Math 5285 Honors abstract algebra
Fall 2007, Vic Reiner
Midterm exam 1- Due Wednesday October 10, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (15 points total) Artin's Chapter 1 Miscellaneous Problems # 7 on p. 37.

2. (16 points total) Let p be a permutation of the numbers $\{1, 2, \ldots, n\}$, and let $\mathrm{inv}(p)$ be the number of pairs (i, j) with $1 \leq i < j \leq n$ for which $p(i) > p(j)$.

Prove that $\mathrm{sign}(p) = (-1)^{\mathrm{inv}(p)}$.

3. (15 points total) Prove that for any matrix $A \in \mathbb{R}^{m \times n}$, there exist matrices $P \in GL_m(\mathbb{R})$ and $Q \in GL_n(\mathbb{R})$ and a nonnegative integer r, such that $D := PAQ$ has exactly r ones down its diagonal and zeroes elsewhere.

In other words, D has

$$d_{1,1} = d_{2,2} = \cdots = d_{r,r} = 1$$

and $d_{i,j} = 0$ for all other values of i, j.

(Hint: putting A into row-reduced echelon form helps.)

4. (4 points each, 24 points total) For each of the following sets and composition laws, indicate whether or not it is a group. If so, explain why, and if not explain why not.

(a) $\{A \in \mathbb{Z}^{n\times n} : \det(A) \in \{+1, -1\}\}$, under matrix multiplication.

(b) $\{A \in \mathbb{Z}^{n\times n} : \det(A) \in 2\mathbb{Z}\}$, under matrix multiplication.

(c) $\{A \in \mathbb{R}^{n\times n} : \det(A) > 0\}$, under matrix multiplication.

(d) Permutations in S_n that have exactly one cycle (of size n) in their cycle notation, under composition.

(e) $\{z = x + iy \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\} : x, y \in \mathbb{Q}\}$, under complex multiplication.

(f) Elementary matrices in $\mathbb{R}^{n\times n}$ of all types ((i), (ii), (iii)) together with the identity matrix $I_{n\times n}$, under matrix multiplication.

5. (5 points each; 15 points total) Let $\phi : G \longrightarrow G'$ be a homomorphism between two groups G, G'.

(a) Prove that if $H' \subset G'$ is a subgroup, then

$$\phi^{-1}(H') := \{g \in G : \phi(g) \in H'\}$$

is a subgroup of G.

(b) Prove that if H' is a *normal* subgroup of G', then $\phi^{-1}(H')$ as defined above is a normal subgroup of G.

(c) Let $H \subset G$ be a subgroup. Does this imply that its image

$$\phi(H) := \{\phi(h) : h \in H\}$$

is a subgroup of G'? Either prove this, or falsify it with an explicit counterexample.

6. (15 points total) Prove that if a matrix $A \in \mathbb{R}^{n\times n}$ has $A^t = -A$ and A is invertible then n must be even.