Math 5285 Honors abstract algebra
Spring 2008, Vic Reiner
Midterm exam 2- Due Wednesday April 16, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points) Let α be the positive real 4^{th} root of 2 and let β be the positive real square root of 3. Find an element γ in $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$ for which $\mathbb{K} = \mathbb{Q}(\gamma)$ (and prove it).

2. (20 points total; 5 points each part) As in Problem 1, let α be the positive real 4^{th} root of 2. Let $\mathbb{K} = \mathbb{Q}(\alpha)$.
(a) Compute the degree $[\mathbb{K} : \mathbb{Q}]$ (with proof).
(b) Prove or disprove: there exists an element σ of the Galois group $G(\mathbb{K}/\mathbb{Q})$ that negates α^2, that is, that sends $\sqrt{2} \overset{\sigma}{\longmapsto} -\sqrt{2}$.
(c) Describe the Galois group $G(\mathbb{K}/\mathbb{F})$.
(d) Is $\mathbb{K}/\mathbb{F}$ a Galois extension? Explain your answer.

3. (30 points total; 15 points each part) Let $f(x)$ in $\mathbb{Q}[x]$ have form as in (a) or (b) below. By computing $\gcd(f(x), f'(x))$ in $\mathbb{Q}[x]$, find an integer polynomial expression $D(b, c)$ (that is, an expression $D(b, c)$ lying in $\mathbb{Z}[b, c]$) with the following property: $f(x)$ has multiple roots when one passes to its spliting field if and only if $D(b, c) = 0$.
(a) $f(x) = x^2 + bx + c$ in $\mathbb{Q}[x]$ is a monic quadratic polynomial.
(b) $f(x) = x^3 + bx + c$ in $\mathbb{Q}[x]$ is a monic cubic polynomial that has zero coefficient on the x^2 term[1].

4. (30 points total; 10 points each part.) Let p be a prime number. Find, as a function of p, the number of irreducible polynomials of degree d in $\mathbb{F}_p[x]$ when
(a) $d = 2$,
(b) $d = 3$,
(c) $d = 4$.
(Hint for all parts: How does the irreducible factorization of $x^{p^d} - x$ in $\mathbb{F}_p[x]$ look?)

[1] which can always be obtained from a monic cubic polynomial $x^3 + ax^2 + bx + c$ by replacing x with $x - \frac{a}{3}$.