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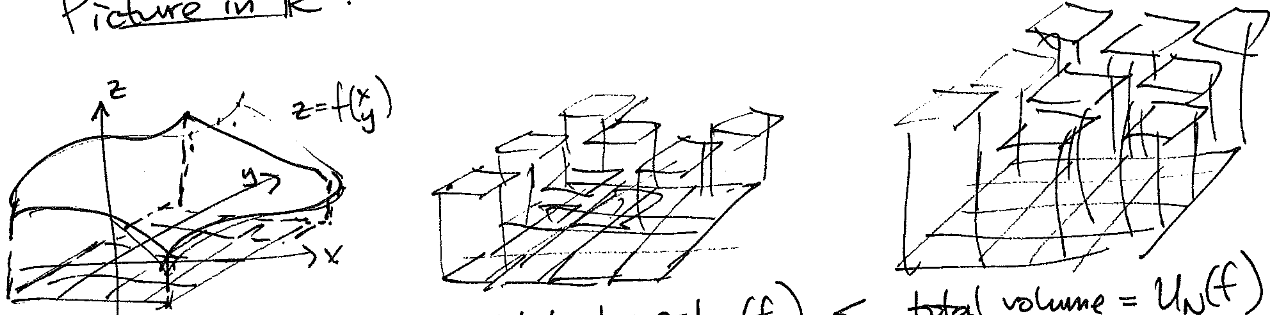
(36)

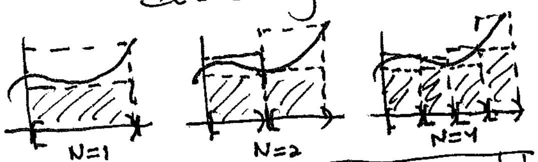
Picture in $\mathbb{R}^2$:



total volume $= L_N(f) \leq$ total volume $= U_N(f)$

It's easy to see that $L_N(f) \leq L_{N+1}(f) \leq L_{N+2}(f) \leq \ldots\ldots \subset \mathbb{R}$, so $\lim\limits_{N\to\infty} L_N(f)$ exists $=: L(f)$

$U_N(f) \geq U_{N+1}(f) \geq U_{N+2}(f) \geq \ldots\ldots \subset \mathbb{R}$, so $\lim\limits_{N\to\infty} U_N(f)$ exists $=: U(f)$



i.e. $L_N(f) \leq L_{N+1} \leq \ldots \leq \lim\limits_{N\to\infty} L_N(f) \leq \lim\limits_{N\to\infty} U_N(f) \leq \ldots \leq U_{N+1}(f) \leq U_N(f)$

$$\underset{L(f)}{\parallel} \quad \overset{?}{=} \quad \underset{U(f)}{\parallel}$$

called the lower and upper integrals of f (DEF'N 4.1.10)

DEF'N 4.1.12: Say $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ which is bounded, of bounded support, is (Riemann) integrable if $U(f) = L(f)$, and then define $\int_{\mathbb{R}^n} f(\vec{x})\,|d^n\vec{x}| := U(f) = L(f)$.

RMK: As the book says, it's hard to find simple enough $f(\vec{x})$'s to compute $\int_{\mathbb{R}^n} f(\vec{x})|d^n\vec{x}|$ directly from this defin, even in one variable, e.g. see $\int_0^1 x\,dx = \frac{1}{2}$

$$\parallel$$

$\int_{\mathbb{R}^1} f(x)\,|d^1x|$ where $f(x) = \begin{cases} x & \text{if } x\in[0,1] \\ 0 & \text{else} \end{cases}$

computed this way in EXAMPLE 4.1.13

More importantly, if we knew ahead of time that $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ is integrable, and if we make any choice of sample points $x_C \in C$ for all the cubes $C \in \mathcal{D}_N(\mathbb{R}^n)$, since $m_C(f) \leq f(x_C) \leq M_C(f)$

$\Rightarrow \quad L_N(f) \leq R(f,N) := \sum\limits_{C\in\mathcal{D}_N(\mathbb{R}^n)}' f(x_C)\,\mathrm{vol}_n(C) \leq U_N(f)$

$\Rightarrow \quad L(f) \leq \lim\limits_{N\to\infty} R(f,N) \leq U(f)$ i.e. $\boxed{\int_{\mathbb{R}^n} f(\vec{x})|d^n\vec{x}| = \lim\limits_{N\to\infty} R(f,N)}$

(37)

How do we know ahead of time some $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$ is integrable?

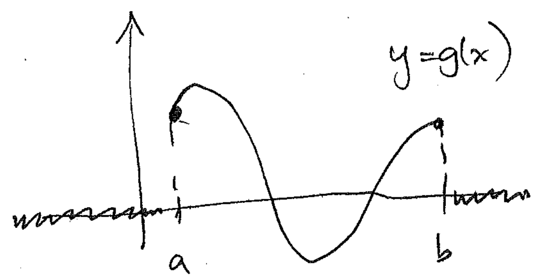
e.g. coming relatively soon is THM 4.3.10: If $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$ is (bounded, bounded support) continuous except on a set with zero volume (e.g. finitely many points), then f is integrable.

This is already reassuring, e.g. in this easy consequence: (Fund. Thm. Calc!)

COR: (not in book until Ch. 6, really) If $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$ is continuous, and $f(x) = \frac{dF}{dx}$ for $\mathbb{R}^1 \xrightarrow{F} \mathbb{R}$

then $\int_a^b f(x)\,dx \left(:= \int_{\mathbb{R}} \underbrace{1_{[a,b]}(x) f(x)}_{g(x):=} |d^1x| \right) = F(b) - F(a) \quad \forall a<b$

proof: Note $g(x) := 1_{[a,b]}(x) f(x)$ satisfies hypotheses of THM 4.3.10, so integrable.
(continuous except at $\{a,b\}$) finite!



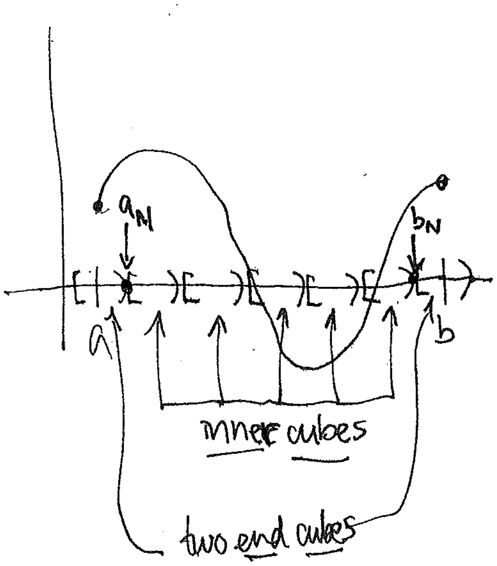
For each N and each cube C in the paving $\mathcal{D}_N(\mathbb{R}^1)$, $C = [a_c, b_c)$
carefully pick x_c via Mean Value Thm for F on $[a_c, b_c]$
to satisfy $F(b_c) - F(a_c) = F'(x_c)(b_c - a_c)$
$= f(x_c)\,\mathrm{vol}_1(C)$

Then $\int_{\mathbb{R}} 1_{[a,b]}(x) f(x) |d^1x| = \lim_{N\to\infty} R(f,N)$

$$= \lim_{N\to\infty} \sum_{\substack{\text{inner cubes} \\ C \in \mathcal{D}_N(\mathbb{R}^1)}} f(x_c)\,\mathrm{vol}_1(C) + \sum_{\substack{\text{two end cubes} \\ C \in \mathcal{D}_N(\mathbb{R}^1)}} f(x_c)\,\mathrm{vol}_1(C)$$

$$= \lim_{N\to\infty} \sum_{\text{inner } C} F(b_c) - F(a_c) + \lim_{N\to\infty} \sum_{\text{two end } C} f(x_c) \cdot \frac{1}{2^N}$$

($f(x_c)$ bounded by $M := \sup\{|f(x)| : x \in [a-\frac{1}{2}, a+\frac{1}{2}] \cup [b-\frac{1}{2}, b+\frac{1}{2}]\}$ — a compact set!)



$$= \lim_{N\to\infty} F(b_N) - F(a_N)$$

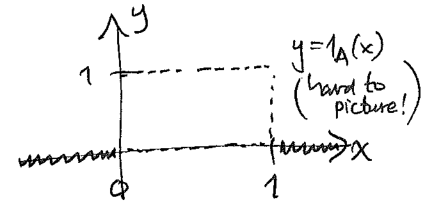
$= F(b) - F(a)$ since F is continuous (why?) and $\lim_{N\to\infty} a_N = a$, $\lim_{N\to\infty} b_N = b$ ∎

How to make a <u>nonintegrable</u> function?

The standard EXAMPLE (4.3.3): Let $A = \{ [0,1] \cap \mathbb{Q} \}$ (bounded ✓, bounded support ✓)

($\mathbb{Q}$: rational numbers $\frac{a}{b}$, $a \in \mathbb{Z}$, $b \in \mathbb{Z} - \{0\}$)

and $\mathbb{R}^n \xrightarrow{f = 1_A} \mathbb{R}$ the <u>indicator function</u> for A



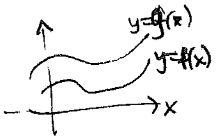
Then $L_N(f) = 0$ for all N (Why?)

$U_N(f) = 1$

so $L(f) \underset{\neq}{\leq} U(f)$.

For a while, we build up properties of integrability and $\int_{\mathbb{R}} f(\vec{x}) |d^n \vec{x}|$.

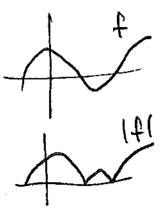
PROP 4.1.14: Assume $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable



(i) Then $f+g$ is also, and $\int_{\mathbb{R}^n} (f+g) |d^n \vec{x}| = \int_{\mathbb{R}^n} f |d^n \vec{x}| + \int_{\mathbb{R}^n} g |d^n \vec{x}|$

(ii) Then af is also for $a \in \mathbb{R}$, and $\int_{\mathbb{R}^n} af |d^n \vec{x}| = a \int_{\mathbb{R}^n} f |d^n \vec{x}|$.

(iii) If $f(\vec{x}) \leq g(\vec{x})$ $\forall \vec{x} \in \mathbb{R}^n$, then $\int_{\mathbb{R}^n} f |d^n \vec{x}| \leq \int_{\mathbb{R}^n} g |d^n \vec{x}|$



(iv) $|f|$ is also integrable, and $\left| \int_{\mathbb{R}^n} f |d^n \vec{x}| \right| \leq \int_{\mathbb{R}^n} |f| |d^n \vec{x}|$

proof: (i): Note every cube $C \in \mathcal{D}_N(\mathbb{R}^n)$ has

$$M_C(f) + M_C(g) \geq M_C(f+g)$$

$$m_C(f) + m_C(g) \leq m_C(f+g)$$

$$\Rightarrow L_N(f) + L_N(g) \leq L_N(f+g) \leq U_N(f+g) \leq U_N(f) + U_N(g)$$

$\downarrow$ take $\lim_{N \to \infty}(-)$, use Squeeze Thm. for limits

$$L(f) + L(g) \leq L(f+g) \leq U(f+g) \leq U(f) + U(g)$$

($L(f)+L(g)$ and $U(f)+U(g)$: equal)

$$\Rightarrow L(f+g) = U(f+g) = \int_{\mathbb{R}^n} (f+g) |d^n \vec{x}|$$

$$L(f) + L(g) = U(f) + U(g) = \int_{\mathbb{R}^n} f |d^n \vec{x}| + \int_{\mathbb{R}^n} g |d^n \vec{x}|.$$