Math 5286 Honors fundamental structures of algebra– 2nd semester
Spring 2019, Vic Reiner
Midterm exam 1- Due Wednesday March 6, in class

Instructions: There are 4 problems. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (40 points total, 5 points each part) True or False?
True assertions must be proven, and false assertions must be disproven.

(a) For any polynomial $f(x)$ in $\mathbb{F}_5[x]$ of degree 1000, the cardinality of the quotient ring $\mathbb{F}_5[x]/(f(x))$ will be 5^{1000}.

(b) The ring $\mathbb{F}_2[x]/(x^{62} + x^{30} + x^{18} + x^8 + 1)$ is a field.

(c) The ring $\mathbb{Q}[x]/(x^{62} + 15x^{30} + 12x^{18} + 81x^8 + 6)$ is a field.

(d) The ring $\mathbb{Z}[i]/(229)$ is a domain.

(e) For every ring R, the units $R[x]^\times$ inside $R[x]$ are exactly the constant units $R^\times$.

(f) One has a ring isomorphism $\mathbb{R}[x]/(x^2 + x + 1) \cong \mathbb{C}$.

(g) One has a ring isomorphism $\mathbb{F}_2[x]/(x^3 + x + 1) \cong \mathbb{F}_2[y]/(y^3 + y^2 + 1)$.

(h) One has a ring isomorphism $\mathbb{Z}[x]/(2x - 1) \cong \mathbb{Z}[y]/(8y - 1)$.

2. (20 points total, 10 points each part)

Let $R = \mathbb{F}_7[x]/(x^3 + x + 1)$.

(a) Prove that R is a field.

(b) Prove that R contains an element whose multiplicative order is 9.

3. (20 points) Let $\mathbb{F}_q$ be a finite field of size $q = 7^3$.
Prove that there are exactly 12 subgroups H with $SL_n(\mathbb{F}_q) \leq H \leq GL_n(\mathbb{F}_q)$.

4. (20 points) For $n \in \{1, 2, 3, \ldots\}$ define

$$f_n(x) := 1 + x + x^2 + \cdots + x^{n-1}.$$

Prove that in $\mathbb{Q}[x]$ one has

$$GCD(f_n(x), f_m(x)) = f_d(x)$$

where $d = GCD(m, n)$ in $\mathbb{Z}$.

(Hint: What is $GCD(x^n - 1, x^m - 1)$?)