(31) 2/9/2017 $\succ$

The book gives (PROP 3.8.10) the more general formula for computing them if the tangent plane $T_{\bar a}M$ is not horizontal:

$$z = f\begin{pmatrix}x\\y\end{pmatrix} = a_1 x + a_2 y + \frac{1}{2}[x \;\; y]^T \begin{bmatrix} a_{2,0} & a_{1,1} \\ a_{1,1} & a_{0,2}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + o\left(\left|\begin{bmatrix}x\\y\end{bmatrix}\right|^2\right)$$

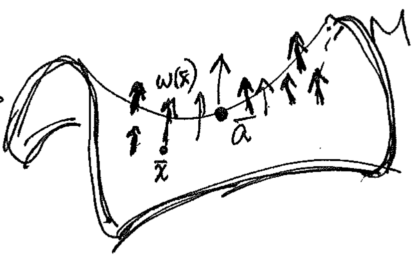
$\Longrightarrow$ mean curvature at $\binom{0}{0}=\bar a$: $$H(\bar a) = \frac{1}{2(1+c^2)^{3/2}}\left((1+a_2^2)a_{2,0} - 2a_1a_2a_{1,1} + (1+a_1^2)a_{0,2}\right)$$

Gaussian curvature at $\binom{0}{0}=\bar a$: $$K(\bar a) = \frac{a_{20}a_{02}-a_{11}^2}{(1+c^2)^2}$$ where $c := \sqrt{a_1^2+a_2^2}$ $(=|\nabla f|)$.

What does mean curvature "mean"?

As the book says, $H(\bar a)$ is telling us roughly how the surface would try to move normal to the tangent plane $T_{\bar a}M$ if we fixed its boundary and let it try to minimize the surface area:

in the $-\vec e_3$ direction in "best coordinates"



boundary fixed

This is made more precise in THM 5.4.4 later — if one fixes the boundary and considers a C^1 normal vector field $\vec w(\bar x)$ to M = graph of $\varphi: U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$

then the family of deformed surfaces
$$M_t = \{\text{graph of } \varphi_t(\bar x) := \bar x + t\vec w(\bar x)\}$$
have
$$\text{surface area}(M_t) = \text{surface area}(M) - 2t\int_M \vec H(\bar x)\cdot w(\bar x)\,|d^2\bar x| + o(t)$$

Define the mean curvature vector at $\bar a$ as $\vec H(\bar a) := H(\bar a)\vec e_3$ where $Z = f\binom{X}{Y}$ with $\begin{pmatrix}X\\Y\\Z\end{pmatrix} = X\vec e_1 + Y\vec e_2 + Z\vec e_3$

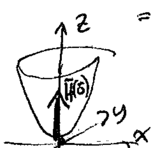
(see example in Fig. 5.4.2 on p. 549)

$H(\bar a)\cdot\vec e_3 \overset{\text{DEF'N}}{=:}$ (normal) mean curvature vector at $\bar x$ — in "best coordinates"

surface integral (ch.5)

Thus minimal surfaces (soap bubbles) spanning some boundary curve should have $H(\bar a) = 0$ at every point $\bar a$.

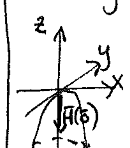
e.g. $z = x^2+3y^2 \rightsquigarrow \begin{bmatrix}2&0\\0&6\end{bmatrix}=A$, $= \frac{1}{2}\bar x^T A \bar x$



$H(\bar 0) = \frac{1}{2}(2+6) = 4 > 0$

$K(\bar 0) = 2\cdot 6 = 12 > 0$

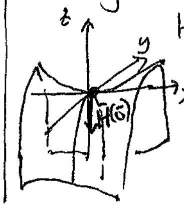
$z = -x^2-3y^2 \rightsquigarrow \begin{bmatrix}-2&0\\0&-6\end{bmatrix} = A$



$H(\bar 0) = \frac{1}{2}(-2-6) = -4 < 0$

$K(\bar 0) = (-2)(-6) = 12 > 0$

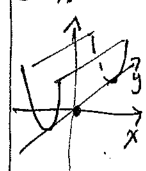
$z = x^2-3y^2 \rightsquigarrow \begin{bmatrix}2&0\\0&-6\end{bmatrix} = A$



$H(\bar 0) = \frac{1}{2}(2-6) = -2 < 0$

$K(\bar 0) = 2(-6) = -12 < 0$

$z = x^2 \rightsquigarrow \begin{bmatrix}2&0\\0&0\end{bmatrix} = A$



$H(\bar 0) = \frac{1}{2}(2+0) = 1 > 0$

$K(\bar 0) = 2\cdot 0 = 0$

$\vec H(\bar 0) = \bar 0$

(32) What does Gaussian curvature "mean"?

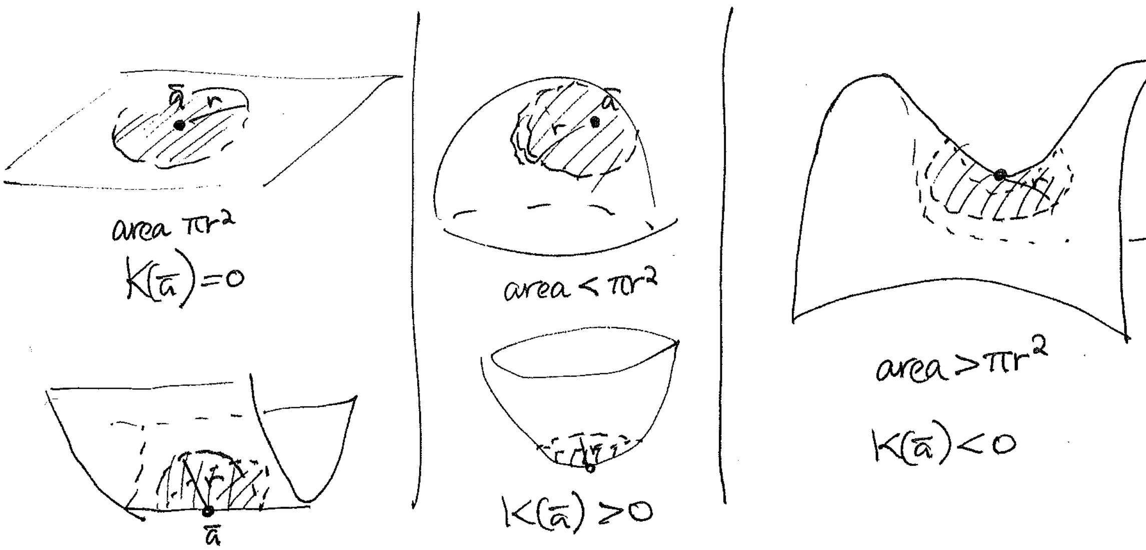
Gauss's "remarkable theorem" gives one answer: Theorema Egregium

THM 3.8.9. (proven as 5.4.1) Let $D_r(\bar{a}) := \{$ points $\bar{x} \in M$: $\exists$ a curve from $\bar{a}$ to $\bar{x}$ inside M of arc length at most $r\} \subset M$

"disk of radius r inside S about ā"

Then
$$\text{surface area}(D_r(\bar{a})) = \underbrace{\pi r^2}_{\text{good 1st approximation}} - \frac{\pi}{12}\overbrace{K(\bar{a})r^4}^{\text{correction}} + O(r^4)$$

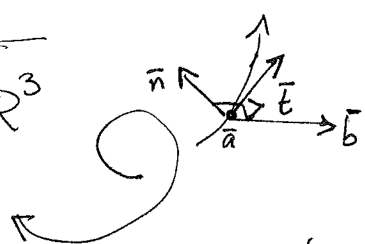
Gaussian curvature at $\bar{a}$ on M



See the 3 billy goats picture on p.378!

One needs to develop techniques for surface area integrals (Chap. 5) to prove this.

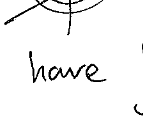
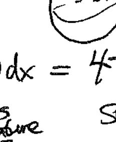
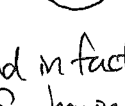
The rest of §3.8 is about curves in $\mathbb{R}^3$



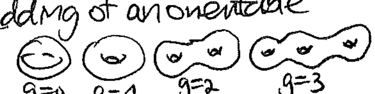
, picking a good coordinate frame, defining curvature, torsion at $\bar{a}$, finding formulas, etc.

REMARK: A cool Math 5378 result:

Gauss-Bonnet theorem: No matter how we embed a 2-dim'l sphere S into $\mathbb{R}^3$

e.g. $S=$  or $S=$  or $S=$ 

one will have $\int_S \underbrace{K(\bar{x})}_{\text{Gauss curvature at } \bar{x}}\,dx = 4\pi$. And in fact for any embedding of an orientable surface S_g having g "handles"



$$\boxed{\int_{S_g} K(\bar{x})\,dx = 2\pi\,\chi(S_g)}$$

Euler characteristic of $S_g = 2-2g$