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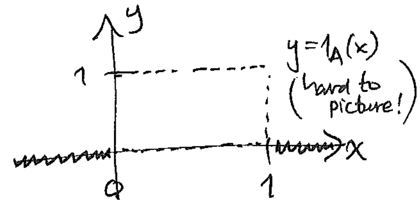
How to make a nonintegrable function?

The standard EXAMPLE (4.3.3): Let $A = \{ [0,1] \cap \underbrace{\mathbb{Q}}_{\text{rational numbers } \frac{a}{b},\ a\in\mathbb{Z},\ b\in\mathbb{Z}-\{0\}} \}$ (bounded ✓, bounded support ✓)

and $\mathbb{R}^1 \xrightarrow{f=1_A} \mathbb{R}$ the indicator function for A



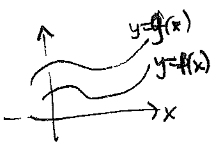
Then $L_N(f) = 0$ for all N (Why?)

$U_N(f) = 1$

so $L(f) \underset{\neq}{<} U(f)$.

For a while, we build up properties of integrability and $\int_{\mathbb{R}} f(\bar{x})\,|d^n\bar{x}|$.

PROP 4.1.14: Assume $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable

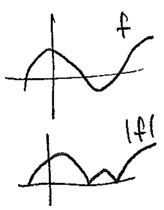


(i) Then $f+g$ is also, and $\int_{\mathbb{R}^n} (f+g)\,|d^n\bar{x}| = \int_{\mathbb{R}^n} f\,|d^n\bar{x}| + \int_{\mathbb{R}^n} g\,|d^n\bar{x}|$

(ii) Then af is also for $a \in \mathbb{R}$, and $\int_{\mathbb{R}^n} af\,|d^n\bar{x}| = a \int_{\mathbb{R}^n} f\,|d^n\bar{x}|$.

(iii) If $f(\bar{x}) \le g(\bar{x})\ \forall \bar{x} \in \mathbb{R}^n$, then $\int_{\mathbb{R}^n} f\,|d^n\bar{x}| \le \int_{\mathbb{R}^n} g\,|d^n\bar{x}|$

(iv) $|f|$ is also integrable, and $\left| \int_{\mathbb{R}^n} f\,|d^n\bar{x}| \right| \le \int_{\mathbb{R}^n} |f|\,|d^n\bar{x}|$



proof: (i): Note every cube $C \in \mathcal{D}_N(\mathbb{R}^n)$ has

$$M_C(f) + M_C(g) \ge M_C(f+g)$$
$$m_C(f) + m_C(g) \le m_C(f+g)$$

$$\Rightarrow L_N(f) + L_N(g) \le L_N(f+g) \le U_N(f+g) \le U_N(f) + U_N(g)$$

$\Big\downarrow$ take $\lim_{N\to\infty}(-)$, use Squeeze Thm. for limits

$$\underbrace{L(f) + L(g) \le L(f+g) \le U(f+g) \le U(f) + U(g)}_{\text{equal}}$$

$$\Rightarrow L(f+g) = U(f+g) = \int_{\mathbb{R}^n} (f+g)\,|d^n\bar{x}|$$

$$\| \qquad\qquad \|$$

$$L(f) + L(g) = U(f) + U(g) = \int_{\mathbb{R}^n} f\,|d^n\bar{x}| + \int_{\mathbb{R}^n} g\,|d^n\bar{x}|.$$

2/15/2017 (ii) Treat the $a>0$ and $a<0$ cases separately

$$U_N(af) = aU_N(f) \qquad\qquad U_N(af) = aL_N(f)$$
$$L_N(af) = aL_N(f) \qquad\qquad L_N(af) = aU_N(f)$$

Why?

... rest is easy from limit laws | ... rest is easy from limit laws

(iii) $U_N(f) \le U_N(g)$ and rest is easy.

Think about this in terms of sups & infs!

(iv) ~~One~~ each cube C, $M_C(|f|) - m_C(|f|) \le M_C(f) - m_C(f)$

$$\Rightarrow \forall N,\ \underbrace{U_N(|f|) - L_N(|f|)}_{\ge 0} \le \underbrace{U_N(f) - L_N(f)}_{\ge 0}$$

$$\Rightarrow 0 \le \lim_{N\to\infty} U_N(|f|) - L_N(|f|) \le \lim_{N\to\infty} U_N(f) - L_N(f) = 0$$

i.e. $U(|f|) \le L(|f|)$, so $|f|$ is integrable

Also on each cube C, $|M_C(f)| \le M_C(|f|)$ Think about this! in terms of sups

$$\Rightarrow |U_N(f)| \le U_N(|f|)\ \forall N$$

$$\Rightarrow \left|\int f\,|d^n\vec{x}|\right| \le \int |f|\,|d^n\vec{x}| \qquad \blacksquare$$

The next fact gets used, e.g., in calculating volumes of boxes

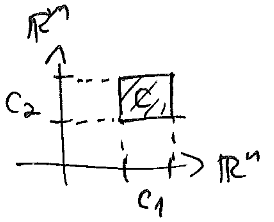
PROP 4.1.16:
$$\left.\begin{array}{l} \mathbb{R}^n_{\vec{x}} \xrightarrow{f_1} \mathbb{R}_{f_1(\vec{x})} \\ \mathbb{R}^m \xrightarrow{f_2} \mathbb{R} \\ \vec{y} \longmapsto f_2(\vec{y}) \end{array}\right\} \text{both integrable} \Rightarrow \begin{array}{l} \mathbb{R}^{n+m} \xrightarrow{f_1 f_2} \mathbb{R} \text{ integrable} \\ \begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix} \longmapsto f_1(\vec{x})f_2(\vec{y}) \end{array}$$

and
$$\int_{\mathbb{R}^{n+m}} f_1 f_2\,|d^n\vec{x}|\,|d^m\vec{y}| = \left(\int_{\mathbb{R}^n} f_1\, d^n\vec{x}\right)\left(\int_{\mathbb{R}^m} f_2\, d^m\vec{y}\right)$$

proof: First prove it in the special case where $f_1(\vec{x}) \ge 0\ \forall \vec{x} \in \mathbb{R}^n$, $f_2(\vec{y}) \ge 0\ \forall \vec{y} \in \mathbb{R}^m$

Then cubes $C \in \mathcal{D}_N(\mathbb{R}^{n+m})$ are products $C = C_1 \times C_2$, $C_1 \in \mathcal{D}_N(\mathbb{R}^n)$, $C_2 \in \mathcal{D}_N(\mathbb{R}^m)$

$\mathrm{vol}_{n+m}(C) = \mathrm{vol}_n(C_1)\cdot \mathrm{vol}_m(C_2)$



and $m_C(f_1f_2) = m_{C_1}(f_1)\,m_{C_2}(f_2)$

$M_C(f_1f_2) = M_{C_1}(f_1)\,M_{C_2}(f_2)$

using $f_1 \ge 0$, $f_2 \ge 0$ here

$$\Rightarrow \begin{array}{l} U_N(f_1f_2) = U_N(f_1)U_N(f_2) \\ L_N(f_1f_2) = L_N(f_1)L_N(f_2) \end{array} \xrightarrow{\lim_{N\to\infty}} \begin{array}{l} U(f_1f_2) = U(f_1)U(f_2) \\ \qquad\qquad\quad \parallel \\ L(f_1f_2) = L(f_1)L(f_2). \end{array}$$

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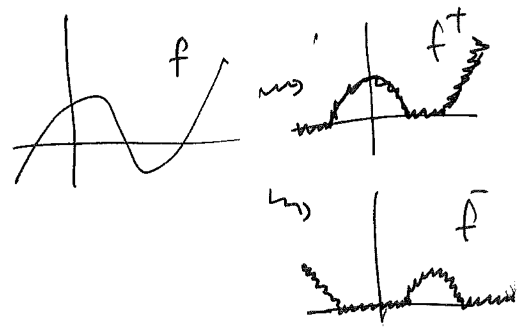
In general case, write $f_1 = f_1^+ - f_1^-$ where $f^+(\bar{x}) := \begin{cases} f(\bar{x}) & \text{if } f(\bar{x}) \geq 0 \\ 0 & \text{if } f(\bar{x}) \leq 0 \end{cases}$

$f_2 = f_2^+ - f_2^-$ $\qquad f^-(\bar{x}) := \begin{cases} -f(\bar{x}) & \text{if } f(\bar{x}) \leq 0 \\ 0 & \text{if } f(\bar{x}) \geq 0 \end{cases}$

and note that f integrable

PROP 4.1.4 (i),(iv)

$$\xRightarrow{\text{PROP 4.1.4 (i),(iv)}} \left\{ \begin{aligned} f^+ &= \tfrac{1}{2}(|f| + f) \\ f^- &= \tfrac{1}{2}(|f| - f) \end{aligned} \right\} \text{ are both integrable, nonnegative}$$



so $\int f_1 f_2 = \int (f_1^+ - f_1^-)(f_2^+ - f_2^-)$

$= \int (f_1^+ f_2^+ - f_1^+ f_2^- - f_1^- f_2^+ + f_1^- f_2^-)$

$= \int f_1^+ f_2^+ - \int f_1^+ f_2^- - \int f_1^- f_2^+ + \int f_1^- f_2^-$

by nonnegative case that we just did above

$\overset{\curvearrowright}{=} \int f_1^+ \int f_2^+ - \int f_1^+ \int f_2^- - \int f_1^- \int f_2^+ + \int f_1^- \int f_2^-$

$= \left(\int f_1^+ - \int f_1^-\right)\left(\int f_2^+ - \int f_2^-\right)$

$= \int f_1 \int f_2$ ∎

Now we can start talking about <u>volume</u> of (some) subsets $A \subset \mathbb{R}^n$

DEF'N 4.1.17: 4.1.18 Say $A \subset \mathbb{R}^n$ is <u>pavable</u> if $1_A(\bar{x}) = \begin{cases} 1 & \text{if } \bar{x} \in A \\ 0 & \text{else} \end{cases}$

is integrable, in which case we say $\mathrm{vol}_n(A) := \int_{\mathbb{R}^n} 1_A(\bar{x}) \, |d^n \bar{x}|$.

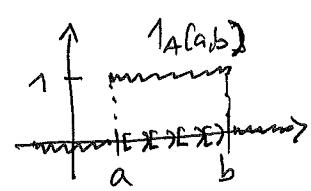
EXAMPLES: ① $A = [0,1] \cap \mathbb{Q}$ is <u>not</u> pavable,

so (for the moment until §4.11) we <u>don't define its volume</u>

② An interval $[a,b] \subset \mathbb{R}^1$ has $\mathrm{vol}_n([a,b]) = b - a$

by direct calculation that $L_N(1_{[a,b]}) \xrightarrow{N \to \infty} b-a$

$U_N(1_{[a,b]}) \xrightarrow{N \to \infty} b-a$



③ Since a box $P = [a_1,b_1] \times [a_2,b_2] \times \ldots \times [a_n,b_n] \subset \mathbb{R}^n$ has $1_P(\bar{x}) = 1_{[a_1,b_1]} 1_{[a_2,b_2]} \cdots 1_{[a_n,b_n]}$,

PROP 4.1.16 shows P is pavable, of volume $(b_1 - a_1) \cdots (b_n - a_n)$.

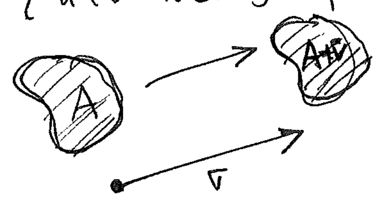
A few more reasonable properties of $\mathrm{vol}_n(-)$:

PROP 4.1.21, 4.1.22, 4.1.24: Assume $A, B \subset \mathbb{R}^n$ are pavable.

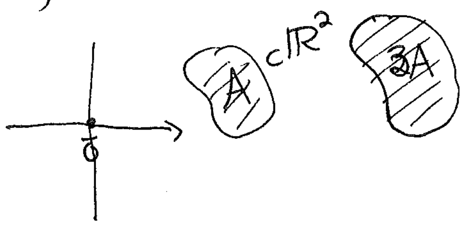
(i) If A, B disjoint ($A \cap B = \emptyset$), then $A \cup B$ is pavable, and $\mathrm{vol}_n(A \cup B) = \mathrm{vol}_n A + \mathrm{vol}_n B$

(In fact, $A_1, \ldots, A_m$ disjoint, pavable $\Rightarrow \bigcup_{i=1}^m A_i$ pavable, $\mathrm{vol}_n\left(\bigcup_{i=1}^m A_i\right) = \sum_{i=1}^m \mathrm{vol}_n A_i$)

(ii) $\forall \bar{v} \in \mathbb{R}^n$, $A + \bar{v} := \{\bar{a} + \bar{v} : \bar{a} \in A\}$ is pavable, and $\mathrm{vol}_n(A + \bar{v}) = \mathrm{vol}_n(A)$



(iii) $\forall t \in \mathbb{R}$, $tA := \{t\bar{a} : \bar{a} \in A\}$ is pavable, and $\mathrm{vol}_n(tA) = t^n \mathrm{vol}_n(A)$

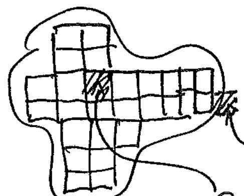


← area $= 3^2$(area of A)

proof: (i): If $A_1, \ldots, A_n$ are disjoint, then $1_{A_1 \cup \cdots \cup A_n} = 1_{A_1} + 1_{A_2} + \cdots + 1_{A_n}$

(so done by result on $\int f+g = \int f + \int g$).

(ii): For each N, $1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \subset A} C} \le 1_A \le 1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \cap A \neq \emptyset} C}$



$C \cap A \neq \emptyset$ (but $C \not\subset A$)

$C \subset A$

and similarly $1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \subset A} C + \bar{v}} \le 1_{A + \bar{v}} \le 1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \cap A \neq \emptyset} C + \bar{v}}$

2/17/2017 >

$\Downarrow$

Because $\bigcup C + \bar{v}$ is a disjoint union of pavable sets $C + \bar{v}$, each a box, and $\mathrm{vol}_n(C + \bar{v}) = \mathrm{vol}_n C$

same reason as on left!

$$L\Big(1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \subset A} C + \bar{v}}\Big) \le L(1_{A+\bar{v}}) \le U(1_{A+\bar{v}}) \le U\Big(1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \cap A \neq \emptyset} C + \bar{v}}\Big)$$

$$L\Big(1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \subset A} C}\Big) = L_N(1_A) \xrightarrow{N \to \infty} L(1_A) = \mathrm{vol}_n A = U(1_A) \xleftarrow{N \to \infty} U_N(1_A) = U\Big(1_{\bigcup_{C \in \mathcal{D}_N(\mathbb{R}^n):\, C \cap A \neq \emptyset} C}\Big)$$

forces equality here!