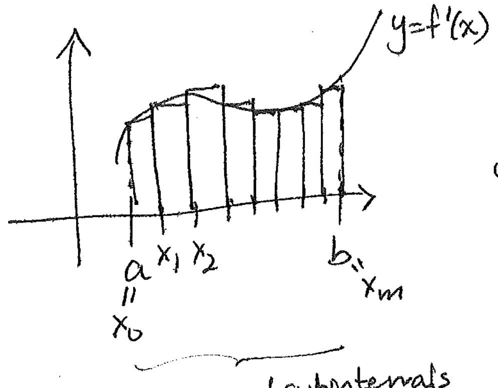
(124) What is slippery here is that the approximations "$\approx$" have small errors, and the errors have to be summed over sums that have more and more terms, so one might worry that they don't stay arbitrarily small!

5/1/2017 REASSURANCE 1:

The book sketches its 2^{nd} proof of Fund'l Thm. of Calculus, and bounds the errors ...



with $f \in C^2(\mathcal{U})$ for some $\mathcal{U} \overset{\text{open}}{\supset} [a,b]$

Informally first, for m large

$$\underbrace{\int_a^b f'(x)\,dx}_{(A)} \approx \underbrace{\sum_{i=0}^{m-1} f'(x_i)(x_{i+1}-x_i)}_{(B_m)} \approx \underbrace{\sum_{i=0}^{m} \big(f(x_{i+1}) - f(x_i)\big)}_{(C_m)} \overset{\text{telescoping sum}}{=} f(b) - f(a)$$

because Mean Value Thm. gives some $c_i \in (x_i, x_{i+1})$ with $f(x_{i+1}) - f(x_i) = f'(c_i)(x_{i+1}-x_i)$, and $f \in C^1 \Rightarrow f'(c_i) \approx f'(x_i)$

To bound the errors, given $\epsilon > 0$, suffices to show $\exists M$ such that $\forall m \geq M$

$$|(A) - (B_m)| < \epsilon$$

and $|(B_m) - (C_m)| < \epsilon$.

For $|(A) - (B_m)| < \epsilon$, this is just Riemann integrability of f', which is continuous on $[a,b]$, since $f \in C^2(\mathcal{U})$ for $[a,b] \subset \mathcal{U}$.

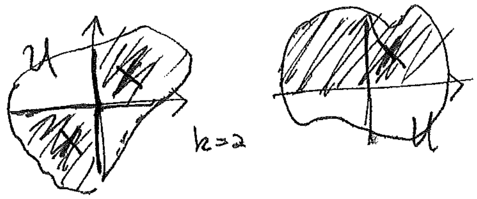
For $|(B_m) - (C_m)| < \epsilon$, use uniform continuity of f' on $[a,b]$ (f' is continuous, $[a,b]$ is compact) to choose M so $m \geq M$ implies $|f'(c_i) - f'(x_i)| \leq \epsilon$

and then
$$|(B_m) - (C_m)| = \left| \sum_{i=0}^{m-1} f'(x_i)(x_{i+1}-x_i) - \underbrace{\big(f(x_{i+1}) - f(x_i)\big)}_{= f'(c_i)(x_{i+1}-x_i)} \right| = \left| \sum_{i=0}^{m-1} \big(f'(x_i) - f'(c_i)\big)(x_{i+1}-x_i) \right|$$
$$\leq \sum_{i=0}^{m-1} |f'(x_i) - f'(c_i)|\,|x_{i+1}-x_i|$$
$$\leq \epsilon \sum_{i=0}^{m-1} |x_{i+1}-x_i| \leq \epsilon(b-a).$$

<u>REASSURANCE 2</u>: The book proves a <u>very</u> special case where one can similarly bound the errors in <u>PROP 6.9.7</u>,

where $k=n$ i.e. $X \subset \mathbb{R}^k$

and X is a union of closed quadrants intersected with $\mathcal{U} \subset \mathbb{R}^k$ bounded



They then use this as a Lemma in the full proof in App. A.26, but there are a lot more technicalities, and one needs <u>pull-backs</u> of differential forms (which are <u>not</u> <u>hard</u>), etc.

§6.12 Closed versus exact forms & potentials

<u>DEF'N</u> (not in book) A k-form $\varphi \in A^k(\mathcal{U})$ for $\mathcal{U}^{open} \subset \mathbb{R}^n$

is called • <u>closed</u> if $d\varphi = 0$

• <u>exact</u> if $\varphi = d\omega$ for some $\omega \in A^{k-1}(\mathcal{U})$

<u>EXAMPLES</u>:

① Since $d(\underbrace{d\omega}_{\varphi}) = 0$, exact $\Rightarrow$ closed <u>always</u>

(but we'll see the converse depends on whether $\mathcal{U}$ has "holes"!)

② For $\bar{F} : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field (or $\mathbb{R}^2 \to \mathbb{R}^2$)

$\varphi = W_{\bar{F}}$ <u>closed</u> means $\operatorname{curl}(\bar{F}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

<u>exact</u> means $\bar{F} = \nabla f$ for some $\overset{\text{potential}}{f} : \mathbb{R}^3 \to \mathbb{R}$ (or $\mathbb{R}^2 \to \mathbb{R}$)

③ For example, if $\bar{F} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ (<u>NOTE</u>: $\varphi = d\theta$ where $\theta\begin{pmatrix} x \\ y \end{pmatrix} = \tan^{-1}\left(\frac{y}{x}\right)$ from polar coords!)

then we claim $\varphi \stackrel{\text{def}}{=} W_{\bar{F}} = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is <u>closed</u>, but <u>not exact</u>

$\in A^2(\mathcal{U})$ where $\mathcal{U} = \mathbb{R}^2 - \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

(126) Check: $d\varphi = d\left(-\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right)$

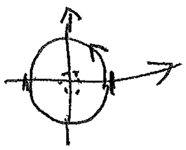
$$= \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2}\, dy\wedge dx + \frac{(x^2+y^2)(+1)-(x)(2x)}{(x^2+y^2)^2}\, dx\wedge y$$

$$= \frac{1}{(x^2+y^2)^2}\Big[\underbrace{2(x^2+y^2)-2x^2-2y^2}_{0}\Big]\, dx\wedge dy = 0 \checkmark$$

So φ is closed.

But $\varphi \neq df$ for some $f \in A^0(U)$, else one would have

$$0 = \int_C \varphi = \int_0^{2\pi} \Big(\underbrace{-\sin t}_{\frac{-y}{x^2+y^2}}\ \underbrace{(-\sin t)}_{\cdot\, dx} + \underbrace{\cos t}_{\frac{x}{x^2+y^2}} \cdot \cos t\Big)\, dt = \int_0^{2\pi} dt = 2\pi$$

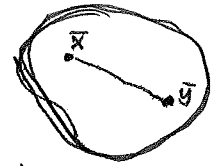
$\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$ $C =$ 

$\uparrow$ $\gamma\uparrow$

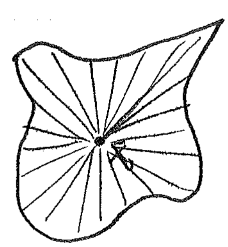
t $[0,2\pi]$ $D\gamma(t) = \begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$

The key thing here was the "hole" at $\binom{0}{0}$ in U.

This can never happen if U is convex, i.e. $\bar x, \bar y \in U \Rightarrow$ the line segment $[\bar x, \bar y] := \{t\bar x + (1-t)\bar y : t \in [0,1]\} \subset U$



or even more generally if U is star-shaped, i.e. $\exists \bar x_0 \in U$ with $[\bar x_0, \bar y] \subset U$ $\forall \bar y \in U$

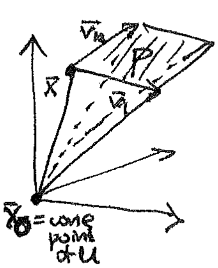


for this reason...

DEF'N 6.12.7 / 6.12.9: Given a star-shaped $U^{open} \subset \mathbb{R}^n$, one can define the cone operator $A^k(U) \xrightarrow{c} A^{k-1}(U)$

$\varphi \longmapsto c\varphi$

by $c\varphi(P_{\bar x}(\bar v_1,\dots,\bar v_{k-1})) := \lim_{h\to 0} \frac{1}{h^{k-1}} \int_{cone(P_{\bar x}(h\bar v_1,\dots,h\bar v_{k-1}))} \varphi$ where $cone(P) =$



(127) and then...

THM 6.12.12 (Poincaré's lemma) For star-shaped $U^{\text{open}} \subset \mathbb{R}^n$,

any k-form $\varphi \in A^k(U)$ has

$$\varphi = d(c\varphi) + c(d\varphi)$$

and hence any closed k-form ($d\varphi = 0$) has $\varphi = d(c\varphi)$ exact.

We don't have time to prove this, but it's not hard – see §6.12.