(81)

3/24/2017 To get a sense of why the formula $\mathrm{vol}_k P(\bar{v}_1, \ldots, \bar{v}_k) = \sqrt{\det A^T A}$ makes sense, let's recall ~~the~~ notion of orthogonal $n\times n$ matrices Q as those for which $Q^TQ = I_n = QQ^T$ i.e. $Q^T = Q^{-1}$

PROPOSITION:

(a) If Q is $n\times n$ orthogonal, then $\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \bar{x} & \longmapsto & Q\bar{x} \end{array}$

preserves dot products: $(Q\bar{x})^T(Q\bar{y}) = \bar{x}^T \underbrace{Q^TQ}_{I_n} \bar{y} = \bar{x}^T\bar{y}$

hence it also preserves lengths $|Q\bar{x}| = \sqrt{(Q\bar{x})^T Q\bar{x}} = \sqrt{\bar{x}^T\bar{x}} = |\bar{x}|$

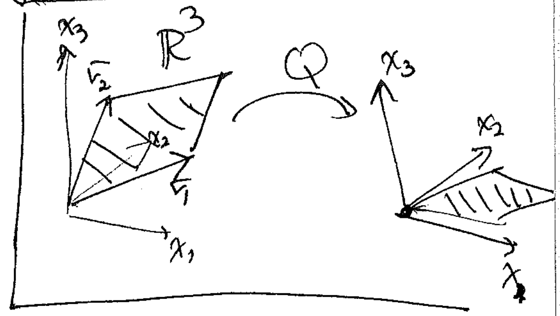
and angles (since $\measuredangle(\bar{x},\bar{y}) = \cos^{-1} \dfrac{\bar{x}^T\bar{y}}{|\bar{x}||\bar{y}|}$)

and preserves the $\mathrm{vol}_k(-)$ formula:

$$\mathrm{vol}_k P(Q\bar{v}_1, \ldots, Q\bar{v}_k) = \sqrt{\det (QA)^T QA} = \sqrt{\det A^T \underbrace{Q^TQA}_{A}} = \sqrt{\det A^TA}$$

(b) Given any $\bar{v}_1, \ldots, \bar{v}_k \in \mathbb{R}^n$, $\exists$ an $n\times n$ orthogonal matrix Q such that $Q\bar{v}_1, \ldots, Q\bar{v}_k \in \mathbb{R}^k \subset \mathbb{R}^n$

$$\mathbb{R}^k = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} n-k \text{ zeroes} \end{matrix} : x_i \in \mathbb{R} \right\}$$



(c) For $\bar{v}_1, \ldots, \bar{v}_k \in \mathbb{R}^k \subset \mathbb{R}^n$, $\quad \det\sqrt{A^TA} = \det\sqrt{B^TB} = |\det B| = \mathrm{vol}_k \underbrace{P(\hat{v}_1, \ldots, \hat{v}_k)}_{\text{in } \mathbb{R}^k}$

$\bar{v}_1 = \begin{bmatrix} \hat{v}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \bar{v}_k = \begin{bmatrix} \hat{v}_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

where $A = \underbrace{\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_k \\ | & & | \end{bmatrix}}_{k} \Big\} n$ where $B = \underbrace{\begin{bmatrix} | & & | \\ \hat{v}_1 & \cdots & \hat{v}_k \\ | & & | \end{bmatrix}}_{k} \Big\} k$

since $\bar{v}_i^T \bar{v}_j$

$$= \underbrace{\begin{bmatrix} -\hat{v}_i^T- \,|\, 0 \cdots 0 \end{bmatrix}}_{n} \begin{bmatrix} \hat{v}_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \hat{v}_i^T \hat{v}_j$$

proof: Only (b) wasn't already proven. Pick an orthonormal basis for $\mathrm{span}(\bar{v}_1, \ldots, \bar{v}_k)$, $\bar{u}_1, \ldots, \bar{u}_m$ so $m \leq k$, and extend them to an orthonormal basis $\bar{u}_1, \ldots, \bar{u}_m, \bar{u}_{m+1}, \ldots, \bar{u}_n$ for $\mathbb{R}^n$.

Then let $Q^{-1} := \begin{bmatrix} | & | & & | & | & & | \\ \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_m & \bar{u}_{m+1} & \cdots & \bar{u}_n \\ | & | & & | & | & & | \end{bmatrix}$, so Q sends $\bar{u}_1, \ldots, \bar{u}_m$ to $\bar{e}_1, \ldots, \bar{e}_m \in \mathbb{R}^k$ and sends $\bar{v}_1, \ldots, \bar{v}_k$ therefore also into $\mathbb{R}^k$ ∎

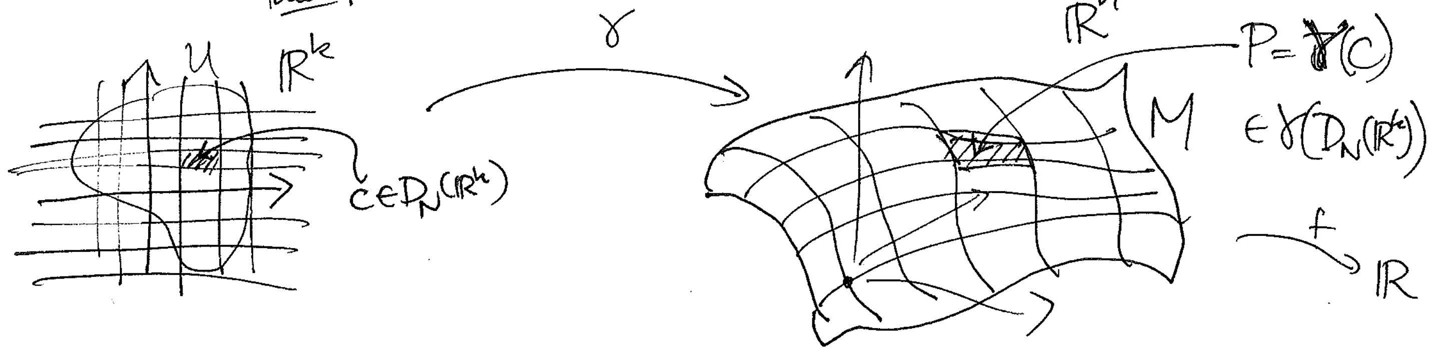
§ 5.2, 5.3 Integrating a function on a parametrized manifold

Given a k-dimensional manifold $M \subset \mathbb{R}^n$ with $k \le n$, how to define $\mathrm{vol}_k(M) = \int_M |d^k\vec{x}|$,

or more generally for a function $M \xrightarrow{f} \mathbb{R}$, how to define $\int_M f(\vec{x})\,|d^k\vec{x}|$?

If we have some "nice" parametrization $\underset{\mathbb{R}^k}{\overset{U}{\cap}} \xrightarrow{\gamma} \underset{\gamma(U)}{\overset{M}{\shortparallel}}$

worry about the fussy (§5.2) details of this later!



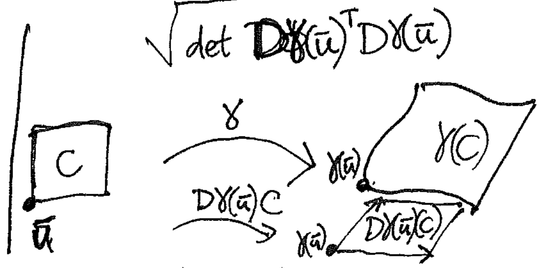
we might heuristically guess

$$\int_M f(\vec{x})\,|d^k\vec{x}| = \lim_{N\to\infty} \sum_{\substack{P \in \gamma(\mathcal{D}_N(\mathbb{R}^k)): \\ \overset{\shortparallel}{\gamma(C)} \quad P\cap M \neq \emptyset}} M_P(f)\,\mathrm{vol}_k(P)$$

$$= \lim_{N\to\infty} \sum_{\substack{C\in \mathcal{D}_N(\mathbb{R}^k): \\ C\cap U\neq\emptyset}} M_C(f\circ\gamma)\,\frac{\mathrm{vol}_k\,\gamma(C)}{\mathrm{vol}_k(C)}\cdot \mathrm{vol}_k(C)$$

$$\text{"="} \sum_{\substack{C\in \mathcal{D}_N(\mathbb{R}^k): \\ C\cap U\neq\emptyset}} M_C(f\circ\gamma)\,\underbrace{\lim_{N\to\infty}\frac{\mathrm{vol}_k\,\gamma(C)}{\mathrm{vol}_k(C)}}_{\sqrt{\det D\gamma(\bar{u})^T D\gamma(\bar{u})}}\cdot \mathrm{vol}_k(C)$$

$$= \int_U (f\circ\gamma)(\bar{u})\sqrt{\det D\gamma(\bar{u})^T D\gamma(\bar{u})}\;|d^k\bar{u}|$$



DEF'N 5.3.1
5.3.2: Given a nice parametrization $\underset{\mathbb{R}^k}{\overset{}{U}} \xrightarrow{\gamma} \underset{\mathbb{R}^n}{M}$ and $M \xrightarrow{f} \mathbb{R}$,

say f is integrable with respect to volume on M if

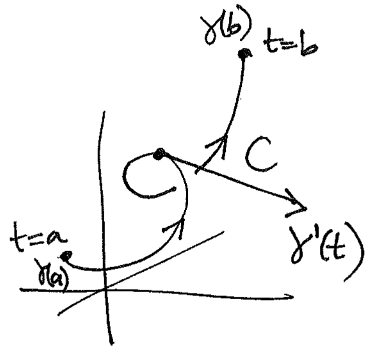
$$\int_M f(\bar{x})\,|d^k\bar{x}| := \int_U f(\bar{\gamma}(\bar{u}))\sqrt{\det D\bar{\gamma}(\bar{u})^T D\bar{\gamma}(\bar{u})}\;|d^k\bar{u}| \text{ exists.}$$

In particular, $\mathrm{vol}_k(M) = \int_M |d^k\bar{x}| := \int_U \sqrt{\det D\bar{\gamma}(\bar{u})^T D\bar{\gamma}(\bar{u})}\;|d^k\bar{u}|$ (if that integral exists.)

EXAMPLES:

① A parametrized curve $C \subset \mathbb{R}^n$

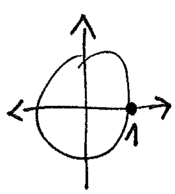
$$[a,b] \xrightarrow{\bar{\gamma}} C$$
$$t \longmapsto \bar{\gamma}(t) = \begin{pmatrix}\gamma_1(t)\\ \vdots \\ \gamma_n(t)\end{pmatrix}$$



has $D\bar{\gamma}(t) = \begin{bmatrix}\gamma_1'(t)\\ \vdots \\ \gamma_n'(t)\end{bmatrix}$, so $\sqrt{\det D\bar{\gamma}(\bar{u})^T D\bar{\gamma}(\bar{u})} = |\gamma'(t)|$

and arclength $\mathrm{vol}_1(C) = \int_{t=a}^{t=b} |\gamma'(t)|\,dt$

e.g. unit circle C



$$\underset{\mathbb{R}^1}{[0,2\pi)} \xrightarrow{\bar{\gamma}} C \qquad \text{has } \mathrm{vol}_1(C) = \int_{t=0}^{t=2\pi} 1\cdot dt = 2\pi \;\checkmark$$
$$t \longmapsto \begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$$

$$\gamma'(t) = \begin{bmatrix}-\sin(t)\\ \cos(t)\end{bmatrix}, \quad |\gamma'(t)| = 1$$

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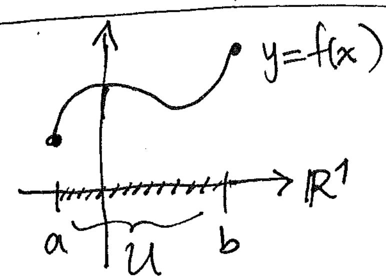
② (EXAMPLE 5.3.4 / 5.3.5) Graph of $y=f(x)$ in $\mathbb{R}^2$

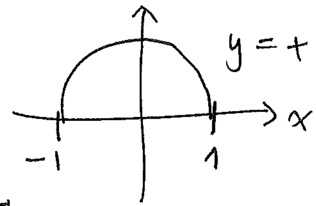


$$\underset{\mathbb{R}^1}{[a,b]} \xrightarrow{\bar{\gamma}} C$$
$$x \longmapsto \begin{pmatrix}x\\ f(x)\end{pmatrix}$$

has $D\bar{\gamma}(t) = \begin{bmatrix}1\\ f'(x)\end{bmatrix}$, so $\mathrm{vol}_1(C) = \int_{x=a}^{x=b} \sqrt{[1\;\, f'(x)]^T \begin{bmatrix}1\\ f'(x)\end{bmatrix}}\;dx$

$$= \int_{x=a}^{x=b} \sqrt{1+f'(x)^2}\;dx$$

e.g. upper semicircle S



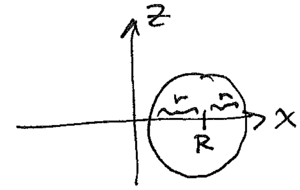
$y=+\sqrt{1-x^2}=f(x)$

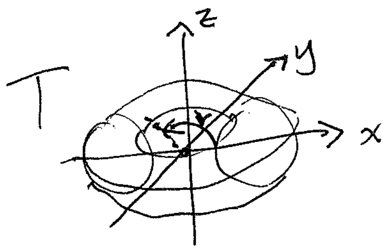
$$f'(x)=\frac{-2x}{2\sqrt{1-x^2}}=\frac{-x}{\sqrt{1-x^2}}$$

$$\mathrm{vol}_1(S)=\int_{x=-1}^{x=1}\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx$$

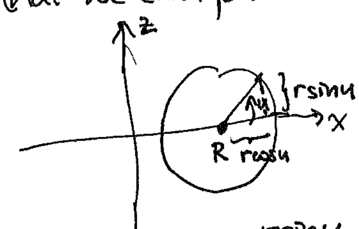
$$=\int_{x=-1}^{x=1}\sqrt{1+\frac{x^2}{1-x^2}}\,dx=\int_{x=-1}^{x=1}\frac{1}{\sqrt{1-x^2}}=\Big[\sin^{-1}(x)\Big]_{x=-1}^{x=1}=\frac{\pi}{2}-\left(\frac{-\pi}{2}\right)=\pi\ \checkmark$$

(3) (EXAMPLE 5.38) Rotating ~~this~~ circle in xz-plane about the z-axis gives a torus T in $\mathbb{R}^3$





that we can parametrize via 2 angles u, v:



$z = r\sin u$

$x=(R+r\cos u)\cos v$

$y=(R+r\cos u)\sin v$

$$\begin{array}{ccc} U & \xrightarrow{\ \gamma\ } & \mathbb{R}^3 \\ \cap & & \\ \mathbb{R}^2 \ \begin{pmatrix}u\\v\end{pmatrix} & \longmapsto & \gamma\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}(R+r\cos(u))\cos(v)\\(R+r\sin(u))\sin(v)\\r\sin(u)\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}\end{array}$$

$$D\gamma\begin{pmatrix}u\\v\end{pmatrix}=\begin{array}{cc} \partial/\partial u & \partial/\partial v \\ \left[\begin{matrix}-r\sin u\cos v \\ -r\sin u\sin v \\ r\cos(u)\end{matrix}\right. & \left.\begin{matrix}-(R+r\cos(u))\sin(v)\\(R+r\cos(u))\cos(v)\\0\end{matrix}\right]\end{array}$$

$$\underbrace{D\gamma\begin{pmatrix}u\\v\end{pmatrix}^T}_{2\times3}\underbrace{D\gamma\begin{pmatrix}u\\v\end{pmatrix}}_{3\times2}\overset{\text{scrap work!}}{=}\underbrace{\begin{bmatrix}r^2&0\\0&(R+r\cos u)^2\end{bmatrix}}_{2\times2}$$

so $$\mathrm{vol}_2(T)=\int_{u=0}^{u=2\pi}\int_{v=0}^{v=2\pi}\underbrace{\sqrt{\det\begin{bmatrix}r^2&0\\0&(R+r\cos u)^2\end{bmatrix}}}_{r(R+r\cos u)}\,du\,dv \overset{\text{Fubini + some scrapwork}}{=} 4\pi^2 rR$$