(104) So why is X not orientable? Assume it was, with orientation $\mathcal{B}(X) \xrightarrow{\Omega} \{\pm 1\}$.

On $\mathcal{U}_1$ we have a strict, bijective parametrization via $\boxed{\text{Why?}}$

$$(\mathbb{R}^2 - \{\bar{0}\}) \times \mathbb{R}^2 \xrightarrow{\bar{\gamma}_1} \mathcal{U}_1$$

$$\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} t \\ u \end{pmatrix}\right) \longmapsto \begin{bmatrix} a_1 & ta_1 & ua_1 \\ b_1 & tb_1 & ub_1 \end{bmatrix}$$

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with
$$D\bar{\gamma}_1 \begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix} = \begin{array}{c} \\ a_1 \\ b_1 \\ ta_1 \\ tb_1 \\ ua_1 \\ ub_1 \end{array} \begin{array}{c} \begin{array}{cccc} \partial/\partial a_1 & \partial/\partial b_1 & \partial/\partial t & \partial/\partial u \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline t & 0 & a_1 & 0 \\ 0 & t & b_1 & 0 \\ \hline u & 0 & 0 & a_1 \\ 0 & u & 0 & b_1 \end{array}\right] \end{array}$$

not both zero

not both zero

always of full rank 4, i.e. injective.

So the map
$$\mathcal{B}\left((\mathbb{R}^2 - \{\bar{0}\}) \times \mathbb{R}^2\right) \xrightarrow{(\bar{\gamma}, D\bar{\gamma})} \mathcal{B}(\mathcal{U}_1) \xrightarrow{\Omega} \{\pm 1\}$$
$$\left(\begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix}, (\bar{v}_1, \ldots, \bar{v}_4)\right)$$

must have constant image $+1$ or -1, since $(\mathbb{R}^2 - \{\bar{0}\}) \times \mathbb{R}^2$ is path-connected.

That is, $\bar{\gamma}_1$ either preserves, or reverses orientation, everywhere.

Similarly on $\mathcal{U}_2$ we have $(\mathbb{R}^2 - \{\bar{0}\}) \times \mathbb{R}^2 \xrightarrow{\bar{\gamma}_2} \mathcal{U}_2$, either always preserving or always reversing orientation

$$\left(\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right) \longmapsto \begin{bmatrix} va_2 & a_2 & wa_2 \\ vb_2 & b_2 & wb_2 \end{bmatrix}$$

Thus on the overlap
$$\mathcal{U}_1 \cap \mathcal{U}_2 = \left\{ \begin{bmatrix} a_1 & ta_1 & ua_1 \\ b_1 & tb_1 & ub_1 \end{bmatrix} = \begin{bmatrix} va_2 & a_2 & wa_2 \\ vb_2 & b_2 & wb_2 \end{bmatrix} \text{ with } \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$
(and hence $t \neq 0$, $v \neq 0$)

we can solve
$$a_2 = ta_1, \quad b_2 = tb_1, \quad v = a_1/a_2 = \frac{1}{t}, \quad w = \frac{ua_1}{a_2} = \frac{u}{t}$$

and get a map $\bar{\gamma}_2^{-1} \circ \bar{\gamma}_1$ sending
$$\begin{pmatrix} a_1 \\ b_1 \\ t \\ u \end{pmatrix} \longmapsto \begin{pmatrix} ta_1 \\ tb_1 \\ \frac{1}{t} \\ \frac{u}{t} \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ v \\ w \end{pmatrix}$$

having
$$\det D(\bar{\gamma}_2^{-1} \circ \bar{\gamma}_1) = \det \begin{array}{c} \begin{array}{cccc} \partial/\partial a_1 & \partial/\partial b_1 & \partial/\partial t & \partial/\partial u \end{array} \\ \left[\begin{array}{cc|cc} t & 0 & a_1 & 0 \\ 0 & t & b_1 & 0 \\ \hline 0 & 0 & -\frac{1}{t^2} & 0 \\ 0 & 0 & -\frac{u}{t^2} & \frac{1}{t} \end{array}\right] \end{array} = -\frac{1}{t} \begin{cases} < 0 & \text{if } t > 0 \\ > 0 & \text{if } t < 0 \end{cases}$$

contradicting the assertions about $\bar{\gamma}_1, \bar{\gamma}_2$ always preserving or reversing Ω!

DEF'N 6.4.2: If $M \subset \mathbb{R}^n$ is a k-dim'l manifold, orientable with $\mathcal{B}(M) \xrightarrow{\Omega} \{\pm 1\}$

then say a parametrization $U \xrightarrow{\bar\gamma} M$, $U \subset \mathbb{R}^k$, with
- $\bar\gamma \in C^1(U)$
- $D\bar\gamma$ injective on $U - X$ for $X \subset U$ with $\mathrm{vol}_k(X) = 0$ and $U - X$ open

is orientation-preserving (o.p.) if $\Omega\big(\gamma(\bar u), \underbrace{D\bar\gamma(\bar u)(\vec e_1), \ldots, D\bar\gamma(\bar u)(\vec e_k)}_{D\bar\gamma(\bar u)}\big) = +1 \quad \forall \bar u \in U - X$

(i.e. the orientation Ω on M is induced by the standard orientation on U, via γ)

Now we can talk about $\int_M \varphi$ for $\varphi \in A^k(M)$

$$:= \int_{[\gamma(U)]} \varphi = \int_{U-X} \varphi\big(P_{\gamma(\bar u)}(D\bar\gamma(\bar u))\big)\, |d^k \bar u|$$

THM 6.4.10: This definition of $\int_M \varphi$ is independent of o.p. parametrization,

i.e. if

$$\begin{tikzcd} U-X \arrow[r, "\bar\gamma_1"] \arrow[d, dashed, "\Phi = \bar\gamma_2^{-1}\circ\bar\gamma_1"'] & M \\ V-Y \arrow[ur, "\bar\gamma_2"] & \end{tikzcd}$$

are both o.p. then

$$\int_{[\gamma_2(V)]} \varphi = \int_{[\gamma_1(U)]} \varphi .$$

Proof: Want to show

$$\int_{V-Y} \varphi\big(P_{\gamma_2(\bar v)}(D\bar\gamma_2(\bar v))\big)\, |d^k \bar v| \overset{?}{=} \int_{U-X} \underbrace{\varphi\big(P_{\bar\gamma_1(\bar u)}(D\bar\gamma_1(\bar u))\big)}_{\sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k}(\gamma_1(\bar u))\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}(D\bar\gamma_1(\bar u))} |d^k \bar u|$$

change of variable $\bar v = \Phi(\bar u)$

$$\begin{tikzcd} & M & \\ U-X \arrow[ur, "\bar\gamma_1"] \arrow[rr, "\Phi = \bar\gamma_2^{-1}\circ\bar\gamma_1"'] & & V-Y \arrow[ul, shift left, "\bar\gamma_2"] \arrow[from=ul, shift left, "\bar\gamma_2^{-1}"] \end{tikzcd}$$

$$= \int_{U-X} \varphi\big(P_{\gamma_2(\Phi(\bar u))}(D\bar\gamma_2(\Phi(\bar u)))\big)\, |\det D\Phi(\bar u)|\, |d^k \bar v|$$

$$= \int_{U-X} \underbrace{\varphi\big(P_{\gamma_1(\bar u)}(D\bar\gamma_2(\Phi(\bar u)))\big) \det D\Phi(\bar u)}_{\sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k}(\gamma_1(\bar u))\, \underbrace{dx_{i_1} \wedge \ldots \wedge dx_{i_k}\big(D\bar\gamma_2(\Phi(\bar u))\big) \det D\Phi(\bar u)}_{= dx_{i_1} \wedge \ldots \wedge dx_{i_k} \underbrace{D(\gamma_2 \circ \Phi)(\bar u)}_{D\bar\gamma_1(\bar u)}}} \cdot \underbrace{\frac{|\det D\Phi(\bar u)|}{\det D\Phi(\bar u)}}_{\operatorname{sgn} \det D\Phi(\bar u) = +1} |d^k \bar v|$$

$= +1$ since $\bar\gamma_1, \bar\gamma_2$ are both o.p., by PROP 6.4.8 (very believable, details to check)

chain rule & $\det(AB) = \det A \det B$

Tricky!

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