

(104) So why is  $X$  not orientable? Assume it was, with orientation  $B(X) \xrightarrow{\Omega} \{\pm 1\}$ .

On  $U_1$  we have a strict, bijection parametrization via

$$(\mathbb{R}^2 - \{0\}) \times \mathbb{R}^2 \xrightarrow{\mathcal{F}_1} U_1 \quad \text{Why?}$$

$$\left( \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} t \\ u \end{pmatrix} \right) \longmapsto \begin{bmatrix} a_1 & ta_1 & ua_1 \\ b_1 & tb_1 & ub_1 \end{bmatrix}$$

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with  $D\mathcal{F}_1 \begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix} =$

	$\partial/\partial a_1$	$\partial/\partial b_1$	$\partial/\partial t$	$\partial/\partial u$	
$a_1$	1	0	0	0	} always of full rank 4, i.e. <u>injective</u> .
$b_1$	0	1	0	0	
$ta_1$	t	0	$a_1$	0	
$tb_1$	0	t	$b_1$	0	
$ua_1$	u	0	0	$a_1$	} not both zero
$ub_1$	0	u	0	$b_1$	

So the map  $\beta: (\mathbb{R}^2 - \{0\}) \times \mathbb{R}^2 \xrightarrow{(\mathcal{F}_1, D\mathcal{F}_1)} B(U_1) \xrightarrow{\Omega} \{\pm 1\}$

$$\left( \begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix}, \begin{pmatrix} \mathbb{R}^2 - \{0\} \\ \mathbb{R}^2 \end{pmatrix} \right)$$

must have constant image  $+1$  or  $-1$ , since  $(\mathbb{R}^2 - \{0\}) \times \mathbb{R}^2$  is path-connected.

That is,  $\mathcal{F}_1$  either preserves, or reverses orientation, everywhere.

Similarly on  $U_2$  we have  $(\mathbb{R}^2 - \{0\}) \times \mathbb{R}^2 \xrightarrow{\mathcal{F}_2} U_2$ , either always preserving or always reversing orientation

$$\left( \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix} \right) \longmapsto \begin{bmatrix} va_2 & a_2 & va_2 \\ vb_2 & b_2 & vb_2 \end{bmatrix}$$

Thus on the overlap  $U_1 \cap U_2 = \left\{ \begin{bmatrix} a_1 & ta_1 & ua_1 \\ b_1 & tb_1 & ub_1 \end{bmatrix} = \begin{bmatrix} va_2 & a_2 & va_2 \\ vb_2 & b_2 & vb_2 \end{bmatrix} \text{ with } \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$   
 (and hence  $t \neq 0, v \neq 0$ )

we can solve  $a_2 = ta_1$   
 $b_2 = tb_1$   
 $v = a_1/a_2 = 1/t$   
 $w = ua_1/a_2 = u/t$

and get a map  $\mathcal{F}_2^{-1} \circ \mathcal{F}_1$  sending  $\begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix} \mapsto \begin{pmatrix} ta_1 \\ tb_1 \\ 1/t \\ u/t \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ v \\ w \end{pmatrix}$

having  $\det D(\mathcal{F}_2^{-1} \circ \mathcal{F}_1) = \det$

	$\partial/\partial a_1$	$\partial/\partial b_1$	$\partial/\partial t$	$\partial/\partial u$	
$a_2$	t	0	$a_1$	0	} $< 0$ if $t > 0$ $> 0$ if $t < 0$
$b_2$	0	t	$b_1$	0	
$v$	0	0	$-1/t^2$	0	
$w$	0	0	$-u/t^2$	$1/t$	

contradicting the assertions about  $\mathcal{F}_1, \mathcal{F}_2$  always preserving or reversing  $\Omega$ .

DEFIN 6.4.2: If  $M \subseteq \mathbb{R}^n$  is a  $k$ -dim'l manifold, orientable with  $\Omega(M) \xrightarrow{\Omega} \pm 1$

then say a parametrization  $U \xrightarrow{\bar{\gamma}} M$  with  $\bar{\gamma} \in C^1(U)$

$$\begin{matrix} \wedge \\ \mathbb{R}^k \end{matrix}$$

- $D\bar{\gamma}$  injective on  $U - X$
- for  $X \subset U$  with  $\text{vol}_k(X) = 0$
- and  $U - X$  open

is orientation-preserving if  $\Omega(\bar{\gamma}(u), \underbrace{D\bar{\gamma}(u)(e_1), \dots, D\bar{\gamma}(u)(e_k)}_{D\bar{\gamma}(u)}) = +1 \quad \forall u \in U - X$

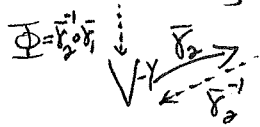
(i.e. ~~any~~ the orientation  $\Omega$  on  $M$  ~~is~~ induced by the standard orientation on  $U$ , via  $\bar{\gamma}$ )

Now we can talk about  $\int_M \varphi$  for  $\varphi \in A^k(M)$

$$:= \int_{\bar{\gamma}(U)} \varphi = \int_{U-X} \varphi(P_{\bar{\gamma}(u)}(D\bar{\gamma}(u))) |d^k u|$$

THM 6.4.10: This definition of  $\int_M \varphi$  is independent of o.p. parametrization,

i.e. if  $U-X \xrightarrow{\bar{\gamma}_1} M$  are both o.p. then

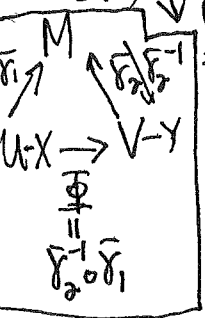


$$\int_{\bar{\gamma}_2(V)} \varphi = \int_{\bar{\gamma}_1(U)} \varphi$$

Proof: Want to show

$$\int_{V-Y} \varphi(P_{\bar{\gamma}_2(v)}(D\bar{\gamma}_2(v))) |d^k v| \stackrel{?}{=} \int_{U-X} \varphi(P_{\bar{\gamma}_1(u)}(D\bar{\gamma}_1(u))) |d^k u|$$

change of variable  $v = \Phi(u)$



$$\int_{U-X} \varphi(P_{\bar{\gamma}_2(\Phi(u))}(D\bar{\gamma}_2(\Phi(u)))) |d\text{et} \Phi(u)| |d^k u|$$

$$= \int_{U-X} \varphi(P_{\bar{\gamma}_1(u)}(D\bar{\gamma}_2(\Phi(u)))) \text{det} \Phi(u) \cdot \left( \frac{|d\text{et} D\Phi(u)|}{|d\text{et} D\bar{\gamma}_1(u)|} \right) |d^k u|$$

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1, \dots, i_k}(\bar{\gamma}(u)) dx_{i_1} \dots dx_{i_k}(D\bar{\gamma}(u))$$

$\frac{|d\text{et} D\Phi(u)|}{|d\text{et} D\bar{\gamma}_1(u)|} = +1$  since  $\bar{\gamma}_1, \bar{\gamma}_2$  are both o.p., by PROP 6.4.8 (very believable, details to check)

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1, \dots, i_k}(\bar{\gamma}(u)) dx_{i_1} \dots dx_{i_k}(D\bar{\gamma}_2(\Phi(u))) \text{det} \Phi(u)$$

$$\stackrel{\text{chain rule}}{\text{tricky!}} \frac{dx_{i_1} \dots dx_{i_k} D(\bar{\gamma}_2 \circ \Phi)(u)}{D\bar{\gamma}_1(u)}$$

