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(k=3) X = a 3-dim'l manifold with boundary surfaces $S_1, S_2, \ldots, S_r$

X = solid torus (filled donut)
∂X = torus (skin/crust of donut)

$$\underbrace{\int_X \overbrace{d\varphi}^{\text{3-form}}}_{\substack{\text{a volume}\\ \text{integral}}} = \underbrace{\int_{\partial X} \overbrace{\varphi}^{\text{2-form}}}_{\substack{\text{a sum of}\\ \text{flux integrals}}} = \sum_{i=1}^{r} \int_{S_i} \varphi$$

Turns out we'll need to integrate more generally over "pieces-with-boundary" $X \subset M$ when M is a manifold ...

Our old notion of boundary $\partial X = \underbrace{\overline{X}}_{\substack{\text{closure}\\ \text{of } X}} - \underbrace{\mathring{X}}_{\substack{\text{interior}\\ \text{of } X}}$ is not what we want, inside of M....

DEF'N 6.6.1: For $X \subset M$ a k-dim'l manifold

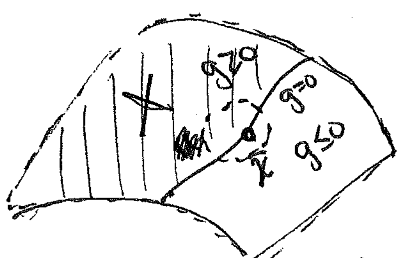
- the boundary of X in M

$$\partial_M X := \left\{ \bar{x} \in M : \text{every open set } U \ni \bar{x} \text{ has both } \begin{array}{l} U \cap X \neq \emptyset \\ U \cap (M \setminus X) \neq \emptyset \end{array} \right\}$$

- the smooth points in $\partial_M X$

$\partial_M^s X := \Big\{ \bar{x} \in M :$ not only is there some $\underset{\mathbb{R}^n}{\overset{\cap}{U}}{}^{\text{open}} \xrightarrow{\bar{f}} \mathbb{R}^{n-k}$ in $C^1(U)$, U containing $\bar{x}$, such that $U \cap M = \bar{f}^{-1}(\bar{0})$, $D\bar{f}(\bar{x})$ surjective

but one can extend it to $U^{\text{open}} \xrightarrow{\binom{\bar{f}}{g}} \mathbb{R}^{n-k+1}$ in $C^1(U)$ such that $g(\bar{x}) = 0$, and $D\binom{\bar{f}}{g}(\bar{x})$ surjective, $U \cap X = \bar{f}^{-1}(\bar{0}) \cap \{g \geq 0\} \Big\}$



$U \cap M = \{\bar{f} = 0\}$

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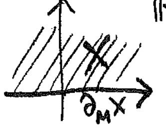
EXAMPLES:

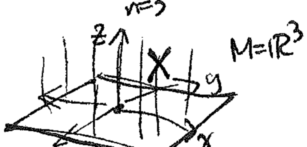
(1) Inside $M = \mathbb{R}^n$ itself,

$$\underset{\cup}{M} \quad X = \left\{ \bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_n \geq 0 \right\}$$

$$\underset{\cup}{X} \quad \partial_M X = \left\{ \bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_n = 0 \right\} = \partial_M^s X$$

i.e. the whole boundary $\partial_M X$ in M is smooth





$\partial_M X$ = xy-plane

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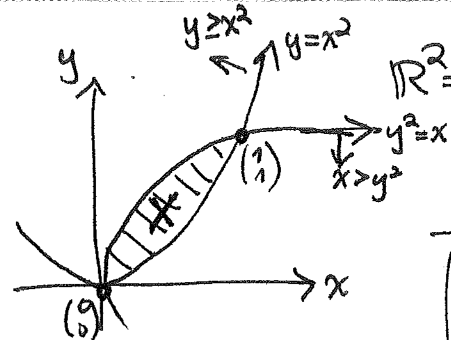
② (EXAMPLE 6.6.3)



$\mathbb{R}^2 = M$

$X = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{matrix} y \geq x^2, \\ x \geq y^2 \end{matrix} \right\}$

$\cup$

$\partial_M X = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X : \begin{matrix} y = x^2 \text{ or} \\ x = y^2 \end{matrix} \right\}$

$\partial_M^s X = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X : \begin{matrix} y = x^2 \text{ or} \\ x = y^2 \end{matrix} \text{ but } \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

see (NON-) EXAMPLE 6.6.4 in book for $X \subset \mathbb{R}^2 = M$ having $\partial_M^s X = \emptyset$!
(called the Koch snowflake, a fractal!)

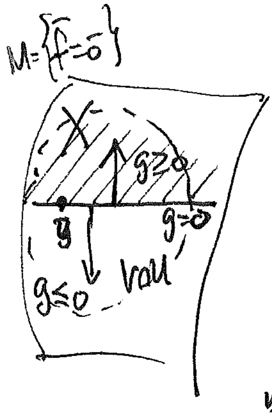
PROP 6.6.5: $\partial_M^s X$ is always a $(k-1)$-dim'l manifold.

proof: Because $\begin{pmatrix} \bar{f} \\ g \end{pmatrix} : U \longrightarrow \mathbb{R}^{\overbrace{n-k+1}^{n-(k-1)}}$ is in $C^1(U)$,

and $D\begin{pmatrix} \bar{f} \\ g \end{pmatrix}(\bar{x})$ is ~~surjective~~ (full rank i.e. $n-(k-1)$),

there is some neighborhood V of $\bar{x}$ with $D\begin{pmatrix} \bar{f} \\ g \end{pmatrix}(\bar{y})$ of full rank (why?),

and hence on $V \cap U$, the locus $Y = \begin{pmatrix} \bar{f} \\ g \end{pmatrix} = \bar{0}$ defines a $(k-1)$-dim'l manifold by Implicit Function Thm.



Then check that since $X \cap U = \{\bar{f} = \bar{0}, g \geq 0\}$, and $D\begin{pmatrix} \bar{f} \\ g \end{pmatrix}(\bar{y})$ has full rank for each $\bar{y} \in Y = \begin{pmatrix} \bar{f} \\ g \end{pmatrix} = \bar{0}$, the set $\partial_M X \cap (V \cap U) = Y$ ∎

not ~~too~~ hard; book suggests showing $T_{\bar{y}} M \xrightarrow{Dg} \mathbb{R}$ is surjective as a 1st step. Still a nontrivial exercise!

DEF'N 6.6.6: Call a compact subset $X \subset M$ (a k-dim'l manifold) a piece-with-boundary in M if
- $\mathrm{vol}_{k-1}(\partial_M^s X) < \infty$
- and $\mathrm{vol}_{k-1}(\underbrace{\partial_M X - \partial_M^s X}_{\text{non-smooth points in } \partial_M X}) = 0$

(Not used in our book, but X is just a plain-old manifold-with-boundary if $\partial_M X = \partial_M^s X$, i.e. its entire boundary in M is smooth)

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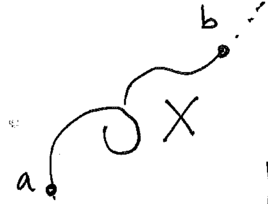
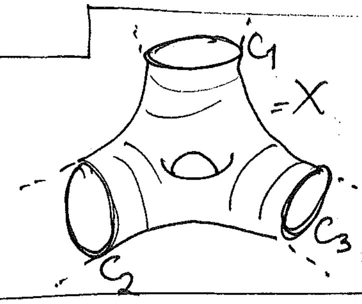
EXAMPLES:

① $k=1$ curves $X \subset C$ a smooth curve with endpoints



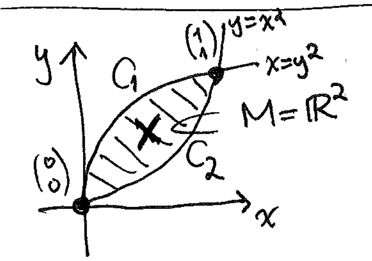
$\partial_C X = \{a,b\} = \partial^s_C X$

② $k=2$ surfaces $X \subset S$ a smooth surface, with boundary curves $C_1, C_2, \dots, C_r$



$\partial_S X = C_1 \sqcup C_2 \sqcup \dots \sqcup C_r = \partial^s_S X$

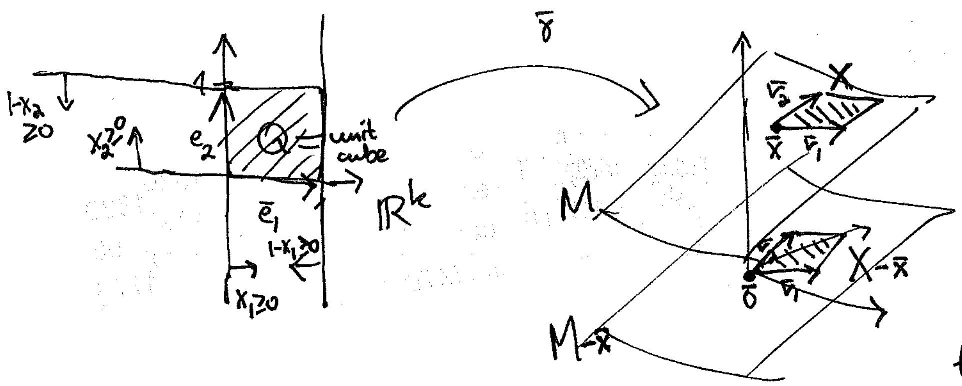
③



$\partial_M X = C_1 \cup C_2$

$\partial^s_M X = C_1 \cup C_2 - \{\binom{0}{0}, \binom{1}{1}\}$ (Why?)

④ Parallelepipeds (EXAMPLE 6.6.10) $X = P_{\bar x}(\bar v_1, \dots, \bar v_k) \subset \mathbb{R}^n$ are always pieces-with-boundary inside the k-dim'l manifold $M = \underbrace{\{\bar x + t_1 \bar v_1 + \dots + t_k \bar v_k : t_i \in \mathbb{R}\}}_{\text{the (affine) } k\text{-dim'l subspace containing } X}$



Not hard to check X is compact (=closed, bounded).

For each $(k-1)$-dim'l "face" of X, to get the appropriate functions

$$\mathbb{R}^n \xrightarrow{\binom{\bar f}{g}} \mathbb{R}^{n-(k-1)}$$
$$\bar y \longmapsto \binom{\bar f(\bar y)}{g(\bar y)}$$

it's probably easier to work with $X - \bar x$ $= P_{\bar 0}(\bar v_1, \dots, \bar v_k)$,

since then if one extends $\bar v_1, \dots, \bar v_k$ to a basis $\bar v_1, \dots, \bar v_k, \bar v_{k+1}, \dots, \bar v_n$ for $\mathbb{R}^n$, then the linear isomorphism

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{T} & \mathbb{R}^n \\ \bar v_1 & \longmapsto & \bar e_1 \\ \vdots & & \\ \bar v_k & \longmapsto & \bar e_k \\ \bar v_{k+1} & \longmapsto & \bar e_{k+1} \\ \vdots & & \\ \bar v_n & \longmapsto & \bar e_n \end{array}$$

lets one define $\bar f(\bar y) = \begin{pmatrix} T(\bar y)_{k+1} \\ \vdots \\ T(\bar y)_n \end{pmatrix}$

to cut out $M - \bar x$ as $\bar f^{-1}(\bar 0)$,

and cut out various faces/half spaces via $g(\bar y) = T(\bar y)_i \geq 0$ for $i = 1, 2, \dots, k$

or $1 - T(\bar y_i) \geq 0$

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⑤ See NON-EXAMPLES 6.6.8, 6.6.9 in book!