(20)

We have Q pos. def. on V_0 since any $\bar{x} \in V_0$ has

$$Q(\bar{x}) = \alpha_1(\bar{x})^2 + \dots + \alpha_k(\bar{x})^2 - \underbrace{\alpha_{k+1}(\bar{x})^2}_{0} - \dots - \underbrace{\alpha_{k+l}(\bar{x})^2}_{0} \geq 0$$

with $Q(\bar{x}) = 0 \iff \alpha_1(\bar{x}) = \dots = \alpha_k(\bar{x}) = 0$

$\iff \alpha_1(\bar{x}) = \dots = \alpha_k(\bar{x}) = \alpha_{k+1}(\bar{x}) = \dots = \alpha_n(\bar{x}) = 0$

since $\bar{x} \in V_0 = \ker \varphi$

$\iff \bar{x} = \bar{0}$

since $\alpha_1, \alpha_2, \dots, \alpha_n$ are a basis for all linear functions $\mathbb{R}^n \to \mathbb{R}$ ∎

2/6/2017 >

This now equips us for...

§3.6 Classifying critical points

THM 3.6.3: Given $\underset{\mathbb{R}^n}{\underset{\cap}{U}} \xrightarrow{f} \mathbb{R}$ (U open) with f differentiable,

if $\bar{x}_0 \in U$ is an extremum (max or min) for f on U

then $\underset{\underset{\nabla f(\bar{x}_0)^T}{\|}}{\underset{\left[\frac{\partial f}{\partial x_1}(\bar{x}_0) \ \dots \ \frac{\partial f}{\partial x_n}(\bar{x}_0)\right]}{\underset{\|}{[Df(\bar{x}_0)]}}} = \bar{0}$, i.e. $\frac{\partial f}{\partial x_1}(\bar{x}_0) = \dots = \frac{\partial f}{\partial x_n}(\bar{x}_0) = 0$

(in which case we call $\bar{x}_0$ a critical point for f and $f(\bar{x}_0)$ a critical value for f; DEF'N 3.6.4)

proof: We've seen this argument before: assume the min case, and

$$\frac{\partial f}{\partial x_i}(\bar{x}_0) = \lim_{t \to 0} \frac{f(\bar{x}_0 + te_i) - f(\bar{x}_0)}{t}$$ exists since f is diff'ble at $\bar{x}_0$

$$= \begin{cases} \lim\limits_{t \to 0^+} \dfrac{\overbrace{f(\bar{x}_0 + te_i) - f(\bar{x}_0)}^{\text{nonnegative}}}{\underbrace{t}_{\text{positive}}} \Rightarrow \geq 0 \\[2ex] \lim\limits_{t \to 0^-} \dfrac{\overbrace{f(\bar{x}_0 + t\bar{e}_i) - f(\bar{x}_0)}^{\text{nonnegative}}}{\underbrace{t}_{\text{negative}}} \Rightarrow \leq 0. \end{cases}$$

so $\frac{\partial f}{\partial x_i}(\bar{x}_0) = 0$ ∎

(21)

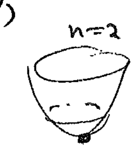
THM 3.6.9. If $\underset{\mathbb{R}^n}{\overset{\text{open}}{U}} \xrightarrow{f} \mathbb{R}$ has a critical point $\bar{a} \in U$

and $f(\bar{a}+\bar{h}) = \underbrace{f(\bar{a}) + \frac{1}{2}\bar{h}^T H \bar{h}}_{P^2_{f,\bar{a}}(\bar{a}+\bar{h})} + o(|\bar{h}|^2)$ with $H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\bar{a})\right]$

the Hessian at $\bar{a}$ of signature (k,ℓ),

then

(i) $k=n$ (i.e. $(k,\ell)=(n,0)$) $\Rightarrow$ $f(\bar{a})$ is a local min

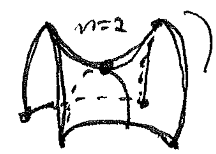


(ii) $\ell=n$ (i.e. $(k,\ell)=(0,n)$) $\Rightarrow$ $f(\bar{a})$ is a local max

(iii) $k>0$ $\Rightarrow$ $f(\bar{a})$ is not a local max since
$\exists\, \bar{b}$ arbitrarily close to $\bar{a}$ with $f(\bar{b})>f(\bar{a})$

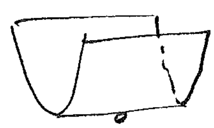
(iv) $\ell>0$ $\Rightarrow$ —"— local min —"—
—"— $f(\bar{b})<f(\bar{a})$

(and if both $k,\ell>0$, $f(\bar{a})$ is called a saddle point)

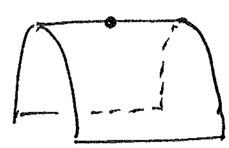


REMARK: $\left\{\begin{matrix}(k,0) \text{ with } k<n \\ (0,\ell) \text{ with } \ell<n\end{matrix}\right\}$ require further Taylor terms to analyze

called degenerate critical points



$(1,0)$
n=2



$(0,1)$
n=2

---

proof of THM 3.6.9: (ii) follows from (i), and (iv) from (iii), replacing f by $-f$.

Note $\frac{f(\bar{a}+\bar{h})-f(\bar{a})}{|\bar{h}|^2} = \frac{1}{2}\frac{\bar{h}^T H \bar{h}}{|\bar{h}|^2} + \underbrace{\frac{o(|\bar{h}|^2)}{|\bar{h}|}}_{\to 0 \text{ as } \bar{h}\to\bar{0}}$

So to prove (i), it's enough to check $\lim_{\bar{h}\to 0} \frac{\bar{h}^T H \bar{h}}{|\bar{h}|^2} > C$ for some positive constant $C>0$.

In fact, if H has orthonormal eigenbasis $\bar{v}_1, \ldots, \bar{v}_n$
with $0 < \lambda_1 \leq \ldots \leq \lambda_n$ as eigenvalues
since $k=n$

then we'll show $\frac{\bar{h}^T H \bar{h}}{|\bar{h}|^2} > \lambda_1$:

(22) Write $\bar h = c_1\bar v_1 + \dots + c_n \bar v_n$ for some $c_i \in \mathbb{R}$.

Then $|\bar h|^2 = \bar h^T \bar h = \left(\sum_{i=1}^n c_i \bar v_i\right)^T \left(\sum_{i=1}^n c_i \bar v_i\right) = c_1^2 + c_2^2 + \dots + c_n^2$

because $\bar v_i^T \bar v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

while
$$\bar h^T H \bar h = \left(\sum_{i=1}^n c_i \bar v_i\right)^T \left(\sum_{i=1}^n c_i H \bar v_i\right)$$
$$= \left(\sum_{i=1}^n c_i \bar v_i\right)\left(\sum_{i=1}^n c_i \lambda_i \bar v_i\right)$$
$$= \lambda_1 c_1^2 + \lambda_2 c_2^2 + \dots + \lambda_n c_n^2$$
$$\geq \lambda_1 c_1^2 + \lambda_1 c_2^2 + \dots + \lambda_1 c_n^2 = \lambda_1 (c_1^2 + \dots + c_n^2)$$

$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

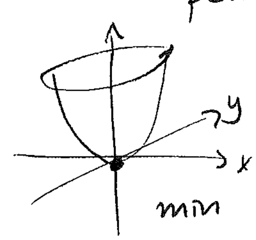
$$\Rightarrow \frac{\bar h^T H \bar h}{\bar h^T \bar h} \geq \frac{\lambda_1\left(\sum_{i=1}^n c_i^2\right)}{\sum_{i=1}^n c_i^2} = \lambda_1 \ (>0).$$

To prove (ii), it's enough to check that if $H\bar v = \lambda \bar v$ with $\lambda > 0$ (and such a $\bar v$ exists since $k>0$)

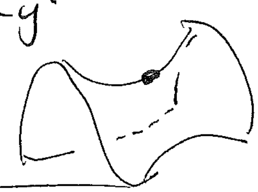
then taking $\bar h = t\bar v$,
$$\frac{\bar h^T H \bar h}{|\bar h|^2} > 0$$
$$\frac{(t\bar v)^T H (t\bar v)}{|t\bar v|^2}$$
$$= \frac{(t\bar v)^T \lambda t \bar v}{|t\bar v|^2} = \lambda \frac{(t\bar v)^T t\bar v}{|t\bar v|^2} = \lambda \ (>0) \quad \blacksquare$$

(Book's) EXAMPLES of degenerate critical points all with $P_{f,0} = x^2$ and $n=2$, signature of H being $(1,0)$:

$f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^4$

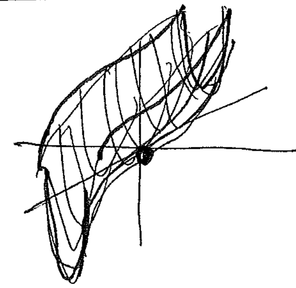


$f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^4$



saddle

$f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^3$



see book's much better plot!

§3.7 Constrained extrema & Lagrange multipliers

How to find the points $\bar{c}$ on a k-dimensional manifold $M \subset \mathbb{R}^n$ that give (local) maxes/mins for some $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$?

If we have a local <u>parametrization</u> $U \xrightarrow{\gamma} \mathbb{R}^n$ of M,

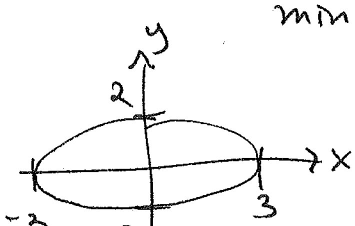
$$U \subset \mathbb{R}^k, \qquad \bar{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \longmapsto \gamma\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \gamma_1(\bar{t}) \\ \vdots \\ \gamma_n(\bar{t}) \end{pmatrix}$$

just find the <u>critical points</u> in $\mathbb{R}^k$ for the composite function $f \circ \gamma$:

$$U \xrightarrow{\gamma} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$$

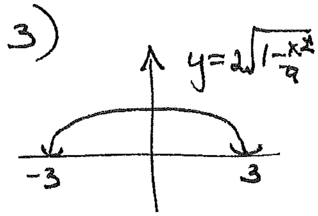
$$U \subset \mathbb{R}^k, \qquad \bar{t} \longmapsto \bar{\gamma}(\bar{t}) \longmapsto f\begin{pmatrix} \gamma_1(\bar{t}) \\ \vdots \\ \gamma_n(\bar{t}) \end{pmatrix}$$

<u>EXAMPLE:</u> What points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ minimize and maximize $f\begin{pmatrix} x \\ y \end{pmatrix} = x + y$?



We can parametrize on this $U = (-3, 3)$

via $y = +2\sqrt{1 - \frac{x^2}{9}} = g(x)$



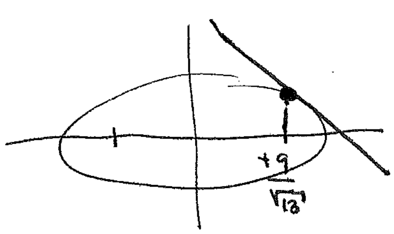
where $f\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} x \\ g(x) \end{pmatrix} = x + g(x)$

$$= x + 2\sqrt{1 - \frac{x^2}{9}}$$

so ~~the~~ critical points are where

$$0 = \frac{d}{dx}\left(x + 2\sqrt{1 - \frac{x^2}{9}}\right) = 1 + 2\left(\frac{1}{2}\right)\left(\frac{-2x}{9}\right)\left(1 - \frac{x^2}{9}\right)^{-\frac{1}{2}}$$

$$0 = 1 - \frac{2x}{9\left(1 - \frac{x^2}{9}\right)^{1/2}}$$



square

$$\frac{2x}{9\left(1 - \frac{x^2}{9}\right)^{1/2}} = 1$$

$$\frac{4x^2}{81\left(1 - \frac{x^2}{9}\right)} = 1, \quad 4x^2 = 81 - 9x^2, \quad 13x^2 = 81, \quad x = \pm\frac{9}{\sqrt{13}}$$