4/21/2017 →

(114) REMARK (similar to PROP 6.6.20)

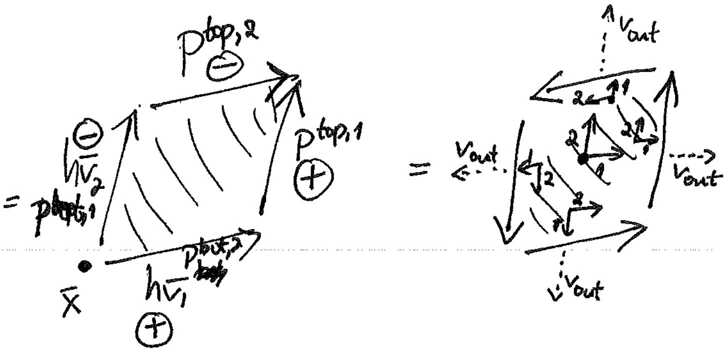
Perhaps before anything else we should remark that in

$$d\varphi(P_{\bar x}(\bar v_1,\dots,\bar v_{k+1})) := \lim_{h\to 0}\frac{1}{h^{k+1}} \underbrace{\sum_{i=1}^{k+1}(-1)^{i-1}\left[+\int_{P^{top,i}_{\bar x}(h\bar v_1,\dots,h\bar v_{k+1})}\varphi - \int_{P^{bot,i}_{\bar x}(h\bar v_1,\dots,h\bar v_{k+1})}\varphi\right]}$$

... this is simply the integral over the various connected components of the smooth boundary of $P_{\bar x}(h\bar v_1,\dots,h\bar v_{k+1})$ as a piece-with-boundary, taken with the induced boundary orientation

e.g. $k=1$

$P_{\bar x}(h\bar v_1, h\bar v_2) =$



Why? On $P^{top,i}$, the vector $h\bar v_i$ points outward

so $\Omega^{\partial}(h\bar v_1,\dots,\widehat{h\bar v_i},\dots,h\bar v_{k+1}) = \Omega(\underbrace{h\bar v_i}_{\bar v_{out}}, h\bar v_1,\dots,\widehat{h\bar v_i},\dots,h\bar v_{k+1})$

$= (-1)^{i-1}\underbrace{\Omega(h\bar v_1,\dots,h\bar v_i,\dots,h\bar v_{k+1})}_{+1}$

$= (-1)^{i-1}$

Similarly, on $P^{bot,i}$, the vector $h\bar v_i$ points inward

so $\Omega^{\partial}(h\bar v_1,\dots,\widehat{h\bar v_i},\dots,h\bar v_{k+1}) = -(-1)^{i-1}$.

(HE) ~~proof of~~ (sketch) THM 6.7.2: Property 2 (linearity) should be pretty clear from the limit of sums of integrals definition of $d\varphi$.

Property 4 is easy enough, but just a special case of **5**, and if we can prove 5 along with the special case of 1 where $\varphi = f(\bar{x})\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$, then the rest of 1 follows by linearity.

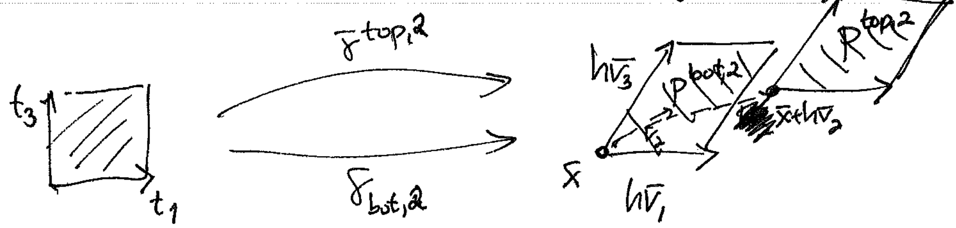
So we examine the i^{th} summand

$$d\varphi(P_{\bar{x}}(\bar{v}_1, \dots, \bar{v}_{k+1})) \overset{(*)}{:=} \lim_{h \to 0} \frac{1}{h^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \left[\int_{P^{top,i}(h\bar{v}_1, \dots, h\bar{v}_{k+1})} \varphi - \int_{P^{bot,i}(h\bar{v}_1, \dots, h\bar{v}_{k+1})} \varphi \right]$$

using parametrizations

$$\bar{t} = \begin{pmatrix} t_1 \\ \vdots \\ \hat{t}_i \\ \vdots \\ t_k \end{pmatrix} \begin{matrix} \in [0,1]^k \xmapsto{\gamma^{top,i}} \bar{x} + h\bar{v}_i + \sum_{j \neq i} t_j h\bar{v}_j \\ \in [0,1]^k \xmapsto{\gamma^{bot,i}} \bar{x} + \sum_{j \neq i} t_j h\bar{v}_j \end{matrix}$$

e.g. $k=3$, $i=2$



and the difference is

$$\int_{\bar{t} \in [0,1]^k} \left(f(\gamma^{top,i}(\bar{t})) - f(\gamma^{bot,i}(\bar{t})) \right) dx_{i_1} \wedge \dots \wedge dx_{i_k}(h\bar{v}_1, \dots, \widehat{h\bar{v}_i}, \dots, h\bar{v}_{k+1}) \, |d^k \bar{t}|$$

multilinearity

$$\overset{\text{multilinearity}}{=} h^k \int_{\bar{t} \in [0,1]^k} \underbrace{\left(f(\bar{x} + h\overbrace{(\bar{v}_i + \sum_{j \neq i} t_j \bar{v}_j)}^{\bar{u}_1}) - f(\bar{x} + h\overbrace{\sum_{j \neq i} t_j \bar{v}_j}^{\bar{u}_2}) \right)}_{= hDf(\bar{x})\bar{v}_i + o(h^2)} dx_{i_1} \wedge \dots \wedge dx_{i_j}(\bar{v}_1, \dots, \hat{\bar{v}}_i, \dots, \bar{v}_{k+1}) |d^k \bar{t}|$$

using Taylor expansion about $\bar{x}$ for f to say $f(\bar{x} + h\bar{u}) = f(\bar{x}) + hDf(\bar{x})\bar{u} + o(h^2)$ since $f \in C^2(\mathcal{U})$

Plugging into $(*)$ gives (being careful about the $\int_{\bar{t} \in [0,1]^k} o(h^2) |d^k \bar{t}|$ – see LEMMA A24.1)

$$d\varphi(P_{\bar{x}}(\bar{v}_1, \dots, \bar{v}_{k+1})) = \sum_{i=1}^{k+1} (-1)^{i-1} Df(\bar{x})\bar{v}_i \cdot dx_{i_1} \wedge \dots \wedge dx_{i_j}(\bar{v}_1, \dots, \hat{\bar{v}}_i, \dots, \bar{v}_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i-1} \sum_{l=1}^{n} \frac{\partial f}{\partial x_l}(\bar{x}) (\bar{v}_i)_l \cdot dx_{i_1} \wedge \dots \wedge dx_{i_j}(\bar{v}_1, \dots, \hat{\bar{v}}_i, \dots, \bar{v}_{k+1})$$

$$= \sum_{l=1}^{n} \frac{\partial f}{\partial x_l}(\bar{x}) \sum_{i=1}^{k+1} (-1)^{i-1} dx_l(\bar{v}_i) \cdot dx_{i_1} \wedge \dots \wedge dx_{i_j}(\bar{v}_1, \dots, \hat{\bar{v}}_i, \dots, \bar{v}_{k+1}) = \sum_{l=1}^{n} \frac{\partial f}{\partial x_l}(\bar{x})\, dx_l \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} (\bar{v}_1, \dots, \bar{v}_{k+1})$$

as desired ∎

(116)

Two more important properties of $\omega \overset{d}{\mapsto} d\omega$:

THM 6.7.7: 6.7.8: (i) For $\varphi \in A^k(U)$ having coefficients in $C^2(U)$, $d(d\varphi)=0$.

(ii) For $\varphi \in A^k(U)$ and $\psi \in A^l(U)$,

$$d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^k \varphi \wedge d\psi$$

proof: (ii) is EXER 6.7.11 on your HW!

For (i), it's enough to check it (by linearity of d) when

$$\varphi = f(\underline{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where one has

$$d(d\varphi) = d\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right)$$

$$= \underbrace{\left(\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i \right)}_{=0 \text{ since } dx_i \wedge dx_j = \begin{cases} 0 & \text{if } i=j \\ -dx_j \wedge dx_i & \text{if } i \neq j \end{cases} \text{ but } \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \qquad \blacksquare$$

4/24/2017 >

EXAMPLE: $\boxed{d(d\, x^2y^3) = d(2xy^3 dx + 3x^2y^2 dy) = 6xy^2 dy\, dx + 6xy^2 dx\, dy = 0}$

§6.8 Div, grad, & curl

There are 3 operations in vector calculus of $\mathbb{R}^3$ and physics that have nice interpretations/unifications via $\omega \mapsto d\omega$:

For $f: \mathbb{R}^3 \to \mathbb{R}$, $\quad \text{grad} f := \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix} = \nabla f$ = gradient of f

For a vector field $\vec{F}: \mathbb{R}^3 \to \mathbb{R}$, $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$, $\quad \text{curl}\, \vec{F} := \text{"}\nabla \times \vec{F}\text{"} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$ = curl of $\vec{F}$

$$= \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ -\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}$$

$\text{div}\, \vec{F} := \text{"}\nabla \bullet \vec{F}\text{"} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \bullet \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ = divergence of $\vec{F}$