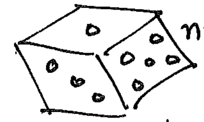
EXAMPLES:

① (discrete) $S = \{1, 2, \ldots, 2n\}$ = rolls of a 2n-sided fair die



has $\mu(x_i) = \frac{1}{2n}$ for each $x_i = 1, 2, \ldots, 2n$

(called the uniform distribution or uniform sample space on S)

If we let $S \xrightarrow{f} \mathbb{R}$

$x_i \longmapsto x_i$ = # value of the roll

then it has mean/expectation $E(f) = \sum_{x_i \in S}' f(x_i)\mu(x_i) = \sum_{i=1}^{2n} i \cdot \frac{1}{2n} = \frac{1}{2n}(1+2+\ldots+2n)$

$$= \frac{1}{2n}\binom{2n+1}{2} = \frac{1}{\cancel{2n}} \frac{\cancel{2n}(2n+1)}{2}$$

$$= \frac{2n+1}{2} = n + \frac{1}{2} \overset{n=3}{\rightsquigarrow} 3\tfrac{1}{2}$$

This is how many \$ you should pay to play if the die roll pays off as many dollars x_i as you rolled.

variance $\mathrm{var}(f) = \sum_{x_i \in S}' (f(x_i) - E(f))^2 \mu(x_i) = E(f - E(f)^2) = E(f^2) - E(f)^2$

$$= \underbrace{\sum_{i=1}^{2n} i^2 \cdot \frac{1}{2n}}_{E(f^2)} - \underbrace{(n+\tfrac{1}{2})^2}_{E(f)^2} = \frac{1}{2n}\sum_{i=1}^{2n} i^2 - (n^2 + n + \tfrac{1}{4})$$

$$= \frac{1}{\cancel{2n}} \frac{\cancel{2n}(2n+1)(4n+1)}{6} - (n^2 + n + \tfrac{1}{4})$$

$$= \frac{8n^2 + 6n + 1}{6} - (n^2 + n + \tfrac{1}{4})$$

$$= \frac{n^2}{3} - \frac{1}{12}$$

standard deviation $\sigma(f) = \sqrt{\mathrm{var}(f)} = \sqrt{\frac{n^2}{3} - \frac{1}{12}} \approx \frac{n}{\sqrt{3}}$ for n large

② EXAMPLE 4.2.12 (discrete) You are flipping a fair coin repeatedly, and seeing how many flips x_i before the 1st heads occurs

$\rightsquigarrow$ sample space $S = \{1, 2, 3, \ldots\}$ ← countable! (~~not finite~~)

with
$\mu(1) = \frac{1}{2}$ (H, ...)
$\mu(2) = \frac{1}{2^2}$ (T, H, ...)
$\mu(3) = \frac{1}{2^3}$ (T, T, H, ...)
⋮
$\mu(i) = \frac{1}{2^i}$ (T, T, ..., T, H, ...) ← ith

(Note that here $\mu(S) = \sum_{i=1}^{\infty} \mu(i) = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$
$= \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$ ✓)

Q: How much should you pay to play for a payoff of

$$S \xrightarrow{f_1} \mathbb{R}, \quad i \longmapsto i = f_1(i) \ ?$$

i.e. you get i\$ if 1st heads occurs on ith flip

What about for a payoff of

$$S \xrightarrow{f_2} \mathbb{R}, \quad i \longmapsto 2^i \ ?$$

$$E(f_1) = \sum_{x_i \in S} f_1(x_i)\mu(x_i) = \sum_{i=1}^{\infty} i \cdot \frac{1}{2^i}$$

$$= \left[\sum_{i=1}^{\infty} \frac{i x^{i-1}}{2^i}\right]_{x=1} = \left[\frac{d}{dx}\sum_{i=0}^{\infty} \frac{x^i}{2^i}\right]_{x=1}$$

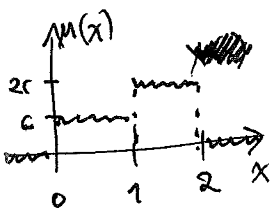
$$= \left[\frac{d}{dx}\, \frac{1}{1-x/2}\right]_{x=1} = \left[\frac{-(-\frac{1}{2})}{(1-x/2)^2}\right]_{x=1} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = \frac{1/2}{1/4} = 2$$

Pay 2\$

$$E(f_2) = \sum_{x_i \in S} f_2(x_i)\mu(x_i) = \sum_{i=1}^{\infty} 2^i \cdot \frac{1}{2^i} = \underbrace{(1+1+\dots)}_{\infty\,!}$$

Pay ∞ \$

(3) (continuous) Suppose $\mu(x)$ is a prob. density on $\mathbb{R}^1$ with $\mu(x) = \begin{cases} 0 & \text{if } x \notin [0,2] \\ c & \text{if } x \in [0,1] \\ 2c & \text{if } x \in [1,2] \end{cases}$



(i) What should c be?

Need $1 = \int_{\mathbb{R}^1} \mu(x)|dx| = \int_0^1 c\,dx + \int_1^2 2c\,dx = c + 2c = 3c$ i.e. $c = \frac{1}{3}$

(ii) For the random variable $\mathbb{R}^1 \xrightarrow{f} \mathbb{R}$, $x \longmapsto f(x) = x$, what is $E(f)$?

$$E(f) = \int_{\mathbb{R}^1} f(x)\mu(x)|dx| = \int_0^1 x \cdot \frac{1}{3}dx + \int_1^2 x \cdot \frac{2}{3}dx$$
$$= \left[\frac{x^2}{6}\right]_0^1 + \left[\frac{x^2}{3}\right]_1^2$$
$$= \frac{1}{6} - \frac{0}{6} + \frac{4}{3} - \frac{1}{3} = \frac{7}{6}$$

make sense?

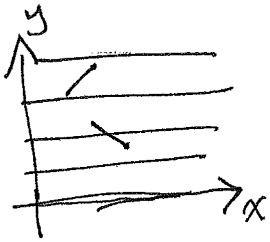
(iii) What are var(f), σ(f)?

$$\mathrm{var}(f) = E(f^2) - (Ef)^2 = \int_0^1 x^2 \cdot \frac{1}{3}dx + \int_1^2 x^2 \cdot \frac{2}{3}dx - \left(\frac{7}{6}\right)^2 = \left[\frac{x^3}{9}\right]_0^1 + \left[\frac{2x^3}{9}\right]_1^2 - \left(\frac{7}{6}\right)^2$$
$$= \frac{1}{9} - \frac{0}{9} + \frac{16}{9} - \frac{2}{9} - \left(\frac{7}{6}\right)^2 = \frac{19}{9} - \frac{49}{36}$$
$$= \frac{76-49}{36} = \frac{27}{36} = \frac{3}{4}$$

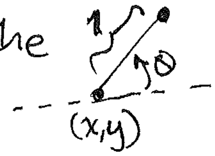
$$\sigma(f) = \sqrt{\mathrm{var}(f)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

(49)

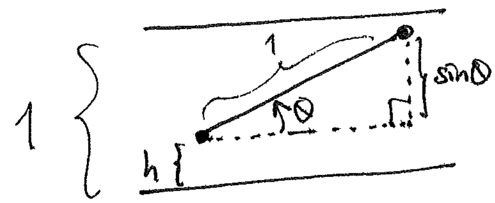
④ (Example 4.2.7) Buffon's needle problem: A needle of length 1 is dropped on lined paper ruled 1 unit apart. What's the probability it intersects a line?



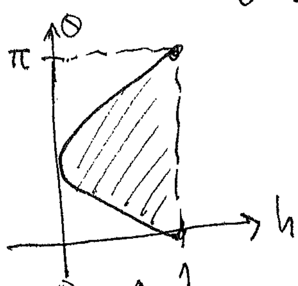
Model this as choosing the



x, y, θ at random with $\theta \in [0,\pi)$. But the x is irrelevant, and we only care about the fractional part $h = y - \underbrace{\lfloor y \rfloor}_{\text{integer part of } y\text{, rounded down}}$ of y with $h \in [0,1)$:



The answer is "yes it intersects a line" if $\boxed{h + \sin\theta > 1}$, so inside $\mathbb{R}^2$ with coordinates (h,θ) and



prob. density $\mu\begin{pmatrix}h\\\theta\end{pmatrix} = \begin{cases} 0 & \text{if } \begin{pmatrix}h\\\theta\end{pmatrix} \notin [0,1)\times[0,\pi) \\ \frac{1}{\pi} \leftarrow \text{why?} & \text{if } \begin{pmatrix}h\\\theta\end{pmatrix} \in [0,1)\times[0,\pi) \end{cases}$

we want Prob(A) where $A = \left\{ \begin{pmatrix}h\\\theta\end{pmatrix} \in [0,1)\times[0,\pi) : \underset{h > 1-\sin\theta}{h+\sin\theta > 1} \right\}$

$\|$

$$\int_{\mathbb{R}^2} 1_A\begin{pmatrix}h\\\theta\end{pmatrix} \mu\begin{pmatrix}h\\\theta\end{pmatrix} \left|d^2\begin{pmatrix}h\\\theta\end{pmatrix}\right| = \frac{1}{\pi}(\text{area shown here}) = \frac{1}{\pi}\left(\int_0^\pi \sin\theta \, d\theta\right) = \frac{1}{\pi}\left[-\cos\theta\right]_0^\pi$$
$$= \frac{1}{\pi}\left[-(-1)-(-1)\right]$$
$$= \frac{2}{\pi}$$

Read about...

- $\underset{\substack{\text{covariance} \\ \text{of } f,g : S \to \mathbb{R}}}{\mathrm{Cov}(f,g)} := E\big((f - Ef)(g - Eg)\big) = E(fg) - E(f)\cdot E(g)$
- and $\underset{\text{correlation coefficient of } f,g}{\mathrm{corr}(f,g)} := \dfrac{\mathrm{Cov}(f,g)}{\sigma(f)\sigma(g)} \in [-1,+1]$

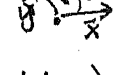
Think:

$\mathrm{var}(f) \longleftrightarrow \vec{x}^T \cdot \vec{x} = \|\vec{x}\|^2$

$\sigma(f) \longleftrightarrow \|\vec{x}\|$

$\mathrm{cov}(f,g) \longleftrightarrow \vec{x}^T\vec{y} = \vec{x}\cdot\vec{y}$

$\mathrm{corr}(f,g) \longleftrightarrow \dfrac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\|\vec{y}\|} = \cos\theta$ if



- Central limit theorem
- Center of mass of $A \subset \mathbb{R}^1$ is $\frac{1}{\mathrm{vol}_1 A}\int_A x\,|dx|$, of $A \subset \mathbb{R}^2$ is $\frac{1}{\mathrm{vol}_2 A}\begin{pmatrix}\int_A x \left|d^2\binom{x}{y}\right| \\ \int_A y \left|d^2\binom{x}{y}\right|\end{pmatrix} = \begin{pmatrix}\text{"expected x coord"} \\ \text{"expected y coord"}\end{pmatrix}$