Math 4707 Intro to combinatorics and graph theory
Fall 2011, Vic Reiner
Midterm exam 1- Due Wednesday Nov. 23, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Prove that a tree having at least one vertex with degree d has at least d distinct leaves (= vertices of degree one).

2. (20 points total) In dominoes, a standard package has $\binom{8}{2} = 28$ dominoes, each with two ends labelled by numbers from $\{0, 1, 2, 3, 4, 5, 6\}$, and with each possible unordered pair of labels occurring once, so the $i - j$ domino is the same as the $j - i$ domino:

$$\begin{array}{ccccccc} 0-0, & 0-1, & 0-2, & 0-3, & 0-4, & 0-5, & 0-6, \\ & 1-1, & 1-2, & 1-3, & 1-4, & 1-5, & 1-6, \\ & & 2-2, & 2-3, & 2-4, & 2-5, & 2-6, \\ & & & 3-3, & 3-4, & 3-5, & 3-6, \\ & & & & 4-4, & 4-5, & 4-6, \\ & & & & & 5-5, & 5-6, \\ & & & & & & 6-6 \end{array}$$

The goal is to lay them out touching two-at-a-time end-to-end in one long cycle, but only touching at ends with matching labels, e.g. the $2 - 5$ and $5 - 4$ dominoes can touch at their ends labelled 5.

Prove this is possible, without exhibiting such a cycle explicitly, by proving this: given a similar pack of $\binom{n+2}{2}$ dominoes having ends labelled with unordered pairs from $\{0, 1, 2, \ldots, n\}$ then this goal is possible if and only if n is even.
(**Hint:** how does this relate to Euler tours in some graph?)

3. (20 points) (Problem 7.2.11 on p. 134 of our text)
Prove that a graph G with *no multiple edges and no self-loops* having n vertices and strictly more than $\binom{n-1}{2}$ edges must be connected.

4. (20 points) Your company has 6 employees $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and 6 tasks to perform $\{y_1, y_2, y_3, y_4, y_5, y_6\}$, but each employee has a different set of tasks they are capable of doing:

employee	tasks they can do
x_1	$\{y_2, y_4, y_5\}$
x_2	$\{y_1, y_2, y_3, y_5, y_6\}$
x_3	$\{y_2, y_4, y_5\}$
x_4	$\{y_2, y_4\}$
x_5	$\{y_2, y_4, y_5\}$
x_6	$\{y_1, y_3, y_5, y_6\}$

Match each employee to at most one task, so that different employees end up doing different tasks, in such a way that the maximum number of tasks are performed. Prove that your answer attains the maximum.

5. (20 points total)
(a) (5 points) Prove that in a tree T on labelled vertex set $V = \{1, 2, \ldots, n\}$, if $\deg_T(i)$ denotes the degree of vertex i, then

$$\sum_{i=1}^{n} (\deg_T(i) - 1) = n - 2.$$

(b) (10 points) Prove that for any positive integers $(d_1, \ldots, d_n)$ satisfying $(d_1 - 1) + \cdots + (d_n - 1) = n - 2$, the number of different labelled trees on vertex set $V = \{1, 2, \ldots, n\}$ with $\deg_T(i) = d_i$ is the multinomial coefficient

$$\binom{n-2}{d_1 - 1,\ d_2 - 1,\ \ldots,\ d_n - 1}.$$

(c) (5 points) Assuming part (b), prove the following for $n \geq 2$:

$$\sum_T x_1^{\deg_T(1)} x_2^{\deg_T(2)} \cdots x_n^{\deg_T(n)} \quad = \quad x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}$$

where the sum runs over all labelled trees on the n vertices $\{1, 2, \ldots, n\}$. For example, here is the picture when $n = 3$:

$1 - 2 - 3 \qquad 1 - 3 - 2 \qquad 2 - 1 - 3$

$$\begin{aligned} x_1^1 x_2^2 x_3^1 \quad + \quad x_1^1 x_2^1 x_3^2 \quad + \quad x_1^2 x_2^1 x_3^1 \quad &= \quad x_1 x_2 x_3 (x_1 + x_2 + x_3)^1 \\ &= \quad x_1 x_2 x_3 (x_1 + x_2 + x_3)^{3-2} \end{aligned}$$