(101)

⑤ PROP 6.3.8 generalizes this:

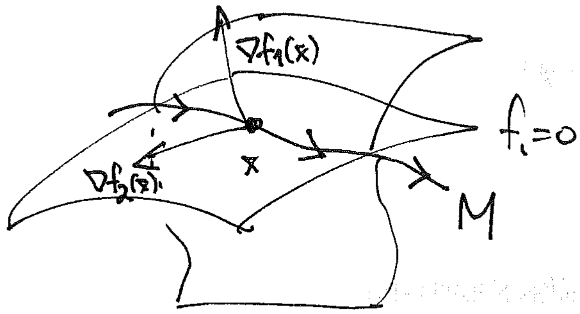
If M is a k-dim'l manifold in $\mathbb{R}^n$ defined as $\bar{f}(\bar{x}) = \bar{0}$ for a C^1 map $\underset{\mathbb{R}^n}{\overset{\cap}{U}} \xrightarrow{\bar{f}} \mathbb{R}^{n-k}$ with $D\bar{f}(\bar{x})$ surjective $\forall \bar{x} \in M$

then decreeing for $(\bar{x}, \underbrace{\bar{v}_1, \ldots, \bar{v}_k}_{\in T_{\bar{x}}M}) \in \mathcal{B}(M)$ that

$$\Omega(\bar{x}, \bar{v}_1, \ldots, \bar{v}_k) = \operatorname{sgn} \det\left(\nabla f_1(\bar{x}), \ldots, \nabla f_{n-k}(\bar{x}), \bar{v}_1, \ldots, \bar{v}_k\right)$$

gives an orientation on M

e.g. $n=3$, $k=1$

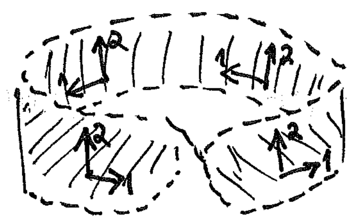


Discuss this first!

NON-EXAMPLES

⑥ Not all manifolds are orientable, e.g.

Möbius band S



It doesn't make sense to do flux integrals over S, although surface area of S makes sense!

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⑦ EXAMPLE 6.4.9: $X := \left\{ \underset{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}}{A} \in \mathrm{Mat}(2,3) : \operatorname{rank}(A) = 1 \right\}$ will turn out to be a 4-dim'l manifold inside $\mathrm{Mat}(2,3) \cong \mathbb{R}^6$, but non-orientable (later).

In showing this, it's helpful to note this fact:

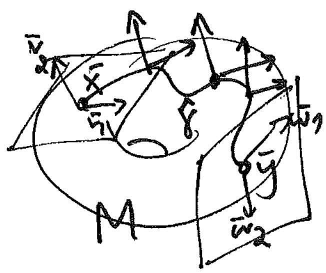
PROP 6.3.10: A (path-) connected manifold M having an orientation Ω will have only two orientations: Ω and $-\Omega$.

(path-) connected: every $\bar{x}, \bar{y} \in M$ have a path $[0,1] \xrightarrow[\text{continuous}]{\gamma} M$, $\gamma(0) = \bar{x}$, $\gamma(1) = \bar{y}$

proof: Enough to show that if 2 orientations Ω, Ω' at some $(\bar{x}, \bar{v}_1, \ldots, \bar{v}_k) \in \mathcal{B}(M)$, they agree on all of $\mathcal{B}(M)$.

(The book's proof of this looks incomplete to me)

Given some other $(\bar{y}, \bar{w}_1, \dots, \bar{w}_k) \in \mathcal{B}(M)$, pick a path $[0,1] \xrightarrow{\gamma} M$,
$\gamma(0) = \bar{x}$
$\gamma(1) = \bar{y}$



and then show one can extend this to

$$[0,1] \xrightarrow{\hat{\gamma}} \mathcal{B}(M) \quad \text{continuously}$$

with
$$\hat{\gamma}(0) = (\bar{x}, \bar{v}_1, \dots, \bar{v}_k)$$
$$\hat{\gamma}(1) = (\bar{y}, \underbrace{\bar{v}_1', \dots, \bar{v}_k'}_{\in T_{\bar{y}}M})$$

not obvious, but not hard using local definition of M as a graph of a function $\bar{f}$

Then we get continuous functions $\Omega \circ \hat{\gamma}, \Omega' \circ \hat{\gamma} : \underset{\mathbb{R}^1}{\underset{\cap}{[0,1]}} \to \underset{\mathbb{R}^1}{\underset{\cap}{\{+1,-1\}}}$

$$[0,1] \xrightarrow{\hat{\gamma}} \mathcal{B}(M) \xrightarrow{\Omega} \{+1,-1\} \subset \mathbb{R}$$
$$[0,1] \xrightarrow{\hat{\gamma}} \mathcal{B}(M) \xrightarrow{\Omega'} \{+1,-1\} \subset \mathbb{R}$$

that agree at 0 (since $\Omega(\bar{x}, \bar{v}_1, \dots, \bar{v}_k) = \Omega'(\bar{x}, \bar{v}_1, \dots, \bar{v}_k)$),
so they must agree at 1 (a continuous function $[0,1] \to \{+1,-1\}$ is constant!)

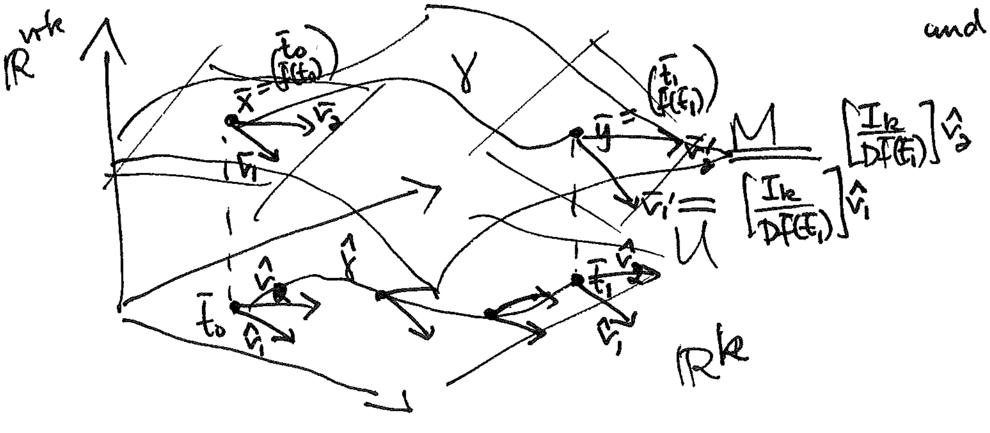
i.e. $\Omega(\bar{y}, \bar{v}_1', \dots, \bar{v}_k') = \Omega'(\bar{y}, \bar{v}_1', \dots, \bar{v}_k')$.

But then $\Omega(\bar{y}, \bar{w}_1, \dots, \bar{w}_k) = \Omega'(\bar{y}, \bar{w}_1, \dots, \bar{w}_k)$ because Ω, Ω' both restrict to orientations on $T_{\bar{y}}M$ ∎

The local picture for extending a path $[0,1] \xrightarrow{\gamma} M$ from $\bar{x}$ to $\bar{y}$ to a path $[0,1] \xrightarrow{\hat{\gamma}} \mathcal{B}_M$ from $(\bar{x}, \bar{v}_1, \dots, \bar{v}_k)$ to some $(\bar{y}, \bar{v}_1', \dots, \bar{v}_k')$:

Locally, $M = \text{graph } \Gamma_{\bar{f}} = \left\{ \begin{pmatrix} \bar{t} \\ \bar{f}(\bar{t}) \end{pmatrix} : \bar{t} \in U \right\}$ for some diff'ble $\underset{\mathbb{R}^k}{\underset{\cap}{U}}{}^{\text{open}} \xrightarrow{\bar{f}} \mathbb{R}^{n-k}$, $\bar{t} \mapsto \bar{f}(\bar{t})$

and $T_{\binom{\bar{t}_0}{\bar{f}(\bar{t}_0)}} M := \text{im} \begin{bmatrix} I_k \\ Df(\bar{t}_0) \end{bmatrix}$



RMK: Same picture shows there is no obstruction to locally orienting a manifold; patching it together globally is the issue.

(103) let's return to our supposedly non-orientable manifold...

EXAMPLE 6.4.9: $X = \left\{ A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \in \mathrm{Mat}(2,3) \text{ of rank } 1 \right\}$

We 1st show it's a 4-dim'l manifold inside $\mathbb{R}^6 = \mathrm{Mat}(2,3)$: (EXER. 6.4.2)

rank $A = 1 \iff A$ has exactly 1 pivot column (not 2, not 0)

$\iff A \neq \begin{bmatrix} 0&0&0\\0&0&0 \end{bmatrix}$ and $\det \underbrace{A_{12}}_{\text{:= columns 1,2 of } A} = 0$, $\det A_{13} = 0$, $\det A_{23} = 0$

i.e. $a_1b_2 - a_2b_1 = 0$, $a_1b_3 - a_3b_1 = 0$, $a_2b_3 - a_3b_2 = 0$

Seems like we need 3 equations, but in fact, on each of three various patches $U_1, U_2, U_3 \subset \mathbb{R}^6$

$U_1 = \left\{ A \text{ with } \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, $U_2 = \left\{ \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, $U_3 = \left\{ \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

one only needs 2 equations,

e.g. on U_1, $\begin{cases} \det A_{12} = 0 \Rightarrow \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \text{ is a mult. of } \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \left(\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\ \det A_{13} = 0 \Rightarrow \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \text{ —''— } \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \left(\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \end{cases}$

so $\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$ are dependent, i.e. $\boxed{\det A_{23} = 0}$

Also on U_1, one can see that as the zero set of $F(A) = \begin{pmatrix} \det A_{12} \\ \det A_{13} \end{pmatrix} = \begin{pmatrix} a_1b_2 - a_2b_1 \\ a_1b_3 - a_3b_1 \end{pmatrix}$

one has $DF(A) = $
$$\begin{array}{cccccc} \partial/\partial a_1 & \partial/\partial b_1 & \partial/\partial a_2 & \partial/\partial b_2 & \partial/\partial a_3 & \partial/\partial b_3 \end{array}$$
$$\begin{bmatrix} b_2 & -a_2 & \boxed{-b_1 \quad a_1} & 0 & 0 \\ b_3 & -a_3 & 0 \quad 0 & \boxed{-b_1 \quad a_1} \end{bmatrix}$$
of full rank 2,

not both 0   not both 0

so on $X \cap U_1$ it looks like a manifold by Implicit Function Thm.

Similarly on U_2, U_3, it's locally a manifold.

(104) So why is X not orientable? Assume it was, with orientation $\mathcal{B}(X) \xrightarrow{\Omega} \{\pm 1\}$.

On U_1 we have a strict, bijective parametrization via [Why?]

$$\left(\mathbb{R}^2 - \{\bar{0}\}\right) \times \mathbb{R}^2 \xrightarrow{\gamma_1} U_1$$

$$\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} t \\ u \end{pmatrix}\right) \longmapsto \begin{bmatrix} a_1 & ta_1 & ua_1 \\ b_1 & tb_1 & ub_1 \end{bmatrix}$$

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with
$$D\gamma_1 \begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix} = \begin{array}{c|c} & \partial/\partial a_1 \quad \partial/\partial b_1 \quad \partial/\partial t \quad \partial/\partial u \\ \begin{matrix} a_1 \\ b_1 \\ ta_1 \\ tb_1 \\ ua_1 \\ ub_1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t & 0 & \boxed{a_1} & 0 \\ 0 & t & \boxed{b_1} & 0 \\ u & 0 & 0 & \boxed{a_1} \\ 0 & u & 0 & \boxed{b_1} \end{bmatrix} \end{array}$$

not both zero

not both zero

always of full rank 4, i.e. injective.

So the map $\mathcal{B}\left(\left(\mathbb{R}^2 - \{\bar{0}\}\right) \times \mathbb{R}^2\right) \xrightarrow{(\gamma, D\gamma)} \mathcal{B}(U_1) \xrightarrow{\Omega} \{\pm 1\}$

$$\left(\begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix}, (\bar{v}_1, \ldots, \bar{v}_4)\right)$$

must have constant image $+1$ or -1, since $\left(\mathbb{R}^2 - \{\bar{0}\}\right) \times \mathbb{R}^2$ is path-connected.

That is, γ_1 either preserves, or reverses orientation, everywhere.

Similarly on U_2 we have $\left(\mathbb{R}^2 - \{\bar{0}\}\right) \times \mathbb{R}^2 \xrightarrow{\gamma_2} U_2$, either always preserving or always reversing orientation

$$\left(\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right) \longmapsto \begin{bmatrix} va_2 & a_2 & wa_2 \\ vb_2 & b_2 & wb_2 \end{bmatrix}$$

Thus on the overlap
$$U_1 \cap U_2 = \left\{ \begin{bmatrix} a_1 & ta_1 & ua_1 \\ b_1 & tb_1 & ub_1 \end{bmatrix} = \begin{bmatrix} va_2 & a_2 & wa_2 \\ vb_2 & b_2 & wb_2 \end{bmatrix} \text{ with } \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and hence } t \neq 0, v \neq 0 \right\}$$

we can solve
$$a_2 = ta_1, \quad b_2 = tb_1, \quad v = a_1/a_2 = \frac{1}{t}, \quad w = \frac{ua_1}{a_2} = \frac{u}{t}$$
and get a map $\gamma_2^{-1} \circ \gamma_1$ sending $\begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix} \longmapsto \begin{pmatrix} ta_1 \\ tb_1 \\ \frac{1}{t} \\ \frac{u}{t} \end{pmatrix}$

having
$$\det D(\gamma_2^{-1} \circ \gamma_1) = \det \begin{array}{c} \partial/\partial a_1 \quad \partial/\partial b_1 \quad \partial/\partial t \quad \partial/\partial u \\ \begin{bmatrix} t & 0 & a_1 & 0 \\ 0 & t & b_1 & 0 \\ 0 & 0 & -\frac{1}{t^2} & 0 \\ 0 & 0 & -\frac{u}{t^2} & \frac{1}{t} \end{bmatrix} \end{array} = -\frac{1}{t} \begin{cases} < 0 & \text{if } t > 0 \\ > 0 & \text{if } t < 0 \end{cases}$$

contradicting the assertions about γ_1, γ_2 always preserving or reversing Ω!