Math 4707 Intro to combinatorics and graph theory
Fall 2011, Vic Reiner
Midterm exam 1- Due Wednesday Dec. 14, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total; 10 points each part)
(a) How many rearrangements are there of the letters in the word STRENGTHENS? For example, EEGHNNRSSTT is one of them.

(b) How many such rearrangements have *all* the properties listed here?:

- the two E's appear adjacent to each other, **but**
- the two N's *do not* appear adjacent to each other, **and**
- the two S's *do not* appear adjacent to each other, **and**
- the two T's *do not* appear adjacent to each other.

2. (20 points total; 10 points each part) Recall that we defined the Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ by $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0, F_1 = 1$.

(a) Find the smallest positive integer m such that one can predict F_n mod 4 (that is, the remainder of F_n upon division by 4) if one knows n mod m (that is, the remainder of n upon division by m).

More precisely, formulate a conjecture which predicts, given the remainder n mod m, what the remainder F_n mod 4 will be, similar to the format of the theorem in Problem 4 on our first midterm exam.

(b) Prove the conjecture that you just stated in (a).

3. (20 points total; 5 points each part) For each of the following families of graphs, write down a formula for the number of *perfect matchings* in the graph, that is, the number of choices of a subset of edges M for which each vertex is incident to *exactly one* edge of M. Note that your answer will be a function of the parameters n or m, n in each case.

(a) The *complete bipartite graph* $K_{m,n}$ on $X \sqcup Y$ with $|X| = m, |Y| = n$.

(b) The *complete graph* K_n on n nodes *(without multiple edges or loops).*

(c) The *cycle* C_n with n edges and n vertices.

(d) The graph $G_{m,n}$ for $n \geq 3$ and $m \geq 1$ which has $m + n$ vertices, $m+n$ edges, and consists of a path with m edges attached at one of its endoints to a cycle with n edges. Formally, define $G_n = (V, E)$ where $V = \{1, 2, \ldots, m + n\}$ and

$$\begin{aligned} E = \{ &\{1, 2\}, \{2, 3\}, \ldots, \{m, m + 1\}, \\ &\{m + 1, m + 2\}, \{m + 2, m + 3\}, \ldots, \{m + n - 1, m + n\}, \\ &\{m + n, m + 1\} \} . \end{aligned}$$

(Hint: Don't even think about doing part (d) until you've drawn the picture of $G_{m,n}$ for a few values of m and n.)

4. (20 points total; 10 points each part) Let $G = (V, E)$ be a bipartite graph with vertices partitioned $V = X \sqcup Y$, and assume

- every x in X has the same degree $d_X \geq 1$, and
- every y in Y has the same degree $d_Y \geq 1$.

(a) Prove that $\frac{d_X}{d_Y} = \frac{|Y|}{|X|}$.

(b) Assuming without loss of generality that $d_X \geq d_Y$, show that there exists at least one matching $M \subset E$ with number of edges $|M| = |X|$.

5. (20 points; 10 points each part) Let G be a *connected, planar, bipartite* graph, with at least 3 edges, having no self-loops, no multiple/parallel edges, and *no cycles of length* 4.

Denote by v, e, f the number of vertices, edges, and faces/regions into which G dissects the plane (including the unbounded region).

(a) Prove that $f \leq \frac{e}{3}$.

(b) Prove that $e \leq \frac{3}{2}v - 3$.