|\begin{pmatrix} x \\ y \end{pmatrix}\|^2)$

has $\left.\frac{\partial f}{\partial x}\right|_{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \left.\frac{\partial f}{\partial y}\right|_{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}} = 0$,

in "best" coordinates it would be

$$Z = g\begin{pmatrix} X \\ Y \end{pmatrix} = X^2 + 2XY + Y^2 + O(\|\begin{pmatrix} X \\ Y \end{pmatrix}\|^2)$$

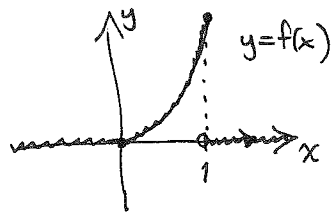
$$\left[= \underbrace{(X+Y)^2}_{\text{a quadratic form of signature } (1,0)} + O(\|\begin{pmatrix} X \\ Y \end{pmatrix}\|^2) \right]$$

$$= \frac{1}{2} [X \; Y] \underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix} + O(\|\begin{pmatrix} X \\ Y \end{pmatrix}\|^2)$$

$$\Rightarrow \text{Gauss curvature } K(\bar{a}) = A_{20}A_{02} - A_{11}^2$$
$$= 2 \cdot 2 - 2^2 = 0$$

2. (12 points; 4 points each part)
For each of the following functions $\mathbb{R} \xrightarrow{f} \mathbb{R}$, say whether f is (Riemann) integrable or not, and explain your reasoning.

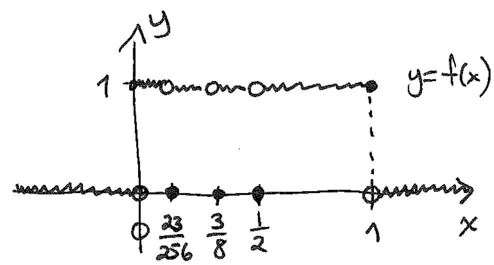
(i) $f(x) = \begin{cases} x^2 & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$



Yes, Riemann integrable since it is continuous except at $x=0$, a set of measure 0.

(ii) $f(x) = \begin{cases} 1 & \text{if } x \in [0,1], \text{ but } x \neq \frac{1}{2}, \frac{3}{8}, \frac{23}{256}, \\ 0 & \text{otherwise.} \end{cases}$

Yes, for same reason: continuous except at $x = 0, \frac{23}{256}, \frac{3}{8}, \frac{1}{2}, 1$



(iii) $f(x) = \begin{cases} 1 & \text{if } x \in [0,1], \text{ and } x \neq \frac{k}{2^m} \text{ for integers } k, m, \text{ with } m \geq 1 \text{ and } 0 \leq k \leq 2^m, \\ 0 & \text{otherwise.} \end{cases}$

No, not Riemann integrable since every dyadic cube $C \in D_N(\mathbb{R}^1)$ will contain points x with $f(x)=0$ and also contain points x with $f(x)=1$,

$C = \left[\frac{\ell}{2^N}, \frac{\ell+1}{2^N}\right)$

so $U_N(f) = \sum_{C \in D_N(\mathbb{R}^1)} M_C(f)\,\mathrm{vol}_1(C) = 1 \quad \forall N$

$L_N(f) = \sum_{C \in D_N(\mathbb{R}^1)} m_C(f)\,\mathrm{vol}_1(C) = 0 \quad \forall N$

$\Rightarrow \lim_{N\to\infty} L_N(f) = 0 \neq 1 = \lim_{N\to\infty} U_N(f)$