(27)

REMARKS:

① There is meaning attached to the Lagrange multipliers $\lambda_1, \dots, \lambda_{n-k}$.
If one imagines changing the constraint values $\bar F(\bar z) = \bar 0$ slightly to $\bar F(\bar z) = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{n-k} \end{pmatrix}$, so $\mathcal{L}\begin{pmatrix} \bar z \\ \bar\lambda \end{pmatrix} = f(\bar z) + \lambda_1(F_1(\bar z) - \epsilon_1) + \dots + \lambda_{n-k}(F_{n-k}(\bar z) - \epsilon_{n-k})$

one sees that $\lambda_i = \frac{\partial \mathcal{L}}{\partial \epsilon_i}$, and since $\mathcal{L}\begin{pmatrix} \bar z \\ \bar\lambda \end{pmatrix} = f(\bar z)$ on the manifold $\bar F(\bar z) = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{n-k} \end{pmatrix}$,

λ_i gives the approximate change in the extremal value of $f(\bar z)$ as one perturbs the constraint $F_i(\bar z) = 0$ to $F_i(\bar z) = \epsilon_i$ (called the shadow price for F_i in economics)

② There is a more complicated version of the Hessian/2nd derivative test, stated as THM 3.7.12 (proven in appendix A.14).

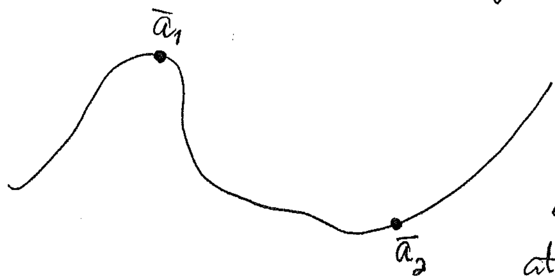
2/8/17

§3.8 Geometry of curves & surfaces

(— a whirlwind tour of some of Math 5378 "Differential geometry"!)

2nd derivatives and Hessians help us quantify curvature on curves and surfaces at a given point.

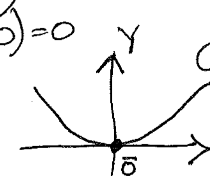
Given a curve $C \subset \mathbb{R}^2$, to quantify greater curvature at $\bar a_1$ than $\bar a_2$,



we put C into best coordinates near $\bar a \in C$ by translating $\bar a$ to $\bar 0$ and rotating to make the tangent line at $\bar a$ horizonal.

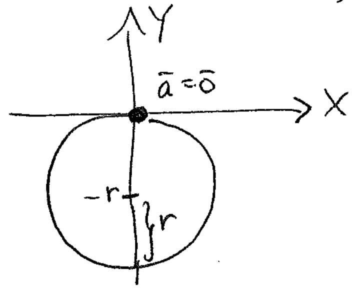
i.e. locally C near $\bar a = \bar 0$ is the graph of $Y = g(X)$ with $g(0) = 0$, $g'(0) = 0$

so $Y = \frac{g''(0)}{2} X^2 + o(|X|^2)$



DEF'N 3.8.1: The curvature $K(\bar a)$ at $\bar a$ on C is $K(\bar a) := |g''(0)|$ (Why absolute value of $g''(0)$?)
and the radius of curvature is $r = \frac{1}{K(\bar a)}$.

(motivating)
<u>EXAMPLE</u>: For a point on a circle of radius r, we may as well translate to this picture:



i.e. $Y = g(X) = \sqrt{r^2 - X^2} - r$

$$= r\left(\sqrt{1-\left(\tfrac{X}{r}\right)^2} - 1\right)$$

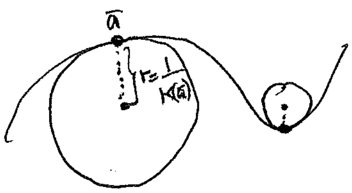
$$= r\left(\left(1 - \left(\tfrac{X}{r}\right)^2\right)^{\frac{1}{2}} - 1\right) = r\left(\cancel{\tbinom{1/2}{0}} + \tbinom{1/2}{1}\left(-\tfrac{X^2}{r^2}\right)^1 + \underbrace{\tbinom{1/2}{2}\left(-\tfrac{X^2}{r^2}\right)^2 + \dots}_{\text{in } \mathcal{O}(|X|^2)} \cancel{-1}\right)$$

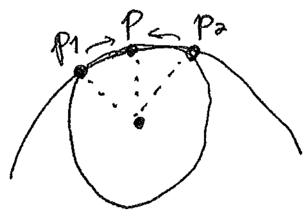
$$= \frac{-X^2}{2r} + \mathcal{O}(|X|^2) \qquad \text{i.e. } \frac{g''(0)}{2} = \frac{-1}{2r}$$

$$\kappa(\bar{a}) = \left|\tfrac{-1}{r}\right| = \frac{1}{r}$$

radius of curvature $= r$

<u>REMARK</u>: One can show that $r = \dfrac{1}{|\kappa(\bar{a})|}$ is the radius of the <u>osculating circle</u> to C at $\bar{a}$, informally defined as the best fit circle to C near $\bar{a}$, and a bit more formally, $\lim\limits_{\substack{p_1, p_2 \to p \\ \text{on } C}} \left(\begin{array}{c}\text{unique circle} \\ \text{containing } \{p_1, p, p_2\}\end{array}\right)$





When C is locally the graph of $y = f(x)$ near $\bar{a} = \binom{a}{f(a)}$, but $f'(a) \neq 0$, one can use this formula:

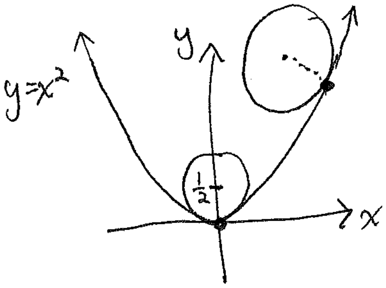
<u>PROP 3.8.2</u>: $$\kappa(\bar{a}) = \frac{|f''(a)|}{(1 + f'(a)^2)^{3/2}} \quad \left(\xrightarrow[\substack{\text{if} \\ y = g(x) \\ \text{with } g'(a)=0}]{} |g''(a)|, \text{ as it should}\right)$$

<u>proof</u>: A bit of a calculation, rotating coordinates; see pp 374-375. Nothing fancy, though. ∎

EXAMPLE (3.8.3): On the parabola $\underset{=f(x)}{y=x^2}$ at $\bar{a}=\begin{pmatrix}a\\a^2\end{pmatrix}$, so $f'(x)=2x$, $f''(x)=2$

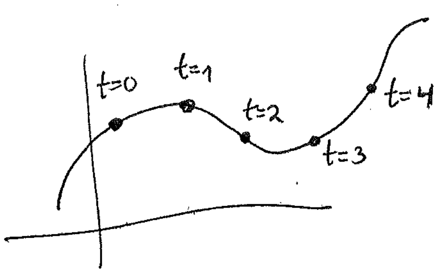
one has $\kappa(\bar{a}) = \dfrac{|f''(a)|}{(1+f'(a)^2)^{3/2}} = \dfrac{|2|}{(1+(2x)^2)^{3/2}} = \dfrac{2}{(1+4a^2)^{3/2}}$ $\xrightarrow{a=0}$ 2, so $r=\frac{1}{2}$

$\xrightarrow{a=\infty}$ 0, so $r=\infty$



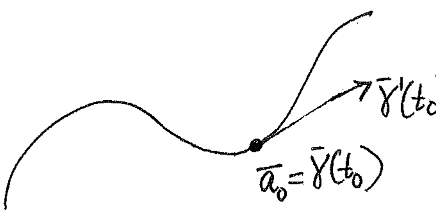
If one parametrizes C near $\bar{a}$ via arc-length, to be ~~analyzed~~ defined more carefully in Chap. 4

i.e. $C=\operatorname{im}\bar{\gamma}$ where $\mathbb{R}^1 \xrightarrow{\bar{\gamma}} \mathbb{R}^2$, $t \longmapsto \bar{\gamma}(t)$ and arc-length from $\gamma(a)$ to $\gamma(b)$ is always $b-a$



then one can show PROP 3.8.6 $\boxed{\kappa(\bar{a}_0) = |\bar{\gamma}''(t_0)|}$ if $\bar{a}_0=\gamma(t_0)$

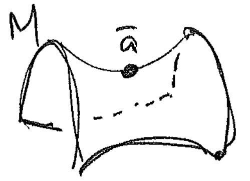
see EXER. 3.8.12 (but it uses other facts proven much later)



$\bar{\gamma}'(t_0)$ = velocity vector at t_0, a unit vector: $|\gamma'(t_0)|=1$

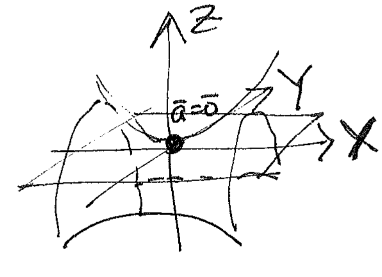
see also McCleary "Geometry from a differentiable viewpoint" p. 81 for a derivation from the osculating circle definition

What about notions of curvature at a point $\bar{a}$ on a surface M?



Again, start by putting a local parametrization of M near $\bar{a}$ into best coordinates: translate $\bar{a}$ to $\bar{0} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ in $\mathbb{R}^3$, and rotate in $\mathbb{R}^3$ to make the tangent plane $T_{\bar{a}}M$ horizontal

i.e. locally M near $\bar{a}$ is the graph of



$$Z = f\begin{pmatrix}X\\Y\end{pmatrix} \quad \text{with } f\begin{pmatrix}X\\Y\end{pmatrix} = 0$$
$$Df\begin{pmatrix}X\\Y\end{pmatrix} = [0 \ \ 0]$$

so
$$Z = f\begin{pmatrix}X\\Y\end{pmatrix} = \frac{1}{2}[X \ Y]\begin{bmatrix}A_{20} & A_{11}\\ A_{11} & A_{02}\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix} + \mathcal{O}\left(\left|\begin{bmatrix}X\\Y\end{bmatrix}\right|^2\right)$$
$$= \frac{1}{2}\bar{x}^T A \bar{x} + \mathcal{O}(|\bar{x}|^2) \quad \left(\text{where } A = \text{Hessian of } f \text{ at } \begin{pmatrix}0\\0\end{pmatrix}\right)$$

We'd like to extract curvature measures from A that don't depend on the orthonormal coordinate system $\begin{pmatrix}X\\Y\end{pmatrix}$ that we chose, e.g. A_{20}, A_{02}, A_{11} would be bad choices.

If we rotate coordinates

via $\begin{bmatrix}X'\\Y'\end{bmatrix} = Q\begin{bmatrix}X\\Y\end{bmatrix}$ for some orthonormal Q, i.e. $Q^T = Q^{-1}$ $\left(\text{so } Q = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\right)$

then it replaces A by $Q^TAQ = Q^{-1}AQ$.

One way to get invariant quantities here are the coefficients of the characteristic polynomial $\varphi_A(t) = \det(tI_{2\times2} - A)$ $\quad (= \varphi_{Q^{-1}AQ}(t)\,!)$

$$= \det\begin{bmatrix}t - A_{20} & -A_{11}\\ -A_{11} & t - A_{02}\end{bmatrix}$$
$$= t^2 - \underbrace{(A_{20} + A_{02})}_{\mathrm{Tr}A}t + \underbrace{(A_{20}A_{02} - A_{11}^2)}_{\det A}$$

DEF'N 3.8.7 & 3.8.8 The mean curvature at $\bar{a}$ on M is $\frac{1}{2}(A_{20} + A_{02}) =: H(\bar{a})$ (Sophie Germain, early 1800's)

The Gaussian curvature at $\bar{a}$ on M is $A_{20}A_{02} - A_{11}^2 =: K(\bar{a})$ (C.F. Gauss, 1820's)