Math 5285 Honors fundamental structures of algebra
Fall 2018, Vic Reiner
Midterm exam 2- Due Wednesday November 14, in class

Instructions: There are 5 problems. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Recall that for $z = a + bi$ in $\mathbb{C}$, so that $a, b \in \mathbb{R}$, its *modulus* or *complex absolute value* is $\|z\| := \sqrt{a^2 + b^2}$.

(a) (5 points) Prove that for z in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, one has $\|z^{-1}\| = 1/\|z\|$.

(b) (5 points) Define $\mathbb{Z}[i] := \{a + bi \in \mathbb{C} : a, b, \in \mathbb{Z}\}$. Show that $\mathbb{Z}[i]$ is closed under addition and multiplication.

(c) (10 points) Defining the multiplicative group $\mathbb{Z}[i]^\times := \{z \in \mathbb{Z}[i] : z^{-1} \in \mathbb{Z}[i]\}$, prove that $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$.

2. (20 points total) Inside $G := GL_2(\mathbb{R})$, consider the subset

$$H := \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} : z \in \mathbb{Z} \right\}.$$

(Note: the $\mathbb{Z}$ inside the definition is *not* a typo, and is not intended to be an $\mathbb{R}$.)

(a) (5 points) Prove that H is a subgroup, and that $H \cong \mathbb{Z}^+$.

(b) (5 points) Prove that $g = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ in G satisfies $gHg^{-1} \subset H$, but $gHg^{-1} \neq H$.

(c) (10 points) Prove this would have been impossible if G had been *finite*, that is, whenever H is a subgroup of a finite group G, and g in G satisfies $gHg^{-1} \subset H$, then it forces $gHg^{-1} = H$.

3. (25 points total, 5 points each) Prove, or disprove via explicit counterexamples, the following assertions.

(a) The dihedral group D_3 of order 6 is isomorphic to the symmetric group S_3.

(b) An infinite group G can never act transitively on a finite set S.

(c) A finite group G can never act transitively on an infinite set S.

(d) Two subgroups H_1, H_2 of a finite group G that have $|H_1| = |H_2| = p^e$ for some prime p and $e \geq 1$ must be *conjugate* within G. That is, there must exist some g in G with $gH_1g^{-1} = H_2$.

(e) There exists a group G of size $|G| = 49$ whose center $Z(G)$ has $|Z(G)| = 7$.

4 (20 points total, 10 points each) A group G is called *simple* if G has no normal subgroups except for $\{1\}$ and G itself.

(a) Prove that a group G with $|G| = 256$ is never simple.

(b) Prove that a group G with $|G| = pq$ for distinct primes p, q is never simple.

5. (15 points total) For $n \geq 3$, let D_n be the dihedral group of order $2n$, the linear symmetries of a regular n-sided polygon,. Within D_n, let r be the clockwise rotation through $\frac{2\pi}{n}$.

(a) (5 points) Prove that for any m in $\mathbb{Z}$, the cyclic subgroup $\langle r^m \rangle$ is normal in D_n.

(b) (10 points) Prove that if $d = \gcd(n, m) \geq 3$, then $D_n / \langle r^m \rangle \cong D_d$.